geht. Es gibt jedoch noch viele andere Varianten, bei denen die Methode der unendlichen Gleichungen (auch kombiniert mit funktionentheoretischen Methoden) sich bewährt.

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Stability of order convergence and regularity in Riesz spaces

by

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Dedicated to Professors
Stanisław Mazur and Władysław Orlicz
on the occasion of the 40th anniversary
of their scientific research

1. Introduction. The years from 1928 until 1936, a period of rapid growth for Banach and Hilbert space theory, were also the time that the foundations were laid for the functional analytic theory of linear vector lattices. This was done, independently, by Riesz [10, 11], Kantorovitch [4, 5] and Freudenthal [1], and it is interesting to observe now, more than thirty years later, the different methods of approach. Riesz was interested primarily in what is now called the order dual space of a given partially ordered vector space, and he presented an extended version of his short 1928 Congress note in a 1940 Annals of Mathematics paper, a translation of a 1937 Hungarian paper. Freudenthal, in 1936, proved a "spectral theorem" for vector lattices, the significance of which is illustrated by the fact that the Radon-Nikodym theorem in integration theory as well as the spectral theorem for Hermitian operators in Hilbert space are corollaries, although it was not until early in the fifties that a direct method was indicated for deriving the spectral theorem for Hermitian operators from the abstract spectral theorem. Finally, around 1935, Kantorovitch began his extensive investigation of the algebraic and convergence properties of vector lattices, with applications to linear operator theory. A few years later, curiously enough between 1940 and 1944, important contributions to the subject were published by Nakano [6, 7, 8], Ogasawara [9], Yosida [13, 14] in Japan and Kakutani [2, 3] in the United States. Of the more recent progress we only mention the work by Kantorovitch, Nakano and their schools. In contrast with Banach and Hilbert space theory, however, where in recent books the main body of the theory has been welded into a unified and elegant whole, there is only a very small number of textbooks on partially ordered vector spaces (and in the existing ones the treatment is mainly restricted to the research of one school). This explains why the present paper is written in a somewhat expository style. There is one chapter on linear vector lattices in the Bourbaki work on integration, and we will follow Bourbaki's example in calling any linear vector lattice a Riesz space. In the present paper we discuss several special properties which a Riesz space may possess (stability of order convergence, regularity, existence of a strong unit) and we prove that certain combinations of these properties are possible only if the space is of finite dimension. We believe that our results in this direction are a little more general than previously known results (cf. e.g. Theorem VI. 4.2 in [12]); our main purpose, however, is to present straightforward proofs.

2. Convergence in Riesz spaces. We recall the definitions of an ordered vector space and of a Riesz space. The real linear space L, with elements $f, g, \ldots$ and with null element 0, is called an *ordered vector space* if L is partially ordered in such a manner that the partial ordering (denoted by $\leqslant$) is compatible with the algebraic structure of L, i.e., $f \leqslant g$ implies $f+h \leqslant g+h$ for every $h \in L$, and $f \geqslant 0$ implies $af \geqslant 0$ for every real $a \geqslant 0$. The ordered vector space L is called a *Riesz space* if L is a lattice with respect to the partial ordering, i.e., if for every pair $f, g \in L$ the supremum $\sup(f, g)$ and the infimum $\inf(f, g)$ with respect to the partial ordering exist in L. It will be assumed throughout the present paper that L is a Riesz space. The notation $f < g$ means that $f \leqslant g, f \neq g$, and the subset $L^+ = \{f : f \in L, f \geqslant 0\}$ is called the *positive cone* of L. The positive cone has the property that if $f, g \in L^+$ and a, b are non-negative real numbers, then $af + bg \in L^+$. Furthermore, if f and $-f$ are simultaneously members of L^+, then $f = 0$.

The following abbreviations are widely used and well-known especially in the case that the elements of L are real functions. Given any $f \in L$, we set

$$f^+ = \sup(f, 0), \quad f^- = \sup(-f, 0), \quad |f| = \sup(f, -f).$$

It is not difficult to prove that f^+, f^- and $|f|$ are members of L^+, $f = f^+ - f^-$, $|f| = f^+ + f^-$ and $\inf(f^+, f^-) = 0$. The formula $f = f^+ - f^-$ shows, in particular, that every element of L is the difference of two elements from the positive cone. As might be expected, the triangle inequality

$$||f| - |g|| \leqslant |f+g| \leqslant |f| + |g|$$

holds; the proof is derived by showing first that

$$(f+g)^+ \leqslant f^+ + g^+ \quad \text{and} \quad (f+g)^- \leqslant f^- + g^-.$$

It is an important fact that L, regarded as a lattice, is a *distributive* lattice. The distributive laws hold even for the infinite case, i.e., if D is a subset of L such that $f_0 = \sup(f : f \in D)$ exists, then

$$\inf(f_0, g) = \sup\{\inf(f, g) : f \in D\}$$

holds for every $g \in L$. Similarly if sup and inf are interchanged.

The elements $f, g \in L$ are called *disjoint* if $\inf(|f|, |g|) = 0$; notation: $f \perp g$. The name is derived from the case that L consists of real functions on a point set X. Disjointness of f and g means now simply that the subsets of X on which f and g differ from zero are disjoint. Disjointness in a Riesz space has some properties analogous to properties of orthogonality in a Euclidean space. If $f_1, \ldots, f_n$ are disjoint to f_0, and $a_1, \ldots, a_n$ are real, then

$$\sum_{i=1}^{n} a_i f_i \perp f_0.$$

Furthermore, if $f_1 \perp f_0$ and $|f_2| \leqslant |f_1|$, then $f_2 \perp f_0$. Finally, if D is a subset of L such that $D \perp f_0$ (i.e., $f \perp f_0$ for every $f \in D$), and if $f_1 = \sup(f : f \in D)$ exists, then $f_1 \perp f_0$. It is an easy corollary that if $f_1, \ldots, f_n$ are all $\neq 0$ and mutually disjoint, then $\{f_1, \ldots, f_n\}$ is a linearly independent system. Indeed, if not, then one of the elements, say f_1, is a linear combination of the remaining elements; $f_1 = a_2 f_2 + \ldots + a_n f_n$. But f_1 is disjoint to $f_2, \ldots, f_n$, so

$$f_1 \perp (a_2 f_2 + \ldots + a_n f_n),$$

i.e., $f_1 \perp f_1$. This implies that $\inf(|f_1|, |f_1|) = 0$, so $|f_1| = 0$, and hence $f_1 = 0$. Contradiction.

Next, note that

$$\sup(f, g) + \inf(f, g) = f + g$$

holds for all f, g. Hence, if $f, g \in L^+$ and $f \perp g$, then $\sup(f, g) = f + g$.

The sequence $\{f_n; n = 1, 2, \ldots\}$ in L is called *increasing* if $f_1 \leqslant f_2 \leqslant \ldots$ and *decreasing* if $f_1 \geqslant f_2 \geqslant \ldots$ This is denoted by $f_n \uparrow$ or $f_n \downarrow$ respectively. If $f_n \uparrow$ and $f = \sup f_n$ exists in L, we will write $f_n \uparrow f$. Similarly, if $f_n \downarrow$ and $f = \inf f_n$ exists, we will write $f_n \downarrow f$. The sequence $\{f_n; n = 1, 2, \ldots\}$ in L is said to *converge in order* to $f \in L$ if there exists a sequence $u_n \downarrow 0$ in L^+ such that $|f - f_n| \leqslant u_n$ holds for all n. This will be denoted by $f_n \to f$. It is evident (triangle inequality) that an order limit is uniquely determined, i.e., if $f_n \to f$ and $f_n \to g$, then $f = g$. Furthermore, the following holds:

(i) If $f_n \uparrow f$ or $f_n \downarrow f$, then $f_n \to f$.

(ii) If $f_n \uparrow$ or $f_n \downarrow$ and $f_n \to f$, then $f_n \uparrow f$ or $f_n \downarrow f$ respectively.

(iii) If $f_n \to f$, $g_n \to g$, and a and b are real, then $af_n + bg_n \to af + bg$, $\sup(f_n, g_n) \to \sup(f, g)$ and $\inf(f_n, g_n) \to \inf(f, g)$. In particular, if $f_n \to f$, then $f_n^+ \to f^+, f_n^- \to f^-$ and $|f_n| \to |f|$.

(iv) If $f_n \to f$, then $f_{n_k} \to f$ for every subsequence such that $n_1 < n_2 < \ldots$

It is not true in general that if $f_n \to f$ in L and $a_n \to a$ in real number space, then $a_n f_n \to af$. If, however, the space L is *Archimedean*, i.e., if it is true for every $u \in L^+$ that the sequence $v_n = n^{-1}u (n = 1, 2, \ldots)$ satisfies $v_n \downarrow 0$, then it follows from $f_n \to f$ and $a_n \to a$ that $a_n f_n \to af$.

There is still another mode of convergence for sequences in L. Given $u \in L^+$, the sequence $\{f_n; n = 1, 2, \ldots\}$ in L is said to *converge u-uniformly* to $f \in L$ whenever, given any $\varepsilon > 0$, there exists N_ε such that $|f - f_n| \leqslant \varepsilon u$ holds for all $n \geqslant N_\varepsilon$. The sequence $\{f_n\}$ is said to *converge relatively uniformly* to f if there exists an element $u \in L^+$ such that f_n converges u-uniformly to f. Notation: $f_n \to f$ (r.u.). In the Soviet literature this is called *convergence with respect to a regulator*. In an Archimedean Riesz space L any relatively uniform limit, if existing, is uniquely determined. This can be seen directly, and it also immediately follows from the fact that in an Archimedean space relatively uniform convergence implies order convergence. Furthermore, relatively uniform convergence in an Archimedean space has the same properties (iii) and (iv) as order convergence.

Relatively uniform convergence in an Archimedean Riesz space is *stable*, i.e., given any sequence $f_n \to 0$ (r.u.), there exists a sequence of positive real numbers $\{\lambda_n\}$ such that $\lambda_n \uparrow \infty$ and $\lambda_n f_n \to 0$ (r.u.). Order convergence is not necessarily stable; in the real sequence space l_∞, for example, order convergence is not stable (let f_n be the element of l_∞ with the first n coordinates zero and all other coordinates equal to 1). The following theorem is known; we present a brief proof.

THEOREM 2.1. *In an Archimedean Riesz space order convergence is stable if and only if order convergence and relatively uniform convergence are equivalent.*

Proof. Assuming first that order convergence is stable, we have to prove that order convergence implies relatively uniform convergence. To this end, let $f_n \to 0$. Since order convergence is stable, there exists a sequence $0 \leqslant \lambda_n \uparrow \infty$ such that $\lambda_n f_n \to 0$. It follows that $\lambda_n |f_n| \to 0$, and hence the sequence $\{\lambda_n |f_n|\}$ is bounded, i.e., there exists $u \in L^+$ such that $\lambda_n |f_n| \leqslant u$ for all n. Then $|f_n| \leqslant \lambda_n^{-1} u$ for all n, and so $f_n \to 0$ (r.u.).

Conversely, if order convergence and relatively uniform convergence are equivalent, then order convergence is stable since relatively uniform convergence is so.

We finally recall that every Riesz space L has the *dominated decomposition property*, i.e., if $0 \leqslant u \leqslant v_1 + v_2$ with $v_1, v_2 \in L^+$, then there exist $u_1, u_2 \in L^+$ such that $u = u_1 + u_2$ and $u_i \leqslant v_i$ for $i = 1, 2$.

3. Projection properties and completeness properties. The subset D of the Riesz space L is called *solid* if $f \in D$, $|g| \leqslant |f|$, implies that $g \in D$. The subset A of L is called an *ideal* (or *order ideal*) if A is a solid linear subspace of L. The ideal A is called a *band* if A is closed under the operation of taking suprema, i.e., if for every subset of A possessing a supremum in L this supremum is a member of A. It easily follows from the remarks on disjointness in the preceding section that for any arbitrary subset D of L the *disjoint complement* D^d of D (i.e., the set of all f such that $f \perp g$ for all $g \in D$) is a band.

It is evident from the definitions that an arbitrary intersection of ideals (or bands) is an ideal (or band). Given the arbitrary subset D of L, the intersection of all ideals including D is called the *ideal generated by D*. The *band generated by D* is defined similarly. If D consists of one element f, it is customary to speak about the *principal ideal* (*principal band*) generated by f. Evidently, the principal ideal generated by f consists of all $g \in L$ such that $|g| \leqslant a_g |f|$ for some positive number a_g (depending on g, therefore). It is a little more difficult to prove that, for every subset D of L, the band generated by D consists of all g such that $|g|$ is the supremum of some subset of the ideal generated by D.

We will prove two simple theorems, particular cases of which are well-known.

THEOREM 3.1. *Let A and B be ideals in the Riesz space L.*

(i) *The algebraic sum $A+B$ is an ideal. Given $f \geqslant 0$ in $A+B$, there exists at least one decomposition $f = f_1 + f_2$ such that $f_1 \in A, f_2 \in B$ and $f_1, f_2 \geqslant 0$.*

(ii) *We have $A \perp B$ if and only if $A \cap B = \{0\}$, where $\{0\}$ is the ideal consisting only of the null element, i.e., if and only if $A+B$ is a direct sum $A \oplus B$. In this case, therefore, the decomposition $f = f_1 + f_2$ with $f_1 \in A$, $f_2 \in B$, is unique, and $f \geqslant 0$ implies $f_1, f_2 \geqslant 0$.*

Proof. (i) In order to show that $A+B$ is an ideal, it is sufficient to prove that $A+B$ is solid, i.e., we have to prove that $f \in A+B$ and $|g| \leqslant |f|$ implies $g \in A+B$. Hence, let $f = f_1 + f_2$ with $f_1 \in A, f_2 \in B$, and let $|g| \leqslant |f|$. Then

$$g^+ \leqslant |g| \leqslant |f| \leqslant |f_1| + |f_2|,$$

so by the dominated decomposition property there exists a decomposition $g^+ = g' + g''$ such that $0 \leqslant g' \leqslant |f_1|$ and $0 \leqslant g'' \leqslant |f_2|$. It follows from $|f_1| \in A$ and $|f_2| \in B$ that $g' \in A$ and $g'' \in B$, and so $g^+ = g' + g'' \in A+B$. Similarly, we have $g^- \in A+B$ and so $g = g^+ - g^- \in A+B$. This is the desired result.

Assume now that $f \geqslant 0$ and $f = f' + f''$ with $f' \in A$ and $f'' \in B$. Then $0 \leqslant f \leqslant |f'| + |f''|$, so by the dominated decomposition property there exists a decomposition $f = f_1 + f_2$ such that $0 \leqslant f_1 \leqslant |f'|$ and $0 \leqslant f_2 \leqslant |f''|$. Since $|f'| \in A$ and $|f''| \in B$, it follows that $f_1 \in A$ and $f_2 \in B$.

(ii) Assume first that $A \perp B$. Given $f \in A \cap B$, we have now that $f \perp f$, so $f = 0$. This shows that $A \cap B = \{0\}$. Conversely, assuming that $A \cap B = \{0\}$, it has to be proved that $\inf(|f_1|, |f_2|) = 0$ for all $f_1 \in A, f_2 \in B$. This is evident because $\inf(|f_1|, |f_2|)$ is a member of $A \cap B$, and so must be the null element.

THEOREM 3.2. *If A and B are ideals in the Riesz space L such that $A \oplus B = L$, then $B = A^d$ and $A = B^d$. In other words, A and B are now bands, each the disjoint complement of the other one.*

Proof. It follows from $A \oplus B = L$ that $A \cap B = \{0\}$, so $B \perp A$, and hence $B \subset A^d$. In order to prove the inverse inclusion, assume that $0 \leqslant u \in A^d$. By hypothesis there exists a decomposition $u = u_1 + u_2$ with $u_1 \in A$ and $u_2 \in B$, and it follows from the preceding theorem that $u_1, u_2 \geqslant 0$. Hence $0 \leqslant u_1 \leqslant u \in A^d$, which implies that $u_1 \in A^d$. But $u_1 \in A$ holds just as well, so $u_1 = 0$. It follows that $u = u_2 \in B$, and it has thus been proved that $A^d \subset B$. The final result is that $B = A^d$.

In view of the last theorem it is now appropriate to call any band A such that $A \oplus A^d = L$ holds a *projection band*. If A is simultaneously a principal band and a projection band, then A is called a *principal projection band*.

THEOREM 3.3. (i) *The band A in the Riesz space L is a projection band if and only if, for any $u \in L^+$, the element*

$$u_1 = \sup(v : v \in A,\ 0 \leqslant v \leqslant u)$$

exists, and in this case u_1 is the component of u in A. Similarly,

$$u_2 = \sup(w : w \in A^d,\ 0 \leqslant w \leqslant u)$$

is then the component of u in A^d. In other words, if u_1 and u_2 are these suprema, then $u = u_1 + u_2$ with $u_1 \in A$ and $u_2 \in A^d$.

(ii) *The principal band A, generated by the element $v \in L^+$, is a projection band if and only if, for any $u \in L^+$, the element*

$$u_1 = \sup_n\{\inf(u, nv)\}, \qquad n = 1, 2, \ldots,$$

exists, and in this case u_1 is the component of u in A.

Proof. (i) Let A be a projection band, so $L = A \oplus A^d$, and let $u \in L^+$ have the decomposition $u = u_1 + u_2$. Define the subset V of L by

$$V = (v;\ v \in A,\ 0 \leqslant v \leqslant u).$$

We have to prove that $u_1 = \sup V$. For any $v \in V$ we have $u - v \geqslant 0$, and $u - v$ has the decomposition $u - v = (u_1 - v) + u_2$, so $u_1 - v \geqslant 0$ by Theorem 3.1 (ii), i.e., $v \leqslant u_1$. This shows that u_1 is an upper bound of V. On the other hand, u_1 is a member of V, and so $u_1 = \sup V$.

Assume now, conversely, that A is a band with the property that

$$u_1 = \sup(v : v \in A,\ 0 \leqslant v \leqslant u)$$

exists for any given $u \in L^+$. Since u_1 is then a member of A (because A is a band), it will be sufficient for the proof of $A \oplus A^d = L$ to show that $u_2 = u - u_1$ is a member of A^d. If not, we have $p = \inf(u_2, z) > 0$ for some $z \in A$. Then $0 < p \in A$, and also $p \leqslant u_2$. Hence $u_1 + p \in A$ as well as $u_1 + p \leqslant u_1 + u_2 = u$, so $u_1 + p$ is a member of the set $(v : v \in A,\ 0 \leqslant v \leqslant u)$. But then $u_1 + p$ is less than or equal to the supremum u_1 of this set, so $u_1 + p \leqslant u_1$, i.e., $p \leqslant 0$, which contradicts $p > 0$.

(ii) Given the principal band A generated by $v \in L^+$ and given $u \in L^+$, we set

$$W = (w : w \in A,\ 0 \leqslant w \leqslant u),$$

$$W' = \big(\inf(u, nv) : n = 1, 2, \ldots\big).$$

It is evident that $W' \subset W$, and so any upper bound of W is an upper bound of W'. In order to prove the converse, note that for any given $w \in W$ there exists a subset D of the ideal generated by v such that $w = \sup D$. Every element of D is majorized by a positive multiple of v, and is also majorized by w and hence by u. It follows that every element of D is majorized by an element of W'. This shows that any upper bound of W' is an upper bound of D, and so of w, and hence of W. It has been proved thus that W and W' have the same upper bounds. In particular, the supremum of W' exists if and only if the supremum of W exists, and these suprema are then the same. By part (i), A is a projection band if and only if $u_1 = \sup W$ exists, i.e., if and only if

$$u_1 = \sup W' = \sup_n \{\inf(u, nv)\}$$

exists, and in this case u_1 is then the component of u in A.

The Riesz space L is said to have the *projection property* if every band is a projection band, and L is said to have the *principal projection property* if every principal band is a projection band. Furthermore, as well-known, L is called *Dedekind complete* (a *K-space* in the Soviet terminology) if every subset which is bounded from above has a supremum, and L is called *Dedekind σ-complete* if every finite or countable subset which is bounded from above has a supremum.

It is obvious that Dedekind completeness implies Dedekind σ-completeness, and it is well-known that Dedekind completeness implies the projection property. It is obvious again that the projection property implies the principal projection property, and it is easy to prove that Dedekind σ-completeness also implies the principal projection property. Indeed, let L be Dedekind σ-complete, and let A be the principal band generated by $v \in L^+$. According to the last theorem, we have to prove that

$$\sup_n \{\inf(u, nv)\}, \qquad n = 1, 2, \ldots,$$

exists for any given $u \in L^+$. Writing $w_n = \inf(u, nv)$ for $n = 1, 2, \ldots$, we have $0 \leqslant w_n \uparrow$ and $w_n \leqslant u$ for all n, so $\sup w_n$ exists on account of the Dedekind σ-completeness of L. Finally, we observe that if L has the principal projection property, then L is Archimedean. For the proof we have to show that if $0 \leqslant nv \leqslant u$ holds for $n = 1, 2, \ldots$, then $v = 0$. It follows from $0 \leqslant nv \leqslant u$ that $\inf(u, nv) = nv$ for all n, and it follows from the principal projection property that $\sup_n \{\inf(u, nv)\}$ exists, i.e., $u_0 = \sup_n nv$ exists. But then

$$2u_0 = \sup 2nv = \sup nv = u_0,$$

so $u_0 = 0$. This implies that $nv = 0$ for all n, so $v = 0$.

4. Atoms. The element $f \neq 0$ in the Riesz space L is called an *atom* whenever it follows from $0 \leqslant u \leqslant |f|$, $0 \leqslant v \leqslant |f|$ and $u \perp v$ that $u = 0$ or $v = 0$. If f is an atom, then af is an atom for every real $a \neq 0$. If f is an atom and $0 \leqslant |g| \leqslant |f|$, then either $g = 0$ or g is an atom. If f is an atom, then either $f > 0$ or $f < 0$. Indeed, since $0 \leqslant f^+ \leqslant |f|$, $0 \leqslant f^- \leqslant |f|$ and $f^+ \perp f^-$, we must have either $f^+ = 0$ or $f^- = 0$. In an Archimedean Riesz space we can say more.

THEOREM 4.1. *In an Archimedean Riesz space the following holds:*

(i) *If f is an atom in L and $0 \leqslant u \leqslant |f|$, then $u = af$ for some real a.*

(ii) *If f and g are atoms in L, then either $f \perp g$ or $f = ag$ for some real $a \neq 0$.*

(iii) *If A is the principal band generated by the atom f in L, then A consists of all real multiples of f, and A is a projection band.*

Proof. (i) This part is well-known; we briefly recall the proof. It may be assumed that $f > 0$ and $0 < u \leqslant f$ (the case $u = 0$ is trivial). The set of numbers $(\beta : \beta u \leqslant f)$ is non-empty and bounded from above since L is Archimedean. Let $\alpha = \sup(\beta : \beta u \leqslant f)$, so $1 \leqslant \alpha < \infty$ and $\alpha u \leqslant f$. We will prove that $f = \alpha u$. If not, we have $v = f - \alpha u > 0$ and so, on account of $(v - \varepsilon u)^+ \uparrow v$ for $\varepsilon \downarrow 0$, there exists a number ε such that $0 < \varepsilon < \alpha$ and $(v - \varepsilon u)^+ > 0$. It follows that

$$0 < (v - \varepsilon u)^+ = \big(f - (\alpha + \varepsilon) u\big)^+ \leqslant f^+ = f,$$

and so, since $\alpha \geqslant 1$, we have that

$$0 < \big(f - (\alpha + \varepsilon) u\big)^+ \leqslant 2\alpha f. \tag{1}$$

Next, note that $0 < (f - (\alpha + \varepsilon) u)^-$ since otherwise $(\alpha + \varepsilon) u \leqslant f$, against the definition of α. Hence

$$0 < \big(f - (\alpha + \varepsilon) u\big)^- = \big((\alpha + \varepsilon) u - f\big)^+ \leqslant (\alpha + \varepsilon) u \leqslant 2\alpha f. \tag{2}$$

But (1) and (2) are contradictory since $2\alpha f$ is an atom. It follows that $f = \alpha u$, where $1 \leqslant \alpha < \infty$ as observed above. Hence $u = \alpha^{-1} f$.

(ii) Assuming that f and g are atoms, we set $u = \inf(|f|, |g|)$. If $u = 0$, then $f \perp g$. If $u > 0$, then part (i) shows that $f = a_1 u$ and $g = a_2 u$ with $a_1 \neq 0$, $a_2 \neq 0$, and so $f = a_1 a_2^{-1} g$.

(iii) By part (i) every element in the ideal generated by the atom f is a real multiple of f. Now assume (as we may) that the atom f is positive, and that $v \in L^+$ is an element in the band A generated by f. Then v is the supremum of a set of elements each of which is a non-negative multiple of f. It follows (since L is Archimedean) that v is also a non-negative multiple of f. Hence, the band A consists of all real multiples of f. For the proof that A is a projection band, it is sufficient to show that

$$\sup\big(\inf(u, nf) \colon n = 1, 2, \ldots\big)$$

exists for every $u \in L^+$ (cf. Theorem 3.3 (ii)). For $n = 1, 2, \ldots$, we have $\inf(u, nf) = \alpha_n f$ for an appropriate increasing sequence $\{\alpha_n\}$ of non-negative numbers. Since $\alpha_n f \leqslant u$ holds for all n, it is impossible that $\alpha_n \uparrow \infty$. Hence $\alpha_n \uparrow \alpha_0 < \infty$, and so

$$\inf(u, nf) = \alpha_n f \uparrow \alpha_0 f,$$

where it has been used again that L is Archimedean. This shows that the desired supremum exists, and so A is a projection band.

THEOREM 4.2. *If L is Archimedean and $\{e_1, \ldots, e_n\}$ is a set of mutually disjoint atoms in L with the property that there exists no non-zero element in L disjoint to $e_1, \ldots, e_n$, then L is n-dimensional and $\{e_1, \ldots, e_n\}$ is a basis of L in the algebraic sense. The algebraic decomposition of any $f \in L$ as a sum of real multiples of the basis elements is exactly the decomposition of f as a sum of components of f in the bands generated by the basis elements.*

Proof. Since $\{e_1, \ldots, e_n\}$ is a linearly independent system, the dimension of L is at least n. The bands $B_1, \ldots, B_n$ generated by $e_1, \ldots, e_n$ are projection bands by the preceding theorem; given $u \in L^+$, let $\alpha_1 e_1, \ldots, \alpha_n e_n$ be the corresponding components of u. Then $0 \leqslant \alpha_i e_i \leqslant u$ for $i = 1, \ldots, n$, and so

$$0 \leqslant \alpha_1 e_1 + \ldots + \alpha_n e_n = \sup(\alpha_1 e_1, \ldots, \alpha_n e_n) \leqslant u,$$

where we have used that the sum and the supremum of a finite number of disjoint non-negative elements are identical. Thus

$$w = u - (a_1 e_1 + \ldots + a_n e_n) \geqslant 0,$$

and so $0 \leqslant w \leqslant u - a_i e_i$ for $i = 1, \ldots, n$. But $u - a_i e_i$ is the component of u in the band B_i^d; it follows that $w \in B_i^d$, i.e., $w \perp e_i$ for $i = 1, \ldots, n$. Hence $w = 0$ by hypothesis, so $u = a_1 e_1 + \ldots + a_n e_n$. The desired results follow immediately.

THEOREM 4.3. *If the Archimedean Riesz space L has the property that any system of mutually disjoint non-zero elements is finite, then L is of finite dimension, say of dimension n, and there exists a basis $\{e_1, \ldots, e_n\}$ of mutually disjoint atoms.*

Proof. Assuming that L does not consist exclusively of the null element, it will be proved first that L contains an atom. Indeed, L contains an element $u > 0$, and if u is an atom we are ready. If not, there exist $u_1, u_2 \in L$ such that $0 < u_1 \leqslant u$, $0 < u_2 \leqslant u$ and $u_1 \perp u_2$. If one of u_1, u_2 is an atom, we are ready; if not, we proceed and obtain non-zero and mutually disjoint elements $u_{11}, u_{12}, u_{21}, u_{22}$. The procedure breaks off after a finite number of steps since by hypothesis there exists no infinite disjoint system of non-zero elements. Hence, L contains an atom e_1; let B_1 be the corresponding principal projection band. If $B_1^d = \{0\}$, the proof is complete. If $B_1^d \neq \{0\}$, it is proved similarly that B_1^d contains an atom e_2. This procedure again breaks off after a finite number of steps, and the desired result follows then from the preceding theorem.

Every Archimedean space L of finite dimension n is of the kind described in the last theorem, and hence there exists a basis $\{e_1, \ldots, e_n\}$ of mutually disjoint atoms. The partial ordering in L is such that $f = a_1 e_1 + \ldots + a_n e_n \geqslant 0$ holds if and only if all coefficients a_i are $\geqslant 0$. We obtain, therefore, as a corollary the known result that L is isomorphic to n-dimensional number space R^n with coordinatewise ordering.

5. Spaces with a strong unit and with stable order convergence. The element $e > 0$ in the Riesz space L is called a *strong unit* if the ideal generated by e is already the whole space L. Evidently, $e > 0$ is a strong unit if and only if for any given $f \in L$ there exists a positive number a_f such that $|f| \leqslant a_f e$. In a space with a strong unit e relatively uniform convergence of a sequence f_n to f is equivalent to e-uniformly convergence of f_n to f.

We will prove now that in an Archimedean Riesz space with a strong unit stability of the order convergence is a severe restriction upon the space. Precisely, the following theorem holds:

THEOREM 5.1. *Let the Archimedean Riesz space L have a strong unit, and let L be either Dedekind σ-complete or have the projection property. Then order convergence in L is stable if and only if L is of finite dimension.*

Proof. If L is Archimedean and of finite dimension, then (according to the remarks in the final paragraph of the preceding section) L is isomorphic to R^n for some n, and hence it is evident that L has a strong unit and order convergence is stable.

Conversely, assume that the Archimedean space L has a strong unit e, and is either Dedekind σ-complete or has the projection property. Assume also that order convergence in L is stable and that in L there exists an infinite system of mutually disjoint non-zero elements. Let $\{f_n; n = 1, 2, \ldots\}$ be a countable subsystem, and denote by B_n the band generated by f_n. The space L has the principal projection property (we recall that Dedekind σ-completeness implies the principal projection property, and so of course does the projection property); for $n = 1, 2, \ldots$, let p_n be the component of e in B_n, and let $s_n = p_1 + \ldots + p_n$. Evidently the sequence $\{s_n\}$ is increasing, and $p = \sup s_n$ exists. If L is Dedekind σ-complete, this is evident; if L has the projection property, and if B is the band generated by the system $\{p_1, p_2, \ldots\}$, then the component p of e in B satisfies $p = \sup s_n$, i.e., $p - s_n \downarrow 0$. Indeed, assume that $0 \leqslant v \leqslant p - s_n$ holds for all n. Then, since $p - s_n$ has the component 0 in the bands $B_1, \ldots, B_n$, the same holds for v. It follows that $v \perp B_n$ for all n, and so $v \perp B$. On the other hand, we have $v \in B$ since $0 \leqslant v \leqslant p$. Hence $v = 0$, i.e., $p - s_n \downarrow 0$. In any case, therefore, s_n converges in order to p. Observing now that order convergence and relatively uniform convergence are equivalent on account of the stability of order convergence, we obtain that $s_n \to p$ (r.u.), which implies (as observed above) that s_n converges e-uniformly to p. Hence, given ε such that $0 < \varepsilon < 1$, there exists a natural number N such that $p - s_N \leqslant \varepsilon e$. Taking components in B_n for any $n > N$, we obtain $p_n \leqslant \varepsilon p_n$, which is impossible on account of $p_n \neq 0$. We have derived, therefore, a contradiction. Hence, every system of mutually disjoint non-zero elements in L is finite. It follows then from the preceding theorem that L is of finite dimension.

In order to illustrate the fact that it is really the existence of a strong unit which forces an Archimedean Riesz space with stable order convergence to be of finite dimension, we present the following example. Let L be the Riesz space of all real sequences $f = (f(1), f(2), \ldots)$ with only finitely many non-zero coordinates, and with pointwise ordering. This space is Dedekind complete and order convergence is stable. There is no strong unit in L, in agreement with the fact that L is not of finite dimension. Note that every principal band in L, considered as a Riesz space on its own, has a strong unit and, in agreement with the last theorem, every principal band is indeed of finite dimension.

The theorem in the present section, under the extra hypothesis that L is Dedekind complete, is known (cf. for example Theorem VI. 4.2 in [12]). The proof is then based on the representation of L as the Riesz space of all real continuous functions on a certain compact topological space. The simple proof presented here avoids this representation theorem.

6. Regular Riesz spaces. The Riesz space L is called *regular* if the following conditions are satisfied:

(i) L is Archimedean.

(ii) Order convergence in L is stable.

(iii) For any sequence $\{u_n\}$ in L^+ there exists a sequence $\{\lambda_n\}$ of positive real numbers such that the sequence $\{\lambda_n u_n\}$ is bounded.

All Archimedean spaces of finite dimension are regular. The space presented as an example in the preceding section satisfies (i) and (ii), but not (iii). The (real) sequence space l_∞ satisfies (i) and (iii), but not (ii). The spaces $l_p(1 \leqslant p < \infty)$ are regular, and the same holds more generally for spaces $L_p(1 \leqslant p < \infty)$ of p-th power summable functions with respect to a countable additive measure.

The notion of a regular Riesz space is due to Kantorovitch [4]. In the original definition the space was assumed to be also Dedekind complete; in the present discussion we will not need this extra assumption. We first recall a simple lemma.

LEMMA 6.1. (i) *If L is regular and if $\{f_n\}$ is an arbitrary sequence in L, then there exists a sequence $\{\lambda_n\}$ of positive real numbers such that $\lambda_n f_n \to 0$.*

(ii) *If L is regular and the double sequence $\{f_{nk}\}$ in L has the property that $f_{nk} \to_k f_n$ for every n, then there exists an element $u > 0$ such that for every n the sequence $\{f_{nk};\, k = 1, 2, \ldots\}$ converges u-uniformly to f_n.*

Proof. (i) Let $\{\mu_n\}$ be a sequence of positive real numbers such that $\{\mu_n |f_n|\}$ is bounded, so $0 \leqslant \mu_n |f_n| \leqslant v$ for some $v \in L^+$ and all n. Let $\lambda_n = n^{-1}\mu_n$ for every n. Then $0 \leqslant \lambda_n |f_n| \leqslant n^{-1} v$ holds for every n, and so $\lambda_n f_n \to 0$.

(ii) Since order convergence and relatively uniform convergence are equivalent, there exists for every n an element $u_n \in L^+$ such that $\{f_{nk};\, k = 1, 2, \ldots\}$ converges u_n-uniformly to f_n. Choose $\lambda_n > 0$ such that the sequence $\{\lambda_n u_n\}$ is bounded, say $\lambda_n u_n \leqslant u$ for all n. It follows that for every n the sequence $\{f_{nk};\, k = 1, 2, \ldots\}$ converges u-uniformly to f_n.

The Riesz space L is said to have the *diagonal property* if it follows from $f_{nk} \to_k f_n$ and $f_n \to f$ that there exists a "diagonal sequence" $\{f_{n,k(n)};\, n = 1, 2, \ldots\}$ with $k(1) < k(2) < \ldots$ such that $f_{n,k(n)} \to f$.

THEOREM 6.2. *The Archimedean Riesz space L is regular if and only if L has the diagonal property.*

Proof. Assume first that L is regular. Let $f_{nk} \to_k f_n$ and $f_n \to f$. By the preceding lemma there exists $u \in L^+$ such that every sequence $\{f_{nk};\, k = 1, 2, \ldots\}$ converges u-uniformly to f_n, and so there is for every n a natural number $k(n)$ such that

$$|f_{n,k(n)} - f_n| \leqslant n^{-1} u .$$

We may assume here that $k(1) < k(2) < \ldots$ It follows easily from $f_{n,k(n)} - f_n \to 0$ and $f_n - f \to 0$ that $f_{n,k(n)} \to f$.

For the converse, assume that L is Archimedean and possesses the diagonal property. In order to prove that order convergence in L is stable, assume that $f_k \to 0$, i.e., $|f_k| \leqslant w_k \downarrow 0$ for an appropriate sequence $\{w_k\}$ in L^+. Let $f_{nk} = n w_k$ for $n, k = 1, 2, \ldots$, Then $f_{nk} \to_k 0$ for $n = 1, 2, \ldots$, and so by the diagonal property there exists a diagonal sequence $f_{n,k(n)} \to 0$ with $k(1) < k(2) < \ldots$ In other words, we have $n w_{k(n)} \to 0$. Now, for any natural number k satisfying $k(n) \leqslant k < k(n+1)$, let $\lambda_k = n$. It is not difficult to see that $\lambda_k \uparrow \infty$ and $\lambda_k w_k \to 0$, so $\lambda_k f_k \to 0$. It remains to prove that for any sequence $\{u_n\}$ in L^+ there exists a sequence $\{\lambda_n : \lambda_n > 0\}$ such that $\{\lambda_n u_n\}$ is bounded. Set $f_{nk} = k^{-1} u_n$ for $n, k = 1, 2, \ldots$ Then $f_{nk} \to_k 0$ for $n = 1, 2, \ldots$, and so $f_{n,k(n)} \to 0$ for appropriate $k(n)$. In other words, setting $\lambda_n = \{k(n)\}^{-1}$, we have $\lambda_n u_n \to 0$, which implies that $\{\lambda_n u_n\}$ is bounded.

Just as in Theorem 5.1 we will assume now that L is either Dedekind σ-complete or L has the projection property. In Theorem 5.1 it was proved that if, in addition, L has a strong unit, then stability of the order convergence implies that L is of finite dimension. We will weaken now the condition that L has a strong unit, and assume only that every principal band in L has a strong unit; we will prove that regularity of L implies now that L is of finite dimension.

THEOREM 6.3. *Let the Riesz space L be either Dedekind σ-complete or have the projection property, and let every principal band in L have a strong unit. Then L is regular if and only if L is of finite dimension.*

Proof. We need only prove that regularity implies finite dimensionality. Observe first that, in view of Theorem 5.1, every principal band in L is of finite dimension. Furthermore, by Theorem 4.3, it will be sufficient to prove that any system of mutually disjoint non-zero elements is finite. Assume, therefore, that there exists a countably infinite system $\{f_n;\, n = 1, 2, \ldots\}$ of mutually disjoint non-zero elements. By the regularity of L there exists a corresponding sequence $\{\lambda_n : \lambda_n > 0\}$ such that $\{\lambda_n |f_n|\}$ is bounded; say $\lambda_n |f_n| \leqslant u \in L^+$ for all n. It follows that all elements f_n are included in the principal band of finite dimension generated by u. On the other hand, $\{f_n; n = 1, 2, \ldots\}$ is a linearly independent system by one of the remarks in section 2. Contradiction. Hence, L must be of finite dimension.

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Invariante Masse positiver Kontraktionen in $C(X)$

von

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1. Einleitung. Wir betrachten einen kompakten Hausdorffraum X, der quasi-stonesch ist, d.h. der Banachverband $C(X)$ ist bedingt σ-ordnungsvollständig, und einen positiven Operator T in $C(X)$, der konstante Funktionen invariant lässt (einen Markov-operator). Bekanntlich erscheinen solche Voraussetzungen oft in der Theorie der messbaren Abbildungen und in der Theorie der Markov-prozesse, wenn man den Körper aller messbaren Mengen (modulo Nullmengen) mit dem Körper aller offen-abgeschlossenen Teilmengen eines kompakten Hausdorffraumes identifiziert.

Extremalpunkte der Menge aller T^*-invarianten Wahrscheinlichkeitsmasse sind sogenannte ergodische Masse. Ein ergodisches Mass mit minimalem Träger ist von Interesse in Zusammenhang mit der letzten Arbeit von Schaefer [4]. Zunächst stellen wir die Frage: Wieviele ergodische Masse können mit einem gegebenen minimal-ergodischen Mass gemeinsamen Träger haben? Hierfür stellt Theorem 1 die "1 oder ∞" Regel auf.

In bezug auf die σ-Ordnungsvollständigkeit von $C(X)$ zeichnen sich ordnungsstetige Masse und Operatoren aus, die wir σ-additiv nennen. Wir stellen dann die Frage: Wann sind alle invariante Masse eines σ-additiven Operators σ-additiv? Theorem 2 antwortet darauf mit dem Mittelergodensatz und der endlichen Dimension der Menge aller invarianten Funktionen.

Umgekehrt behandeln wir auch die Frage: Wann kann kein σ-additives Mass invariant sein? Eine Antwort darauf ergibt sich aus der Charakterisierung (Theorem 3) des von allen σ-additiven, invarianten Massen annullierten Bandes. Aus Theorem 3 folgt auch die von Ito [2] bewiesene

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