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The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces

by

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Introduction. Let $\mathscr{A}$ be a class of Banach spaces. A Banach space X is said to be *universal* (resp. *isometrically universal*) for $\mathscr{A}$ if each member of $\mathscr{A}$ is isomorphic, i.e. linearly homeomorphic (resp. isometrically isomorphic), to a closed linear subspace of X. The Banach-Mazur theorem (cf. [1], p. 187, th. 10) asserts that the space C of all continuous scalar-valued functions on the unit interval is isometrically universal for all separable Banach spaces. Other examples of separable Banach spaces which are universal for the class of all separable Banach spaces have been exhibited in [8], [9], [15], [16] and [17]. The space constructed in in [8] and [9] has the interesting additional property that all its finite-dimensional subspaces are "universally located". In [14] and [4] it is proved that the Fréchet space (= locally convex complete linear metric space) of all continuous functions on the real line is universal for all separable Fréchet spaces. In [10] it is shown that the space of all infinitely differentiable functions on the real line is universal for all nuclear Fréchet spaces.

The purpose of the present paper is to prove certain negative results. We show that if X is a Banach space universal for all separable reflexive Banach spaces, then X^*, the dual to X, is non-separable. Consequently, there is no separable reflexive Banach space universal for all separable reflexive Banach spaces. This has solved negatively the problem posed by Banach and Mazur ([18], Problem 49; cf. also [11]). Our results strengthen, in the separable case, Theorem 3.1 (i)-(iii) of [13], where the non-existence of separable reflexive spaces isometrically universal for all separable reflexive spaces is proved. Let us mention finally that the non-existence of a finite-dimensional space universal for all two-dimensional Minkowski spaces was established in [2] (cf. also [11] and [7]).

The main results of the present paper are contained in Section 3. Sections 1 and 2 are devoted to the study of the concept of the index of a Banach space. The idea of this concept goes back to some constructions of Zalcwasser [19] and Gillespie and Hurwitz [6].

Notation. Throughout this paper the capital letters $X, Y, \ldots$ denote Banach spaces. By X^* we denote the dual space to X. By "subspace" we mean "closed linear subspace". If $u: X \to Y$ is a linear operator, then $u^*: Y^* \to X^*$ denotes the linear operator adjoint to u (cf. [5], p. 478). A Banach space Y is said to be *isomorphic* to a Banach space X if there exists an isomorphism (= linear homeomorphism) $u: Y \to X$ that is a one-to-one bicontinuous linear operator from Y onto X. By the *w-topology* of X we mean the X^*-topology of X (cf. [3], p. 19, and [5], p. 149). By the *w^*-topology* of X^* we mean the X-topology of X^* (cf. [3], p. 17, and [5], p. 462). The symbol $x_n \overset{w}{\to} x_0$ (resp. $f_n \overset{w^*}{\to} f_0$) denotes that the sequence (x_n) of elements of X is w-convergent to x_0, i.e. (x_n) is convergent to x_0 in the w-topology or equivalently

$$\lim_n f(x_n) = f(x_0)$$

for every f in X^* (resp. the sequence (f_n) of elements of X is w*-convergent to f_0, i.e. (f_n) is convergent to f_0 in the w*-topology of X^* or equivalently

$$\lim_n f_n(x) = f_0(x)$$

for every $x \in X$).

The Greek letters $\alpha, \beta, \ldots$ will be reserved for denoting ordinal numbers. By ω_1 we shall denote the first uncountable ordinal number. If α and α' are ordinals such that

$$\alpha = \alpha' + 1,$$

then α' is called the *predecessor* of α.

1. Now we give

1.1. Definition. Let G and Γ denote bounded sets in a Banach space X and in its dual X^* respectively. Let us assume that Γ is w*-compact. Let $\varepsilon > 0$. To each countable ordinal α we assign by transfinite induction a set $P_\alpha(\varepsilon; G, \Gamma)$ in X as follows:

(1.1.1) $P_0(\varepsilon; G, \Gamma) = \Gamma$.

(1.1.2) If α' is the predecessor of α, then $P_\alpha(\varepsilon; G, \Gamma) = \{f \in X^*$: there exist $x_m \in G$ and $f_m \in P_{\alpha'}(\varepsilon; G, \Gamma)$ for $m = 1, 2, \ldots$ such that $f_m \overset{w^*}{\to} f$; $x_m \overset{w}{\to} 0$, $\limsup_m |f_m(x_m)| \geqslant \varepsilon\}$.

(1.1.3) If α has no predecessor, then

$$P_\alpha(\varepsilon; G, \Gamma) = \bigcap_{\xi<\alpha} P_\xi(\varepsilon; G, \Gamma).$$

Let us set

$$\eta(\varepsilon; G, \Gamma) = \sup\{\alpha < \omega_1 \mid P_\alpha(\varepsilon; G, \Gamma) \neq \emptyset\}. \tag{1.1.4}$$

We leave it to the reader to prove the next two propositions.

1.2. PROPOSITION. *If $\varepsilon_1 > \varepsilon_2 > 0$, $G_1 \subset G_2 \subset X$, $\Gamma_1 \subset \Gamma_2 \subset X^*$ (G_1, G_2 are bounded and Γ_1, Γ_2 w*-compact), then for every α*

$$P_\alpha(\varepsilon_1; G_1, \Gamma_1) \subset P_\alpha(\varepsilon_2; G_2, \Gamma_2), \tag{1.2.1}$$

$$\eta(\varepsilon_1; G_1, \Gamma_1) \subset \eta(\varepsilon_2; G_2, \Gamma_2). \tag{1.2.2}$$

1.3. PROPOSITION. *If ε, G, Γ are as in definition 1.1 and if $u: X \to Y$ is an isomorphism, then*

$$P_\alpha(\varepsilon; G, \Gamma) = u^*\big(P_\alpha(\varepsilon; uG, (u^*)^{-1}\Gamma)\big) \quad \textit{for} \quad 0 \leqslant \alpha < \omega_1, \tag{1.3.1}$$

$$\eta(\varepsilon; G, \Gamma) = \eta\big(\varepsilon; u(G), (u^*)^{-1}\Gamma\big). \tag{1.3.2}$$

In particular, if $a > 0$ and $b > 0$, then

$$\eta(\varepsilon; aG, b\Gamma) = \eta(\varepsilon/ab, G, \Gamma)\ (^1). \tag{1.3.3}$$

1.4. PROPOSITION. *If ε, G, Γ are as in definition 1.1, and if X^* is separable, then*

$$\eta = \eta(\varepsilon; G, \Gamma) < \omega_1 \quad \textit{and} \quad P_\eta(\varepsilon; G, \Gamma) \neq \emptyset.$$

Proposition 1.4 is an immediate consequence of the next two lemmas.

1.5. LEMMA. *Under the assumptions of Proposition 1.4 the set $P_\alpha(\varepsilon) = P_\alpha(\varepsilon; G, \Gamma)$ has the following properties:*

(1.5.1) *$P_\alpha(\varepsilon)$ is w*-compact;*

(1.5.2) *$P_{\alpha+1}(\varepsilon)$ is nowhere dense in $P_\alpha(\varepsilon)$;*

(1.5.3) *$P_{\alpha+1}(\varepsilon) \subset P_\alpha(\varepsilon)$.*

Proof. We shall prove by transfinite induction that $P_\alpha(\varepsilon)$ is w*-compact and that $P_{\alpha+1}(\varepsilon)$ is a closed nowhere dense subset of $P_\alpha(\varepsilon)$ ($0 \leqslant \alpha < \omega_1$). That will clearly imply conditions (1.5.1)-(1.5.3).

The inductive hypothesis is obviously true for $\alpha = 0$, and it is clear that if it is true for all $\beta < \alpha$ and if α has no predecessor, then $P_\alpha(\varepsilon)$ is w*-compact.

Now suppose that, for an arbitrary ordinal number α with $0 \leqslant \alpha \leqslant \omega_1$, the set $P_\alpha(\varepsilon)$ is w*-compact. Let $f^{(m)} \in P_{\alpha+1}(\varepsilon)$, $m = 1, 2, \ldots$, and let $f^{(m)} \overset{w^*}{\to} f$. Then by (1.1.2) there exist sequences $(x_{n,m})$ in G and $(f_{n,m})$ in $P_\alpha(\varepsilon)$ such that $f_{n,m} \overset{w^*}{\to} f^{(m)}$, $x_{n,m} \overset{w}{\to} 0$ and

$$\limsup_n |f_{n,m}(x_{n,m})| \geqslant \varepsilon \quad \text{for} \quad m = 1, 2, \ldots$$

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(1) $aG = \{x: x = ax', x' \in G\}$, $b\Gamma = \{f: f = bf', f' \in \Gamma\}$.

Since the w-convergence in G and the w*-convergence in $P_a(\varepsilon)$ are metrisable (cf. [5], p. 426), one can choose a sequence of the pair of integers (n_k, m_k), $k = 1, 2, \ldots$, such that $f_{n_k,m_k} \xrightarrow{w^*} f$, $x_{n_k,m_k} \xrightarrow{w} 0$ and

$$|f_{n_k,m_k}(x_{n_k,m_k})| \geqslant \varepsilon - k^{-1} \quad \text{for} \quad k = 1, 2, \ldots$$

Let $f_k = f_{n_k,m_k}$ and let $x_k = x_{n_k,m_k}$ $(k = 1, 2, \ldots)$. Then $f_k \xrightarrow{w^*} f$, $x_k \xrightarrow{w} 0$ and

$$|f_k(x_k)| \geqslant \varepsilon - k^{-1} \quad \text{for} \quad k = 1, 2, \ldots$$

Therefore, by (1.1.2), $f \in P_{a+1}(\varepsilon)$. Hence $P_{a+1}(\varepsilon)$ is closed. Therefore $P_{a+1}(\varepsilon) \subset P_a(\varepsilon)$ because, by (1.1.2), $P_a(\varepsilon)$ is dense in $P_{a+1}(\varepsilon)$ in the w*-topology. Finally, we shall show that $P_{a+1}(\varepsilon)$ is nowhere dense in $P_a(\varepsilon)$. Otherwise there would exist in $P_a(\varepsilon)$ a closed neighbourhood F which is contained in $P_{a+1}(\varepsilon)$. Let $(f^{(m)})_{m=1}^{\infty}$ be a sequence of elements of F which is dense in F. Such a sequence exists because F is w*-separable. By (1.1.2) there are sequences $(f_{n,m})_{n=1}^{\infty}$ in $P_a(\varepsilon)$ and $(x_{n,m})_{n=1}^{\infty}$ in G such that $f_{n,m} \xrightarrow{w^*} f^{(m)}$, $x_{n,m} \xrightarrow{w} 0$ and

$$\limsup_n |f_{n,m}(x_{n,m})| \geqslant \varepsilon \quad (m = 1, 2, \ldots).$$

Let us choose a sequence $(n_m)_{m=1}^{\infty}$ so that the sequence (x_m) obtained by the standard process from the double sequence $(x_{i+n_m,m})$, $i, m = 1, 2, \ldots$, is weakly convergent to zero (this is possible because the w-topology on the bounded set G is metrisable). Then (x_m) may be regarded as a sequence of continuous functions on the compact metric space $P_a(\varepsilon)$ tending pointwise to zero. Let

$$Q(\varepsilon) = \{f \in P_a(\varepsilon)\colon \text{there exists in } P_a(\varepsilon) \text{ a sequence } (g_m) \text{ such that } g_m \xrightarrow{w^*} f \text{ and } \limsup_n |g_m(x_m)| \geqslant \varepsilon\}.$$

Then $Q(\varepsilon)$ may be regarded as a set of points at which the oscillation of the sequence (x_m) of continuous functions is $\geqslant \varepsilon$. Since (x_m) pointwise tends to zero, $Q(\varepsilon)$ is closed and nowhere dense in $P_a(\varepsilon)$ (cf. [19] and [6]). Since $f^{(m)} \in Q(\varepsilon)$ for $m = 1, 2, \ldots$ and since the sequence $(f^{(m)})$ is dense in F, the set F is contained in $Q(\varepsilon)$. But this contradicts the fact that $Q(\varepsilon)$ is nowhere dense in $P_a(\varepsilon)$. This completes the proof.

1.6. Lemma. *Under the assumptions of Proposition* 1.4 *there exists an ordinal number* $a < \omega_1$ *such that*

$$P_a(\varepsilon; G, \Gamma) \neq \emptyset \quad \text{and} \quad P_{a+1}(\varepsilon, G, \Gamma) = \emptyset.$$

The proof follows from Lemma 1.5 and the Cantor-Baire theorem (cf. [12], p. 146 and 150).

2. Let $K = \{x \in X\colon \|x\| \leqslant 1\}$ and let $K^* = \{f \in X^*\colon \|f\| \leqslant 1\}$ be unit balls in X and X^* respectively.

2.1. Definition. Let us define, for $\varepsilon > 0$,

$$\eta(\varepsilon; X) = \eta(\varepsilon; K, K^*), \qquad \eta(X) = \sup_{\varepsilon>0} \eta(\varepsilon; X).$$

The ordinals $\eta(\varepsilon; X)$ and $\eta(X)$ are called the *ε-index* of X and the *index* of X respectively. The next proposition is an immediate consequence of Propositions 1.2 and 1.4.

2.2. Proposition. *If* X^* *is separable, then*

$$\eta(X) = \sup_n \eta(n^{-1}; X) < \omega_1.$$

The next proposition shows that $\eta(X)$ is a linear topological invariant.

2.3. Proposition. *If* u *is an isomorphism from a Banach space* Y *onto a subspace of a separable Banach space* X, *then* $\eta(X) \geqslant \eta(Y)$. *Moreover, if* u *is an isometric isomorphism, i.e.* $\|u(y)\| = \|y\|$ *for every* $y \in Y$, *then* $\eta(\varepsilon; X) \geqslant \eta(\varepsilon; Y)$ *for* $\varepsilon > 0$.

Let $X_1 = u(Y)$ and let K, K^*, K_1, K_1^*, S, S^* denote unit balls of X, X^*, X_1, X_1^*, Y, Y^* respectively.

2.4. Lemma. *If* $f \in P_a(\varepsilon; K_1, K_1^*)$, *then there exists an* f' *in* $P_a(\varepsilon; K, K^*)$ *such that* f' *is an extension of* f.

Proof. For $a = 0$ the assertion of the Lemma is an immediate consequence of the Hahn-Banach extension principle. Let $0 < a < \omega_1$ and let us assume that the assertion of the lemma holds for $0 \leqslant \beta < a$. If $a = a'+1$ and if $f \in P_a(\varepsilon; K_1, K_1^*)$, then there are $f_m \in P_{a'}(\varepsilon; K_1, K_1^*)$ and $x_m \in K_1$ such that $f_m \xrightarrow{w^*} f$, $x_m \xrightarrow{w} 0$ and

$$\limsup_n |f_m(x_m)| \geqslant \varepsilon$$

(we may suppose that $\liminf |f_m(x_m)| \geqslant \varepsilon$). By the inductive hypothesis there exists an $f_m' \in P_a(\varepsilon; K, K^*)$ such that f_m' is an extension of f_m $(m = 1, 2, \ldots)$. The separability of X implies that the bounded sequence $(f_m')_{m=1}^{\infty}$ contains a w*-convergent subsequence, say $(f_{m_k}')_{k=1}^{\infty}$. Let $f_{m_k}' \xrightarrow{w^*} f'$. Then clearly f' is an extension of f and $f' \in P_a(\varepsilon; K, K^*)$. If a has no predecessor, then the inductive step for a is an immediate consequence of (1.1.2) and the inductive hypothesis.

Proof of Proposition 2.3. It follows immediately from Lemma 4.2 that $\eta(\varepsilon; X) \geqslant \eta(\varepsilon; X_1)$ for $\varepsilon > 0$. Since u is an isomorphism from Y onto X_1, there are $a, b > 0$ such that

$$u(aS) \subset K_1 \subset u(bS).$$

Hence

$$u^*(aK_1^*) \subset S^* \subset u^*(bK_1^*)$$

(if u is an isometric isomorphism, then $a = b = 1$). Thus by Propositions 1.2 and 1.3, we get

$$\eta(\varepsilon;\, Y) = \eta(\varepsilon;\, S,\, S^*) = \eta\big(\varepsilon;\, u(S),\, (u^*)^{-1}(S^*)\big)$$

$$\leqslant \eta(\varepsilon;\, a^{-1}K_1,\, bK_1^*) = \eta\left(\frac{a}{b}\,\varepsilon;\, K_1,\, K_1^*\right) = \eta\left(\frac{a}{b}\,\varepsilon;\, X_1\right).$$

Hence

$$\eta(Y) = \sup_{\varepsilon>0} \eta(\varepsilon;\, Y) \leqslant \sup_{\varepsilon>0} \eta\left(\frac{a}{b}\varepsilon;\, X_1\right) = \eta(X_1) \leqslant \eta(X),$$

and this completes the proof.

3. By $(X\times Y)_1$ (respectively $(X\times Y)_\infty$) we denote the Cartesian product of Banach spaces X and Y with the norm $\|(x, y)\|_1 = \|x\|+\|y\|$ (resp. $\|(x, y)\|_\infty = \max(\|x\|, \|y\|)$). Observe that $[(X\times Y)_1]^* = (X^*\times Y^*)_\infty$. If $\{X_t\}_{t\in T}$ is a family of Banach spaces, then the symbol $l_2\{X_t\}_{t\in T}$ denotes the Banach space of all functions $x(\cdot)$ from T into the product $\prod_{t\in T} X$ such that

$$\|x(\cdot)\|_2 = \Big(\sum_{t\in T} \|x(t)\|^2\Big)^{1/2} < +\infty.$$

Let us observe that if each X_t is reflexive, then $l_2\{X_t\}_{t\in T}$ is reflexive and if each X_t is separable and the index set T is countable, then $l_2\{X_t\}_{t\in T}$ is separable.

By l_2 we shall denote the Hilbert sequence space.

3.1. LEMMA. *If X^* is separable, then*

$$\eta\big(\varepsilon;\, (X\times l_2)_1\big) \geqslant \eta(\varepsilon;\, X)+1 \quad \textit{for} \quad 0<\varepsilon<1.$$

Proof. Let $K, K^*, K_1, K_1^*, S, S^*$ denote the unit balls of $X, X^*, l_2, l_2^*, (X\times l_2)_1$ and $[(X\times l_2)_1]^*$ respectively. Let $(e_n)_{n=1}^\infty$ be an orthonormal system in l_2 and let e_n^*, for $n = 1, 2, \ldots$, be the linear functional determined by $e_n^*(e) = \langle e, e_n\rangle$ for $e\in l_2$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in l_2. We shall show by transfinite induction that for each e_n^* $(n = 1, 2, \ldots)$ and each $\alpha<\omega_1$

(3.1.1) *If $f\in P_\alpha(\varepsilon;\, K, K^*)$, then $(f, e_n^*)\in P_\alpha(\varepsilon;\, S, S^*)$.*

(3.1.1) is obviously true for $\alpha = 0$, because $P_0(\varepsilon, K, K^*) = K^*$, $P_0(\varepsilon, S, S^*) = S^*$, $e_n^*\in K^*$, and $[(X\times l_2)_1]^* = (X^*\times l_2)_\infty$. Assuming (3.1.1) for an ordinal $\alpha<\omega_1$ we shall show that it is true for $\alpha+1$. Let $f\in P_{\alpha+1}(\varepsilon, K, K^*)$. Then for $m = 1, 2, \ldots$ there exist $f_m\in P_\alpha(\varepsilon, K, K^*)$ and $x_m\in K$ such that $f_m \xrightarrow{w^*} f$, $x_m \xrightarrow{w} 0$ and

$$\limsup_m |f_m(x_m)| \geqslant \varepsilon.$$

Let us put, for $m = 1, 2, \ldots$, $\varphi_m = (f_m, e_n^*)$ and $y_m = (x_m, 0)$. Then from the respective properties of the sequences (f_m) and (x_m)

$$\varphi_m \xrightarrow{w^*} (f, e_n^*), \qquad y_m \xrightarrow{w} 0, \qquad y_m\in S,$$

and, since $\varphi_m(y_m) = f_m(x_m)$ for $m = 1, 2, \ldots$,

$$\limsup_m |\varphi_m(y_m)| \geqslant \varepsilon.$$

By the inductive hypothesis, $\varphi_m\in P_\alpha(\varepsilon, S, S^*)$, $m = 1, 2, \ldots$ Thus $(f, e_n^*)\in P_{\alpha+1}(\varepsilon, S, S^*)$. Finally, if $\alpha<\omega_1$ has no predecessor and if we assume (3.1) for every $\beta<\alpha$, then (3.1.1) follows trivially from (1.1.3), and this completes the proof of (3.1.1) for every $\alpha<\omega_1$. To complete the proof of the lemma, choose an f in $P_{\alpha_0}(\varepsilon;\, K, K^*)$, where $\alpha_0 = \eta(\varepsilon;\, X)$. Observe that according to Proposition 1.4 the set $P_{\alpha_0}(\varepsilon;\, K, K^*)$ is non-empty. By (3.1.1)

$$\psi_n = (f, e_n^*)\in P_{\alpha_0}(\varepsilon;\, S, S^*) \quad \text{for} \quad n = 1, 2, \ldots$$

Clearly, $\psi_n \xrightarrow{w^*} (f, 0)$, because $e_n^* \xrightarrow{w^*} 0$. Let $y_n = (0, e_n)$ for $n = 1, 2, \ldots$ Then $y_n\in S$, $y_n \xrightarrow{w} 0$ (because $e_n \xrightarrow{w} 0$) and

$$\psi_n(y_n) = e_n^*(e_n) = 1 \quad \text{for} \quad n = 1, 2, \ldots$$

Therefore $(f, 0)\in P_{\alpha_0+1}(\varepsilon;\, S, S^*)$. Hence for $0<\varepsilon<1$

$$\eta\big(\varepsilon;\, (X\times l_2)_1\big) \geqslant \alpha_0+1 = \eta(\varepsilon;\, X)+1.$$

This completes the proof.

3.2. PROPOSITION. *For every countable ordinal α there exists a separable reflexive Banach space X_α such that*

(3.2.1) $$\eta(X_\alpha) \geqslant \alpha.$$

Proof. Let us set $X_0 = l_2$, $X_{\alpha+1} = (X_\alpha\times l_2)_1$ for $0\leqslant\alpha<\omega_1$, $X_\alpha = l_2\{X_\beta\}_{\beta<\alpha}$ for $\alpha<\omega_1$ having no predecessors. Clearly, the spaces X_α defined above are reflexive and separable. We shall show that they satisfy (3.2.1) by proving by transfinite induction that for $0\leqslant\alpha<\omega_1$

(3.2.2) $$\eta(\varepsilon;\, X_\alpha) \geqslant \alpha \quad \text{for} \quad 0<\varepsilon<1.$$

Obviously, (3.2.2) holds for $\alpha = 0$. If α has no predecessor and if (3.2.2) holds for $0\leqslant\beta<\alpha$, then using the fact that the space $X_\alpha =$

$= l_2\{X_\beta\}_{\beta<\alpha}$ contains a subspace isometrically isomorphic to X_β $(0 \leqslant \beta < \alpha)$ and applying Proposition 2.3 we get

$$\eta(\varepsilon; X_\alpha) \geqslant \sup_{\beta<\alpha} \eta(\varepsilon; X_\beta) \geqslant \sup_{\beta<\alpha} \beta = \alpha.$$

Finally, if (3.2.2) holds for an ordinal $\alpha < \omega_1$, then in view of Lemma 3.1 and the definition of $X_{\alpha+1}$ we derive (3.2.2) for $\alpha+1$.

The next result is an obvious consequence of Propositions 2.2, 2.3 and 3.2.

3.3. THEOREM. *If X is a Banach space which is universal for all separable reflexive Banach spaces, then X^* is non-separable.*

3.4. COROLLARY. *Let $\mathscr{A}$ be one of the following classes:*

(i) *of all separable reflexive Banach spaces;*

(ii) *of all Banach spaces with separable duals.*

Then there is no member of $\mathscr{A}$ which is universal for $\mathscr{A}$.

Finally, let us observe that Corollary 3.4 (i) implies (by the standard duality arguments) the following

3.5. COROLLARY. *There is no separable reflexive space X with the property that for every separable reflexive space Y there is a bounded linear operator from X onto Y.*

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