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On duality between (L_1F)-spaces and (L_2F)-spaces*

by

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Dedicated to Professors
Stanisław Mazur and Władysław Orlicz
on the 40-th anniversary
of their scientific carreer

The (L_1F)-spaces and (L_2F)-spaces are generalizations of the (LF)-spaces defined in [2]. The (L_1F)-spaces defined in this paper are a little more special than the (L_1F)-spaces defined in [11]. This difference is inessential and for easier references we prefer to keep the same name.

In this paper a theory of duality is presented for (L_1F)-spaces and (L_2F)-spaces, which is so conceived that it contains an essential part of Grothendieck duality theory for (F)-spaces and (DF)-spaces (cf. [3] and also [6]), and includes (LF)-spaces of Dieudonné and Schwarz [2]. We prove that the functor of taking the adjoint equipped with the strongest topology [9] carries the (L_1F)-class into the (L_2F)-class and vice versa, the class (L_2F) into the class (L_1F) (cf. Propositions 3.1 and 3.2). The space $\mathscr{D}$ of all distributions [8] is an (L_2F)-space, but it is not an (L_1F)-space. This is worth mentioning since the space $\mathscr{D}$ has so far stayed outside any reasonable classification.

The problem started its history with Grothendieck's question about a class of spaces which is closed under a number of operations and within which an analogue of the closed graph theorem is still valid (cf. [4], p. 18-19). A certain aspect of this was clarified in [10], and afterwards widely discussed by Raikov [7]. In connection with the closed graph theorem, Słowikowski started to investigate in [12] the so called (L_1F)-, and (L_2F)-inductive families. However, no definition of the corresponding (L_1F)- and (L_2F)-spaces as used in this paper was given. The definition of (L_1F)-space comes later in [10] and subsequently in [11] in a slightly different form which would be less convenient for our purpose. The

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spaces (L_2F) have not been defined by name before. They appear, though, in [9] simply as adjoints to (L_1F)-spaces.

Let X be a linear space. We denote by $\{X, \pi\}$ the linear topological space X equipped with the topology π. We write

$$\{X, \pi\} \geqslant \{Y, \varrho\},$$

if there exists a one-one linear mapping $A: X \to Y$ which is π, ϱ-continuous.

1. (L_1F)**-decompositions and** (L_1F)**-spaces.** Let $U_{k,n}$ be vector spaces and let $\|\cdot\|_{k,n}^U$ be pseudonorms defined on them $(k, n = 1, 2, \ldots)$.

A double sequence $(\{U_{k,n}, \|\cdot\|_{k,n}^U\})$ is said to be an (L_1F)-*decomposition*, if

(1.1) All $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$ are complete pseudonormed spaces such that

$$\{U_{k,n}, \|\cdot\|_{k,n}^U\} \leqslant \{U_{k+1,n}, \|\cdot\|_{k+1,n}^U\},$$
$$\{U_{k,n}, \|\cdot\|_{k,n}^U\} \geqslant \{U_{k,n+1}, \|\cdot\|_{k,n+1}^U\}$$

for $k, n = 1, 2, \ldots$

We put

$$U_n = \bigcap_{k=1}^{\infty} U_{k,n}, \qquad U = \bigcup_{n=1}^{\infty} U_n.$$

(1.2) For every n, if $u \in U_n$ and $\|u\|_{k,n}^U = 0$ for all k, then $u = 0$.

(1.3) For every natural k and n, $U \cap U_{k,n}$ is a dense subspace of $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$.

We say that $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$ is an (L_1F)-*decomposition of* U, and we denote by π_n the topology of U_n which is induced by the sequence of pseudonorms $(\|\cdot\|_{k,n}^U : k = 1, 2, \ldots)$. It follows from (1.1) and (1.2) that $\{U_n, \pi_n\}$ is for each n an (F)-space. In the following, we assume that ι denotes the inductive topology of U which is induced by the decomposition $\{U_n, \pi_n\}$. This topology ι of U is said to be *induced* by the (L_1F)-decomposition $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$ of U.

A linear topological space $\{X, \tau\}$ is said to be an (L_1F)-*space*, if there exists an (L_1F)-decomposition of X such that the topology induced on X by this decomposition is a Hausdorff topology and coincides with τ.

It is to be noted that if condition (1.3) is not satisfied by a decomposition $(\{U_{k,n}, \|\cdot\|_{k,n}^U\})$ of U, then we can always produce a new decomposition $(\{U_{\bar{k},n}, \|\cdot\|_{\bar{k},n}^U\})$ setting $U_{\bar{k},n}$ = the closure of $U \cap U_{k,n}$ in $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$, which satisfies this condition.

Denote by $\mathfrak{N}$ the directed set of all sequences of natural numbers partially ordered coordinatewise. Let $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$ be an (L_1F)-decomposition of a linear space U. To every $\mathfrak{k} = (k_n) \in \mathfrak{N}$ we assign a sequence of pseudonormed spaces $(\{U_{k,n}, \|\cdot\|_{k,n}^U\})$ in the following way. We put

$$U_{\mathfrak{k},n} = \operatorname{Lin}\Big(\bigcup_{i=1}^{n} U_{k_i,i}\Big).$$

Then, we define there pseudonorms $\|\cdot\|_{\bar{k},n}$ setting

$$\|u\|_{\bar{k},n} = \inf\Big\{\sum_{i=1}^{n} \|u_i\|_{k_i,i}^U : u = \sum_{i=1}^{n} u_i,\ u_i \in U_{k_i,i}\Big\}.$$

Lemma 1.1. *The pseudonormed spaces* $\{U_{\mathfrak{k},n}, \|\ \|_{\bar{\mathfrak{k}},n}\}$ *are complete.*

Proof. For a given $\mathfrak{k} \in \mathfrak{N}$ and $n \in N$, consider the cartesian product

$$\boldsymbol{U}_{\mathfrak{k},n} = U_{k_1,1} \times \ldots \times U_{k_n,n}$$

with topology given by the pseudonorm

$$|\boldsymbol{u}|_{\mathfrak{k},n} = \sum_{i=1}^{n} \|u_i\|_{k_i,i}^U,$$

where $\boldsymbol{u} = (u_1, \ldots, u_n) \in \boldsymbol{U}_{\mathfrak{k},n}$. Clearly $\{\boldsymbol{U}_{\mathfrak{k},n}, |\cdot|_{\mathfrak{k},n}\}$ is complete. We put

$$\boldsymbol{L} = \{(u_1, \ldots, u_n) \in \boldsymbol{U}_{\mathfrak{k},n} : u_1 + \ldots + u_n = 0\}.$$

Of course $\boldsymbol{L}$ is a linear subspace of $\boldsymbol{U}_{\mathfrak{k},n}$ and the quotient space $\{\boldsymbol{U}_{\mathfrak{k},n}, |\cdot|_{\mathfrak{k},n}\}/\boldsymbol{L}$ is complete. In order to prove the Lemma, it is now enough to notice that the correspondence

$$\boldsymbol{U}_{\mathfrak{k},n}/\boldsymbol{L} \ni u/\boldsymbol{L} \leftrightarrow u_1 + \ldots + u_n \in U_{\mathfrak{k},n}$$

gives a linear mapping of $\boldsymbol{U}_{\mathfrak{k},n}/\boldsymbol{L}$ onto $U_{\mathfrak{k},n}$ which is an isometry with respect to the pseudonorm $|u|_{\mathfrak{k},n}$. This way the Lemma has been proved.

Write

$$U_{\mathfrak{k}} = \bigcup_{n=1}^{\infty} U_{\mathfrak{k},n}$$

and let $U_{\mathfrak{k}}'$ be the set of all linear functionals on $U_{\mathfrak{k}}$ which are continuous in every $\{U_{\mathfrak{k},n}, \|\cdot\|_{\mathfrak{k},n}\}$, $n = 1, 2, \ldots$ Let us define the subspace L by

$$L = \{u \in U_{\mathfrak{k}} : u'u = 0 \text{ for every } u' \in U_{\mathfrak{k}}'\},$$

and let us define the pseudonorms $\|\cdot\|_{\mathfrak{k},n}^U$ setting

$$\|u\|_{\mathfrak{k},n}^U = \inf\{\|u+z\|_{\bar{\mathfrak{k}},n} : z \in L \cap U_{\mathfrak{k},n}\}.$$

Proposition 1.1. *The pseudonormed spaces* $\{U_{\mathfrak{k},n}, \|\cdot\|_{\mathfrak{k},n}\}$ *are complete.*

Proof. It follows directly from Lemma 1.1.

Denote by $\iota_{\mathfrak{k}}$ the inductive topology of $U_{\mathfrak{k}}$ induced by the decomposition $\{U_{\mathfrak{k},n}, \|\cdot\|_{\mathfrak{k},n}^U\}$. It does not have to be a Hausdorff topology.

LEMMA 1.2. *For every $\mathfrak{k}\in\mathfrak{N}$, U is dense in $\{U_{\mathfrak{k}}, \iota_{\mathfrak{k}}\}$.*

Proof. It follows directly from the definition of the (L_1F)-decompositions, condition 1.3, that $U \cap U_{k_i,i}$ is dense in $\{U_{k_i,i}, \|\cdot\|_{k_i,i}\}$ for every i, and hence U must be dense in the inductive limit which is $\{U_{\mathfrak{k}}, \iota_{\mathfrak{k}}\}$. The Lemma has been proved.

PROPOSITION 1.2. *The projective limit of $\{U_{\mathfrak{k}}, \iota_{\mathfrak{k}}\}$ coincides with the space $\{U, \iota\}$, where ι is the topology induced on U by the considered (L_1F)-decomposition.*

Proof. We know from lemma 1.2 that U is dense in the projective limit of $\{U_{\mathfrak{k}}, \iota_{\mathfrak{k}}\}$. If we prove that the projective limit topology of $\{U_{\mathfrak{k}}, \iota_{\mathfrak{k}}\}$ is equivalent to the topology of U induced by the given (L_1F)-decomposition, then the proposition is established. But this is an immediate consequence of the following lemma:

LEMMA 1.3. *A pseudonorm defined on U is continuous in the topology ι induced on U by its (L_1F)-decomposition if and only if there exists $\mathfrak{k}\in\mathfrak{N}$ such that this pseudonorm is continuous in $\iota_{\mathfrak{k}}$-topology.*

Proof. If a pseudonorm $\|\cdot\|$ defined on U is continuous in every $\{U_n, \pi_n\}$, then to every n there corresponds k_n such that $\|\cdot\|$ is continuous with respect to $\|\cdot\|_{k_n,n}^U$ for every n. Hence, it is continuous with respect to every $\|\cdot\|_{\mathfrak{k},n}^U$, where $\mathfrak{k} = (k_n)$ and it is therefore continuous in $\iota_{\mathfrak{k}}$-topology. Conversely, for every $\mathfrak{k}\in\mathfrak{N}$ the identical injection of $\{U,\iota\}$ into $\{U_{\mathfrak{k}}, \iota_{\mathfrak{k}}\}$ is continuous, because each pseudonorm which is continuous with respect to every $\|\cdot\|_{k_n,n}^U$ is continuous in $\{U, \iota\}$. This way the lemma has been proved, and the proof of Proposition 1.2 has been completed.

Consider a sequence $\{X_n, \|\cdot\|_n\}$ of pseudonormed spaces such that X_n are subspaces of a linear space Z. Let $X = \bigcup_{n=1}^{\infty} X_n$ and denote by X' the space of all linear functionals over X which are continuous when restricted to every $\{X_n, \|\cdot\|_n\}$. We define

$$\|x\| = \inf\Big\{\sum_{n=1}^{\infty}\|x_n\|_n : x = \sum_{n=1}^{\infty} x_n,\ x_n\in X_n,\ \text{a. a. } x_n = 0\Big\},$$

where the greatest lower bound is extended over all such decompositions of x into sums of x_n that almost all x_n vanish and the infinite summation makes sense. Further, we define

$$\|x'\|_n' = \sup\{|x'x| : x\in X_n, \|x\|_n \leqslant 1\},$$

$$\|x'\|' = \sup\{|x'x| : x\in X, \|x\| \leqslant 1\},$$

$$\|x'\|_{\sup}' = \sup\{\|x'\|_n' : n = 1, 2, \ldots\}.$$

LEMMA 1.4 (cf. [15], lemma 5.1, p. 218). *The following identity holds:*

$$\|x'\|_{\sup}' = \|x'\|'.$$

Proof. We have $|x'x| \leqslant \|x'\|_n'\|x\|_n$ which implies $|x'x| \leqslant \|x'\|_{\sup}'\|x\|_n$ and consequently $|x'x| \leqslant \|x'\|_{\sup}'\|x\|$ which in turn gives $\|x'\|' = \|x'\|_{\sup}'$. Conversely, since $\|x\| \leqslant \|x\|_n$, we have $\|x'\|' \geqslant \|x'\|_n'$ and therefore $\|x'\|' \geqslant \|x'\|_{\sup}'$ and the lemma follows.

We shall now recall the definition of the strongest topology of the adjoint to a locally convex space (cf. [9]). Consider a locally convex space $\{X, \tau\}$ and let X' denote the space of all linear τ-continuous functionals on $\{X, \tau\}$. To every continuous pseudonorm $\varrho(\cdot)$ on X, we assign a Banach space $\{X', \|\cdot\|_\varrho'\}$ setting

$$\|x'\|_\varrho' = \sup\{|x'x| : \varrho(x) \leqslant 1\}$$

and

$$X_\varrho' = \{x'\in X' : \|x'\|'\}.$$

The family of all τ-continuous pseudonorms ϱ is directed strengthwise by inclusion. To every $x'\in X'$ there corresponds at least one ϱ such that $x'\in X_\varrho'$. The inductive limit of the family $\{X', \|\cdot\|_\varrho'\}$ provides X' with a topology which is called the *strongest topology* of X' (cf. [9]). The neigbourhoods of zero in X' in the strongest topology are all those sets in X' which absorb the polars of open sets of X. It is obvious that the strongest dual is ultrabornologic (cf. [1], p. 34) and it is therefore bornologic and barrelled.

Let us consider an (L_1F)-space $\{U, \iota\}$ and let $\{U_{k,n}, \|\cdot\|_{k,n}^U\}$ be its (L_1F)-decomposition. Let us denote by U' the adjoint to $\{U, \iota\}$, i.e. the space of all continuous linear functionals on $\{U, \iota\}$. For a given $u'\in U'$ we write

$$\|u'\|_{k,n}' = \sup\{|u'u| : u\in U \cap U_{k,n}, \|u\|_{k,n} \leqslant 1\}$$

and for a sequence $\mathfrak{k} = (k_n)\in\mathfrak{N}$ we write

$$U_{\mathfrak{k}} = \{u'\in U' : \|u'\|_{k_n}', n < \infty \text{ for } n = 1, 2, \ldots\},$$

and we denote by $\pi_{\mathfrak{k}}'$ the topology of $U_{\mathfrak{k}}'$ induced by pseudonorms $\|\cdot\|_{\mathfrak{k},n}'$, $n = 1, 2, \ldots$

PROPOSITION 1.2. *We have the identity*

$$U' = \bigcup_{\mathfrak{k}\in\mathfrak{N}} U_{\mathfrak{k}}'$$

and the inductive limit of $\{U_{\mathfrak{k}}', \pi_{\mathfrak{k}}'\}$ provides U' with the strongest topology of the adjoint.

Proof. Every $u'\in U'$ is continuous in at least one $\{U_{\mathfrak{k}}, \pi_{\mathfrak{k}}\}$ and therefore it is continuous in every $\{U \cap U_{k_n,n}, \|\cdot\|_{k_n,n}\}$, which means that

$\|u'\|'_{k_n,n}$ are all finite. Conversely, if for a given u' the relation $\|u'\|'_{k,n} < \infty$ holds for every n, then u' is continuous in $\{U_k, \pi_k\}$. This proves that $U' = \bigcup_{\mathfrak{k}\in\mathfrak{N}} U'_{\mathfrak{k}}$.

Each (F)-space $\{U'_{\mathfrak{k}}, \pi'_{\mathfrak{k}}\}$ is the inductive limit of Banach spaces $\{U'_{\mathfrak{k},\mathfrak{p}}, \|\cdot\|'_{\mathfrak{k},\mathfrak{p}}\}$, where $\mathfrak{k}, \mathfrak{p} \in \mathfrak{N}$ and

$$\|u'\|'_{\mathfrak{k},\mathfrak{p}} = \sup\{p_n^{-1}\|u'\|'_{k_n,n} : n = 1, 2, \ldots\},$$

$$U'_{\mathfrak{k},\mathfrak{p}} = \{u' \in U'_{\mathfrak{k}} : \|u'\|'_{\mathfrak{k},\mathfrak{p}} < \infty\}.$$

Hence, in place of the inductive limit of $\{U'_{\mathfrak{k}}, \pi'_{\mathfrak{k}}\}$ we can consider equivalently the inductive limit of $\{U'_{\mathfrak{k},\mathfrak{p}}, \|\cdot\|'_{\mathfrak{k},\mathfrak{p}}\}$. Applying lemma 1.4, we have

$$\|u'\|'_{\mathfrak{k},\mathfrak{p}} = \sup\{|u'u| : \|u\|_{\mathfrak{k},\mathfrak{p}} \leqslant 1\},$$

where

$$\|u\|_{\mathfrak{k},\mathfrak{p}} = \inf\Big\{\sum_{n=1}^{\infty} p_n\|u_n\|k_n, n : u = \sum_{n=1}^{\infty} u_n, u_n \in U_{k_n}, \text{ a.a. } u_n = 0\Big\},$$

and the greatest lower bound is extended on all decompositions of u into sums of u_n which almost all vanish, so that the occurring sum is in fact finite and makes sense.

To conclude the proof it is enough to notice that for every $\mathfrak{k}$ the pseudonorms $\|\cdot\|_{\mathfrak{k},\mathfrak{p}}$ from a basis in $\{U_{\mathfrak{k}}, \pi_k\}$ and therefore the whole family of pseudonorms

$$(\|\cdot\|_{\mathfrak{k},\mathfrak{p}} : \mathfrak{k}, \mathfrak{p} \in \mathfrak{N})$$

is a basis in (U, ι) and by the same token the inductive limit of $\{U'_{\mathfrak{k},\mathfrak{p}}, \|\cdot\|'_{\mathfrak{k},\mathfrak{p}}\}$ induces the strongest topology of the adjoint space.

PROPOSITION 1.3. *Every (LF)-space is an (L_1F)-space.*

Proof. Suppose that U is an (LF)-space (cf. [2]). Then there is a sequence $\{U_n, \tau_n\}$ such that $U_n \subset U_{n+1}$, $U = \bigcup_n U_n$ and such that the topology τ_n coincides with the topology induced on U_n by the topology τ_{n+1}. Suppose that τ_n is given by a pointwise non-decreasing sequence of pseudonorms $\|\cdot\|_{k,n}$, $k = 1, \ldots$ To every two natural numbers k and n there correspond q and M such that

$$\|x\|_{k,n} \leqslant M\|x\|_{q,n+1} \quad \text{for } x \in U_n.$$

Hence one can produce double sequences $k_{m,n}$ and $M_{m,n} > 0$ such that

1. $M_{m,n}\|x\|_{k_{m,n},n} \leqslant M_{m,n+1}\|x\|_{k_{m,n+1},n+1}$ for $x \in U_n$, $n = 1, 2, \ldots$;

2. for every n, the sequence of pseudonorms $\|\cdot\|_{k_{m,n},n}$, $m = 1, 2, \ldots$, induces the topology τ_n on U_n.

We put

$$|||x|||_m = \inf\Big\{\sum_{n=1}^{\infty} M_{m,n}\|x_n\|_{k_{m,n},n} : x = \sum_{n=1}^{\infty} x_n, x_n \in U_n\Big\},$$

where almost all x_n are zero, so that the infinite sums are always finite. We can always make $|||x|||_m$ pointwise non-decreasing. Then, for every n the topology τ_n is induced by pseudonorms $|||x|||_m$ restricted to U_n.

There exist complete pseudonormed spaces $\{U^m, |||\cdot|||_{\tilde{m}}\}$ such that

a. $\{U^m, |||\cdot|||_{\tilde{m}}\} \geqslant \{U^{m+1}, |||\cdot|||_{\widetilde{m+1}}\}$;

b. U is dense in every $\{U^m, |||x|||_{\tilde{m}}\}$;

c. $|||\cdot|||_{\tilde{m}}$ coincides with $|||\cdot|||_m$ on U for $m = 1, 2, \ldots$

We can now define:

$$U_{k,n} = \text{the closure in } \{U^k, |||\cdot|||_{\tilde{m}}\} \text{ of } U_n;$$

$$|||\cdot|||_{k,n} = \text{the restriction of } |||\cdot|||_k \text{ to } U_{k,n}.$$

It is easy to see that $\{U_{k,n}, |||\cdot|||_{k,n}\}$ is an (L_1F)-decomposition of U such that the topology ι inducing it on U coincides with the original topology τ. This way the proposition has been proved.

2. (L_2F)-decompositions and (L_2F)-spaces. A double sequence $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$ is said to be an (L_2F)-*decomposition*, if

(2.1) All $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$ are complete pseudonormed spaces such that

$$\{V_{k,n}, \|\cdot\|^V_{k,n}\} \geqslant \{V_{k+1,n}, \|\cdot\|^V_{k+1,n}\}$$

and

$$\{V_{k,n}, \|\cdot\|^V_{k,n}\} \leqslant \{V_{k,n+1}, \|\cdot\|^V_{k,n+1}\}.$$

We put

$$V_n = \bigcup_{k=1}^{\infty} V_{k,n}, \qquad V = \bigcap_{n=1}^{\infty} V_n.$$

(2.2) For a given $v \in V_{k,n}$, if $\|v\|^V_{p,n} = 0$ for $p > k$, then $\|v\|^V_{k,n} = 0$. For a given $v \in V$, if to every n there corresponds some k such that $\|v\|_{k,n} = 0$, then $v = 0$.

(2.3) For every natural k and n, $V \cap V_{k,n}$ is a dense subspace of $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$.

If these conditions are satisfied, we say that $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$ is an (L_2F)-*decomposition of* V.

We note that in the case when condition (2.3) is not satisfied by a decomposition $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$ of V, we can always produce a new decomposition $\{V_{\bar{k},n}, \|\cdot\|^v_{b,n}\}$ setting

$$V_{\bar{k},n} = \text{the closure in } \{V_{k,n}, \|\cdot\|^V_{k,n}\} \text{ of } V \cap V_{k,n}$$

and this new decomposition satisfies condition (2.3).

In each V_n we introduce the inductive topology ι_n induced by the decomposition $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$, $n = 1, 2, \ldots$ In the following, let $\{V, \pi\}$ denote the projective limit of $\{V_n, \iota_n\}$. We call π the *topology of* V *induced by its* (L_2F)*-decomposition.*

A linear topological space X is said to be an (L_2F)*-space*, if there exists an (L_2F)-decomposition of X such that the topology π induced on X by this decomposition is a Hausdorff topology and coincides with τ.

Let $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$ be an (L_2F)-decomposition of a linear space V. To every $\mathfrak{k} = (k_n) \in \mathfrak{N}$ we assign a sequence of pseudonormed spaces $\{V_{\mathfrak{k},n}, \|\cdot\|^V_{\mathfrak{k},n}\}$ in the following way. We put

$$V_{\mathfrak{k},n} = \bigcap_{i=1}^{n} V_{k_i,i}$$

and

$$\|v\|^V_{\mathfrak{p},n} = \max\{\|v\|^V_{k_i,i} : i = 1, \ldots, n\}.$$

This way we obtain an (F)-sequence $\{V_{\mathfrak{k},n}, \|\cdot\|^V_{\mathfrak{k},n}\}$ (cf. [13]), where every space $\{V_{\mathfrak{k},n}, \|\cdot\|^V_{\mathfrak{k},n}\}$ is complete. Setting

$$V_{\mathfrak{k}} = \bigcap_{n=1}^{\infty} V_{\mathfrak{k},n}$$

and

$\pi_{\mathfrak{k}} =$ the topology induced on $V_{\mathfrak{k}}$ by all pseudonorms $\|\cdot\|^V_{\mathfrak{k},n}$ $(n = 1, 2, \ldots)$,

we obtain a Fréchet space $\{V_{\mathfrak{k}}, \pi_{\mathfrak{k}}\}$. It is obvious that

$$V = \bigcup_{\mathfrak{k}\in\mathfrak{N}} V_{\mathfrak{k}}.$$

Using the terminology of [14], the family of (F)-spaces $\{V_{\mathfrak{k}}, \pi_{\mathfrak{k}}\}$ is an inductive family, and this particular kind of inductive family is called a σ^2*-family* (cf. [14], p. 3).

The following proposition holds:

PROPOSITION 2.1. *The inductive limit of* $\{V_{\mathfrak{k}}, \pi_{\mathfrak{k}}\}$ *coincides with the space* $\{V, \pi\}$, *where* π *is the topology induced on* V *by the* (L_2F)*-decomposition* $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$.

Proof. For $\mathfrak{k} = (k_n) \in \mathfrak{N}$ we have

$$\{V_n, \iota_n\} \leqslant \{V_{k_n,n}, \|\cdot\|^V_{k_n,n}\} \leqslant \{V_{\mathfrak{k}}, \pi_{\mathfrak{k}}\}.$$

Passing to the inductive limit on the right-hand side we obtain

$$\{V_n, \iota_n\} \leqslant \{V, \iota^\sim\},$$

where $\iota^\sim$ denotes the inductive limit of the topologies $\pi_{\mathfrak{k}}$. Passing to the projective limit on the left-hand side we obtain

$$\{V, \pi\} \leqslant \{V, \iota^\sim\},$$

where π denotes the topology of V which is induced by the (L_2F)-decomposition of it.

The converse relation is a little more difficult to prove. We shall first need the following

LEMMA 2.1. *If a pseudonorm* $\|\cdot\|$ *is continuous in* $\{V, \iota^\sim\}$, *then to every* $\mathfrak{k}\in\mathfrak{N}$ *there corresponds a natural number* n *such that* $\|\cdot\|$ *is continuous in* $\{V \cap V_{\mathfrak{k},n}, \|\cdot\|^V_{\mathfrak{k},n}\}$.

Proof. First, we note that

$$\{V \cap V_{k,n}, \|\cdot\|^V_{k,n}\}$$

is a σ^2-family (cf. [14]). Then, for every $\mathfrak{k}$, $\{V \cap V_{\mathfrak{k},n}, \|\cdot\|^V_{\mathfrak{k},n}\}$ is an (F)-sequence such that $\|\cdot\|$ restricted to the first of these spaces, for $n = 1$, $V \cap V_{\mathfrak{k},1}$ satisfies the continuity condition stated in proposition 6, γ of [13], p. 289, 282. Using that proposition, it follows from ε that there exists n such that $\|\cdot\|$ is continuous in $\{V \cap V_{\mathfrak{k},n}, \|\cdot\|^V_{\mathfrak{k},n}\}$, which concludes the proof of the lemma.

Going back to the proof of Proposition 2.1, we still have to prove that for every pseudonorm $\|\cdot\|$ which is continuous in V, there exists such n_0 that

$$\|v\| \leqslant N_k \|v\|^V_{k,n_0}$$

for some constant N_k and all $v \in V_{k,n_0}$, $k = 1, 2, \ldots$ It follows from Lemma 2.1 that for every $\mathfrak{k}$ there exist a natural number n_k and a constant $M_k > 0$ such that

$$\|v\| \leqslant M_k \|v\|^V_{k,n_k}$$

for all $v \in V'_{k,n_k} \cap V$ and $k = 1, 2, \ldots$

By the definition of (L_2F)-space $\|\cdot\|$ is continuous and the proof of the proposition has been completed.

Consider an (L_2F)-space $\{V, \pi\}$ with an (L_2F)-decomposition $\{V_{k,n}, \|\cdot\|^V_{k,n}\}$. Denote by V' the adjoint to $\{V, \pi\}$. For $v' \in V'$ we define a sequence of pseudonorms

$$\|v'\|'_{k,n} = \sup\{|v'v| : v \in V \cap V_{k,n}, \|v\|_{k,n} \leqslant 1\},$$

and next we define the subspaces

$$V'_n = \{v' \in V' : \|v'\|'_{k,n} < \infty,\ k = 1, 2, \ldots\}.$$

We denote by π'_n the topology of V'_n induced by pseudonorms $\|\cdot\|'_{k,n}$, $k = 1, 2, \ldots$

PROPOSITION 2.2. *We have the identity*

$$V' = \bigcup_{n=1}^{\infty} V'_n$$

and the inductive limit of $\{V'_n, \pi'_n\}$ *provides* V' *with the strongest topology of the adjoint.*

Proof. For $v' \in V'$ there exists n such that v' is continuous in $\{V_n, \iota_n\}$ and then it belongs to V'_n. Conversely, if $v' \in V'_n$ for some natural n, then v' is continuous in $\{V_n, \iota_n\}$, and therefore in $\{V, \pi\}$, too. This proves that $V' = \bigcup_{n=1}^{\infty} V'_n$. To prove the rest, we note that every $\{V'_n, \pi'_n\}$ is the inductive limit of the spaces $\{V'_{\cdot,n,\mathfrak{p}}, \|\cdot\|'_{\cdot,n,\mathfrak{p}}\}$ which are defined in the following way. First we define the pseudonorms $\|v'\|'_{\cdot,n,\mathfrak{p}}$. For $\mathfrak{p} \in \mathfrak{N}$ we set

$$\|v'\|'_{\cdot,n,\mathfrak{p}} = \sup\{p_k^{-1}\|v'\|'_{k,n} : k = 1, 2, \ldots\}$$

and then we set

$$V'_{\cdot,n,\mathfrak{p}} = \{v' \in V'_n : \|v'\|'_{\cdot,n,\mathfrak{p}} < \infty\}.$$

Therefore, the inductive limit of $\{V'_n, \pi'_n\}$ can be substituted by the inductive limit of

$$\{V'_{\cdot,n,\mathfrak{p}}, \|\cdot\|'_{\cdot,n,\mathfrak{p}}\},$$

where $(n, \mathfrak{p}) \in \{1, 2, \ldots\} \times \mathfrak{N}$. We are now able to use lemma 1.4. We have

$$\|v'\|'_{\cdot,n,\mathfrak{p}} = \sup\{|v'v| : \|v\|_{\cdot,n,\mathfrak{p}} \leqslant 1\},$$

where

$$\|v\|_{\cdot,n,\mathfrak{p}} = \inf\Big\{\sum_{k=1}^{\infty} p_k\|v_k\|_{k,n} : v = \sum_{k}^{\infty} v_k,\ v_k \in V_{k,n},\ \text{a. a. } v_k = 0\Big\},$$

the greatest lower bound being extended over all decompositions of v into a sum,

$$v = \sum_{k=1}^{\infty} v_k,$$

where almost all terms v_k vanish. These pseudonorms $\|v\|_{\cdot,n,\mathfrak{p}}$ form a basis for the topology of $\{V, \pi\}$. The proposition has been proved.

A slight modification of example II, [14], p. 4, yields an interesting (L_2F)-space of continuous functions which is adjoint to no (L_1F)-space.

Consider a normal topological space R and a double sequence $R_{k,n}$ of open subsets of R such that

$$R_{k+1,n} \subset R_{k,n} \subset R_{k,n+1}, \qquad k, n = 1, 2, \ldots,$$

$$R_n = \bigcap_{k=1}^{\infty} R_{k,n}, \qquad R = \bigcup_{n=1}^{\infty} R_n.$$

For every scalar-valued function f defined on R we introduce a pseudonorm

$$\|f\|_{k,n} = \sup\{|f(r)| : r \in R_{k,n}\},$$

and we use these pseudonorms to define subspaces $C_{k,n}(R)$ as

$$C_{k,n}(R) = \{f \in C(R) : \|f\|_{k,n} < \infty\},$$

where $C(R)$ denotes the space of all continuous functions on R. We set

$$C_n(R) = \bigcup_{k=1}^{\infty} C_{k,n}(R), \qquad C^{\sim}(R) = \bigcap_{n=1}^{\infty} C_n(R).$$

Since R is normal, to every $f \in C_{k,n}(R)$ there corresponds a bounded $\bar{f} \in C(R)$ such that

$$\|f - \bar{f}\|_{k,n} = 0.$$

Hence, the double sequence

$$\{C_{k,n}(R), \|\cdot\|_{k,n}\}$$

is an (L_2F)-decomposition of $C^{\sim}(R)$. The topology induced by this decomposition on $C^{\sim}(R)$ is separated, because every functional $F_t(f) = f(t)$ is continuous in this topology and these functionals separate points.

We note that if R is separable and σ-compact, then we can produce $(R_{k,n})$ in such a way that every R_n is compact and $R_{k,n}$, $k = 1, 2, \ldots$, form a complete system of neighbourhoods on R_n. Then, having in mind that every continuous function must be bounded in a neighbourhood of every point and using compactness of R_n, we conclude that every continuous function is bounded in some $R_{k,n}$, for every n, and therefore $C^{\sim}(R) = C(R)$.

3. Adjoint decompositions and spaces. Consider a double sequence $\{X_{k,n}, \|\cdot\|_{k,n}\}$ which is an (L_1F)-decomposition (an (L_2F)-decomposition) of a linear space X. First, we define the adjoint double sequence.

Let X^* be the algebraic adjoint of X. We define complete pseudonormed spaces $\{X'_{k,n}, \|\cdot\|'_{k,n}\}$, where $X'_{k,n} \subset X^*$. For every $x' \in X^*$ we set

$$\|x'\|'_{k,n} = \sup\{|x'x| : x \in X \cap X_{k,n}, \|x\|_{k,n} \leqslant 1\}.$$

The $\|x'\|'_{k,n}$ are pseudonorms and they are used to determine the subspaces $X'_{k,n} \subset X^*$:

$$X'_{k,n} = \{x' \in X^* : \|x'\|'_{k,n} < \infty\}.$$

PROPOSITION 3.1. *Subject to corrections for conditions* (1.3) *or* (2.3), *the adjoint to an* (L_1F)-*decomposition is an* (L_2F)-*decomposition and the adjoint to an* (L_2F)-*decomposition is an* (L_1F)-*decomposition.*

Proof. It is easy to see that passing to the adjoint decompositions, conditions (1.1) and (2.1) turn one into the other. Suppose now that a sequence $\{U_{k,n}, \|\cdot\|_{k,n}\}$ is an (L_1F)-decomposition of a linear space U. Then, condition (1.3) guarantees that U is dense in every linear topological space $\{U_{k,n}, \|\cdot\|_{k,n}\}$. Therefore, if a functional which is continuous in the space $\{U_{k,n}, \|\cdot\|_{k,n}\}$ vanishes on $U \cap U_{p,n}$ for all $p > k$, then it vanishes on $U \cap U_{k,n}$. This proves the first part of 2.2. If for a given u' we have $\|u'\|'_{k,n} = 0$ for some k and all n, it means that u' vanishes on every $U_n = \bigcap_{k=1}^{\infty} U_{k,n}$, and so v' is identically equal to zero.

Suppose now that $\{V_{k,n}, \|\cdot\|_{k,n}\}$ is an (L_2F)-decomposition of V. Then $\|v'\|'_{k,n}$ vanishes for every k only for such functionals v' which vanish on the whole space

$$V \cap V_n = \Big(\bigcup_{k=1}^{\infty} V_{k,n}\Big) \cap V,$$

which proves Proposition 1.2. The proposition has therefore been proved.

Consider an (L_1F)-space (an (L_2F)-space) $\{X, \tau\}$ with a corresponding decomposition $\{X_{k,n}, \|\cdot\|_{k,n}\}$. Let $\{X'_{k,n}, \|\cdot\|'_{k,n}\}$ denote the corresponding adjoint decomposition which has already been corrected for condition (2.3) ((1.3)).

Proposition 3.2. *The adjoint sequence $\{X'_{k,n}, \|\cdot\|'_{k,n}\}$ decomposes the space X', the adjoint space to $\{X, \tau\}$, and the topology it induces on X' is the strongest topology of the adjoint.*

Proof. Suppose that X is an (L_1F)-space. It follows from Proposition 1.2 that the strongest topology of X' is given by the inductive limit $\{X'_k, \|\cdot\|'_k\}$ which by virtue of Proposition 2.1 is the topology which is induced in X' by the decomposition $\{X'_{k,n}, \|\cdot\|'_{k,n}\}$.

Suppose now that $\{X, \tau\}$ is an (L_2F)-space. It follows from Proposition 2.2 that the strongest topology for X' is just the topology which is given in X' by the inductive limit of $\{X'_n, \|\cdot\|'_n\}$. By virtue of the definition of the (L_1F)-topology, this inductive limit topology is exactly the (L_1F)-topology induced by the adjoint decomposition. The proposition has been proved.

4. The closed graph theorem for (L_1F)- and (L_2F)-spaces. Every (L_1F)-space and (L_2F)-space is an inductive limit of (F)-spaces. It follows directly from the definition of (L_1F)-space and from Proposition 2.1 for (L_2F)-spaces. Hence, they are also inductive limits of Banach spaces and we have the following

Proposition 4.1. *Both (L_1F)- and (L_2F)-spaces are ultrabornologic* (cf. [1], p. 34).

We shall recall some definitions of [11] and [12]. Let $\{X, \tau\}$ be a linear topological space.

A sequence (x_n), $x_n \in X$, is said to be *inductively convergent in* X (cf. [11], p. 100), if there exists an (F)-space $\{Y, \varrho\} \geqslant \{X, \tau\}$ such that $x_n \in Y$ and $\{x_n\}$ is convergent in $\{Y, \varrho\}$.

A sequence (x_n) is said to be *co-convergent to zero* (cf. [10], p. 21, and [5], p. 385), if there exists a sequence (t_n) of scalars such that $\lim |t_n| = \infty$ and the set $\{t_n x_n\}$ is bounded in $\{X, \tau\}$, which means that it is absorbed by every neighbourhood of zero in $\{X, \tau\}$.

A sequence (x_n) is *co-convergent to x*, if $(x_n - x)$ is co-convergent to zero.

A sequence (x_n) is said to be *co-fundamental*, if there exists a sequence t_n of scalars with $\lim |t_n| = \infty$ such that the set $\{t_n(x_n - x_m):$ $n < m\}$ is bounded in $\{X, \tau\}$.

A linear topological space $\{X, \tau\}$ is said to be *co-complete*, if every co-fundamental sequence in $\{X, \tau\}$ has a limit point in $\{X, \tau\}$.

We can as well characterize the co-complete spaces in the following way. A space $\{X, \tau\}$ is co-complete, if every bounded absorbing closed convex set in $\{X, \tau\}$ spans in X a Banach space which is continuously injected in $\{X, \tau\}$.

It is easy to see that the following proposition holds:

Proposition 4.2. *A space $\{X, \tau\}$ is co-complete if and only if to every bounded subset B of $\{X, \tau\}$ there corresponds an (F)-space $\{Y, \varrho\} \geqslant \{X, \sigma\}$ such that B is contained and bounded in $\{Y, \varrho\}$.*

It follows from this proposition that for co-complete spaces the inductive convergence and co-convergence are the same. However, (L_1F)-spaces or (L_2F)-spaces need not be co-complete.

Let $\{Y, \varrho\}$ be an ultrabornologic space and let $\{X, \tau\}$ be either an (L_1F)-space or (L_2F)-space. A linear mapping

$$T: \{Y, \varrho\} \to \{X, \tau\}$$

is said to be *inductively closed* (*co-closed*), if for every two sequences (y_n) and (x_n), $y_n \in Y$, $x_n \in X$, which are inductively convergent (co-convergent) to y and x, respectively, in $\{Y, \varrho\}$ and $\{X, \tau\}$

$$\text{if } x_n = Ty_n \text{ for every } n, \text{ then } x = Ty.$$

Proposition 4.3. *Every linear sequentially co-closed transformation from an ultrabornologic space $\{Y, \varrho\}$ into an (L_iF)-space $\{X, \tau\}$, $i = 1, 2$, is continuous.*

Proof. We note first that $\{X, \varrho\}$ is the inductive limit of either an inductive sequence of (F)-spaces in the case of (L_iF)-space or of the inductive family used in proof of Proposition 2.1. In both cases the inductive families admit an overhelming set of components (cf. [14], p. 4).

We denote by $\mathcal{Y}$ the inductive family of all Banach spaces that are subspaces of Y with continuous identical injection in $\{Y, \varrho\}$. The inductive family $\mathcal{Y}$ decomposes $\{Y, \varrho\}$. Applying theorem 4 of [14] we find that to every $\{Z, \zeta\} \epsilon \mathcal{Y}$ there corresponds an $\{S, \sigma\} \epsilon \mathfrak{X}$ such that $TZ \subset S$. But $\{Y, \varrho\}$ is ultrabornologic, and so it is the inductive limit of $\{U, \zeta\} \epsilon \mathfrak{h}$. The transformation T is therefore a continuous transformation from $\{Y, \varrho\}$ into $\{X, \tau\}$. The proposition has been proved.

COROLLARY 4.1. *Consider a locally convex space $\{X, \tau\}$ which is either (L_1F)-space or (L_2F)-space. Then the topology τ of X is the coarsest ultrabornologic topology of X among those topologies of X which preserve limits of inductively convergent sequences in $\{X, \tau\}$.*

This means that if ϱ is an ultrabornologic topology of X such that for every sequence (x_n), $x_n \epsilon X$, which is inductively convergent to u in $\{X, \tau\}$ and v in $\{X, \varrho\}$ it is always $u = v$, then the topology ϱ is finer than τ.

The question which is the finest ultrabornologic topology of a given linear locally convex space is a trivial one. The finest locally convex ultrabornologic topology of a given locally convex space $\{X, \tau\}$ is the inductive limit topology of all finite-dimensional linear subspaces of X. Clearly, a sequence of elements $x_n \epsilon X$ is inductively convergent to x in this topology if and only if it is contained in a finite-dimensional subspace of X and convergent to x in this subspace. Such sequence must therefore be convergent to the same limit in any reasonable topology of X. This shows that Corollary 4.1 cannot be strengthened.

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