STUDIA MATHEMATICA, T. XXXI. (1968)

Some remarks on Dragilev's theorem

by

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M. M. Dragilev has shown that in many nuclear Fréchet spaces ([1]) all the bases are quasisimilar (see Definition 1). The most recent theorem on this subject [3] is a strengthening of former results of Dragilev [2] and Mitiagin [5], nevertheless its proof, based on the technique of approximative invariants, is much simpler than the former proofs. This theorem can be splitted into two parts:

(A) If X is a nuclear Fréchet space with a regular (see Definition 3) basis (x_n), and if (y_n) is another basis in X, then there is a permutation of indices (r_n) such that (y_{r_n}) is a regular basis in X.

(B) Under some additional conditions on X, any two regular bases (x_n) and (y_n) in X are semisimilar, i.e. there are positive numbers c_n such that the series $\sum\limits_n t_n x_n$ is convergent if and only if $\sum\limits_n t_n c_n y_n$ is convergent.

The main result of the present paper, Theorem 2.2, is a strengthening of (A) above; in our setting the basis (x_n) of X need not to be regular, and (y_n) is a basis in a complemented subspace Y of X, not necessarily in the whole of X. The assertion gives a relationship between (y_n) and (x_n).

This result can be used to give isomorphical characterization of complemented subspaces in some spaces. For instance, if Y is a Fréchet space which is isomorphic to a complemented subspace of the space of all entire functions of one complex variable, then Y is isomorphic to a subspace spanned on a subsequence (z^{j_n}) of the step system (z^n).

In section 2 we prove Theorem 2.2 (mentioned above), and derive some corollaries; one of them is the Dragilev's theorem. The proof is a combination of Dragilev's argument with a lemma of [1], here — 1.7.

([1]) Following Bourbaki, by a *Fréchet space* we mean a locally convex complete linear metric space; these spaces have been introduced by S. Mazur and W. Orlicz and called by them B_0-*spaces*. In fact, the Dragilev's theorems are also valid for some non-metrizable linear topological spaces, but, for simplicity, we shall restrict our attention to the case of Fréchet spaces.

For completeness we present this proof here, although a big part of it is literally the same as in [3].

In section 3 we discuss a relationship between Theorem 2.1 and the problem of isomorphic characterization of complemented subspaces.

In Preliminaries we introduce the terminology and state the propositions which are used in proofs of the results of sections 2 and 3. These propositions are either well-known or are reformulations of known results in terms of matrix representation of nuclear spaces.

1. Preliminaries. In the sequel X will denote an infinite-dimensional Fréchet space over the field of real or complex scalars. By a *seminorm* on X we mean a continuous non-negative functional $|\cdot|$ on X such that $|x+y| \leqslant |x|+|y|$ and $|ty| = |t|\,|x|$ for all vectors x, y in X and all scalars t. A system of seminorms $(|\cdot|_p)$, $p = 1, 2, \ldots$, on X is said to be *admissible*, if $\lim\limits_n x_n = x$ in the topology of X if and only if $\lim\limits_n |x_n - x|_p = 0$ for $p = 1, 2, \ldots$ and, moreover, $|x|_1 \leqslant |x|_2 \leqslant |x|_3 \leqslant \ldots$

Further we shall use the following well-known facts:

1.0. *If $(|\cdot|_p)$ is an admissible system of seminorms on X, and $|\cdot|$ is any seminorm on X, then there exist an index q and a constant $C > 0$ such that $|x| \leqslant C|x|_q$ for all $x \in X$.*

1.1. *If $|\cdot|_p$, for $p = 1, 2, \ldots$, are seminorms on X and, for all x in X, $\|x\| = \sup\limits_p |x|_p < \infty$, then $\|\cdot\|$ is a seminorm.*

We recall that a sequence (x_n) of vectors in X is said to be a *basic sequence*, if there is a closed subspace Y of X such that any y in Y can be uniquelly expressed in a form

$$y = \sum_{n=1}^{\infty} t_n x_n;$$

(x_n) is a *complemented basic sequence* (briefly: CBS), if the subspace Y is complemented in X; (x_n) is a *basis* in X, if $Y = X$. It is well known that the coefficients $t_n = f_n(x)$ of the expansion with respect to the basis are continuous linear functionals: they will be called the *coefficient functionals* of the basis (x_n).

Let (x_n) be a basis in X. A seminorm $\|\cdot\|$ on X is said to be (x_n)-*normal*, if $\|f_j(x)x_j\| \leqslant \|x\|$ for every $x \in X$ and every j.

We recall also the following proposition:

1.2. *For any admissible system $(|\cdot|_p)$ of seminorms on X, there is an admissible system $(\|\cdot\|_p)$ of (x_n)-normal seminorms.*

Proof. Let $\|x\|_p = \sup\limits_{j,k} \sum\limits_{i=j}^{j+k} |f_i(x)x_i|_p$.

1.3. *If (y_n) is a CBS in X, then there is a system (g_n) of continuous linear functionals (g_n) biorthogonal to (y_n) such that for any seminorm $|\cdot|$ on X,*

$$\sup_n |g_n(x)y_n| < \infty \quad \textit{for all } x \textit{ in } X. \tag{1}$$

Proof. Let $Y = \overline{\operatorname{span}}(y_n)$, the closed subspace of X spanned on (y_n), and let $P: X \to Y$ be a continuous linear projection onto Y. Let $h_n \in Y^*$ be coefficient functionals of (y_n). Then $g_n = h_n P$ have the required property.

It can also be shown that, in the case where X is nuclear, any basic sequence admitting biorthogonal functionals (g_n) satisfying condition (1), is a CBS.

Definition 1. Basic sequences (x_n) and (y_n) are *similar*, if the convergence of the series $\sum\limits_n t_n x_n$ implies that of $\sum\limits_n t_n y_n$ and vice versa. (x_n) and (y_n) are *semisimilar*, if there exist scalars a_n making (x_n) and $(a_n y_n)$ similar. (x_n) and (y_n) are *quasisimilar*, if there are permutations (r_n) and (s_n) of positive integers such that (x_{r_n}) and (y_{s_n}) are basic sequences and are semisimilar.

It is well-known that

1.4. *Let X and Y be Fréchet spaces. If there exist bases (x_n) and (y_n) in X and in Y, respectively, such that (x_n) and (y_n) are similar, then the spaces X and Y are isomorphic under the map: $\sum\limits_{n=1}^{\infty} t_n x_n \to \sum\limits_{n=1}^{\infty} t_n y_n$. Hence the quasisimilarity of the bases also implies isomorphism of the spaces.*

Let U and V be convex symmetric neighbourhoods of zero in X. The *n-dimensional Kolmogorov diameter* of V with respect to U is

$$d_n(V, U) = \inf_{\dim L \leqslant n} \inf\{t > 0 \colon L + tU \supset V\},$$

the first inf taken over all at most n-dimensional linear subspaces of X.

The space X is *nuclear* if for every U there exists a V such that $\sum\limits_{n=1}^{\infty} d_n(V, U) < \infty$. (By Mitiagin [5], this definition is equivalent to the standard definition of nuclear spaces by means of nuclear operators, see e.g. Pietsch [1].) We recall that *every closed subspace of a nuclear space is nuclear.*

Definition 2. A basic sequence (x_n) in a nuclear Fréchet space X is said to be *represented* by a matrix $[a_{pn}]$, $p, n = 1, 2, \ldots$ (briefly: $(x_n) \sim [a_{pn}]$), if there is an admissible system $(|\cdot|_p)$ of (x_n)-balanced seminorms on the space $Y = \overline{\operatorname{span}}(x_n)$ such that $|x_n|_p = a_{np}$. If X has a basis represented by $[a_{pn}]$, we will say also that X is *representable* by the matrix $[a_{pn}]$.

The classical Dynin-Mitiagin [4] theorem states that every basis (x_n) in any nuclear Fréchet space is *absolute*, i.e. $\sum |f_n(x)x_n| < \infty$ for any x in X and any seminorm $|\cdot|$ on X. Each of the next two propositions is equivalent to this theorem:

1.5. *Let (y_n) be a basic sequence in a nuclear Fréchet space X. Then for every seminorm $|\cdot|$ on X there is a seminorm $\|\cdot\|$, with $|x| \leqslant \|x\|$ and $\sum_n |y_n|/\|y_n\| < \infty$ (here we assume $0/0 = 0$).*

1.6. *If basic sequences (x_n) and (y_n) in nuclear Fréchet spaces X and Y can be represented by the same matrix $[a_{pn}]$, then (x_n) and (y_n) are similar (of course, provided that both spaces are over the same field of scalars).*

The following lemma is a consequence of 1.5:

1.7. *If (x_n) is a basis in a nuclear Fréchet space X, then (x_n) can be reordered in such a way that for any seminorm $|\cdot|$ there is a seminorm $\|\cdot\|$ such that*

$$\|x_n\| \geqslant n^2 |x_n| \quad \text{for } n = 1, 2, \ldots \tag{2}$$

The proof of this lemma is given in [1], but for completeness we present it also here.

By 1.5, there is an admissible system of seminorms $(|\cdot|_p)$ on X such that

$$\sum_{n=1}^{\infty} |x_n|_p/|x_n|_{p+1} < \infty \quad \text{for all } p.$$

Let $c_{pn} = |x_n|_p/|x_n|_{p+1}$. We can easily define by induction a sequence (c_n) of positive numbers such that

$$\sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad c_n \geqslant c_{pn}$$

for all but finitely many values of n. Let (r_n) be a permutation of positive integers such that (c_{r_n}) is non-increasing; then $K = \sup n c_{r_n} < \infty$. Hence, for each p, $n c_{pr_n} \leqslant n c_{r_n} \leqslant K$ for all but finitely many n, and therefore there exist constants K_p such that $n c_{pr_n} \leqslant K_p$ for all n. The last inequality means that

$$K_p |x_{r_n}|_{p+1} \geqslant n |x_{r_n}|_p.$$

Now let $|\cdot|$ be an arbitrary seminorm on X. Since the system $(|\cdot|_p)$ is admissible, there exist a constant $C > 0$ and an index p_0 such that $|x| \leqslant C|x|_{p_0}$. Letting $\|x\| = C\cdot K_{p_0}\cdot K_{p_0+1}|x|_{p_0+2}$, we obtain

$$\|x_{r_n}\| \geqslant C\cdot n\cdot K_{p_0}|x_{r_n}|_{p_0+1} \geqslant C\cdot n^2 |x_{r_n}|_{p_0} \geqslant n^2 |x_{r_n}|,$$

which completes the proof of the lemma.

Definition 3. A matrix $[a_{pn}]$ is of type (D_0), if all the numbers a_{pn} are positive, and for each p, the sequence $(a_{pn}/a_{p+1\,n})$ is non-increasing with respect to n. $[a_{pn}]$ is of type (D_1), if it is of type (D_0), and, moreover, $a_{1n} = 1$ for all n, and for every p there is a q, with $a_{qn} \geqslant a_{pn}^2$ for all n. $[a_{pn}]$ is of type (D_2), if it is of type (D_0) and, moreover, $\lim_p a_{pn} = 1$ for all n, and for every p there is a q with $\lim_n a_{pn}/a_{qn}^2 = 0$.

A basic sequence (x_n) in a nuclear Fréchet space X is said to by of type (D_i), if there is a matrix of the type (D_i) representing it. Bases of type (D_0) are also called *regular* [3].

We obviously have

1.8. *If $[a_{pn}]$ is a matrix of a type (D_i) and (k_n) is a non-decreasing sequence of positive integers, with $\lim_n k_n = \infty$, then the matrix $[a_{pk_n}]$ is of the same type (D_i).*

The classical examples of nuclear Fréchet spaces are:

1° The spaces of analytic functions defined on an open domain D in the complex r-dimensional space, with the topology of uniform convergence on compact subsets of D.

2° The spaces of infinitely differentiable functions defined on a compact C^∞-manifold (with or without boundary), under the topology of uniform convergence of the functions and all their derivatives.

We shall list some of these spaces:

$A(\mathscr{C})$ — the space of all entire functions of one complex variable. The sequence (z^{n-1}) is a basis in this space and is represented by the matrix $[(e^{n-1})^{p-1}]$, which is of type (D_1).

$A(\mathscr{C}^r)$ — the space of all entire functions of r complex variables. The system of mononomials $(z_1^{n_1}\cdot z_2^{n_2}\ldots z_r^{n_r})_{n_1,\ldots,n_r=1}^{\infty}$ is a basis in this space; this basis, when properly ordered, is represented by the matrix $[(e^{n-1})^{p-1}]$ (see [8]), which is of type (D_1).

$A(\mathscr{D})$ — the space of all analytic functions of r complex variables defined on a convex bounded open domain $\mathfrak{D}$ is representable by the matrix $[e^{-\sqrt[r]{n}/p}]$, of type $(\mathscr{D}_2)$, see [10], and for the special case of $\mathscr{D} = \mathscr{C}_0^r$, the r-cylinder — [8] and [9].

$C^\infty(T)$ — the space of all infinitely differentiable periodic functions defined on the interval $[-\pi, \pi]$. The trigonometric system $1, \cos t, \sin t, \cos 2t, \ldots$ (and also the system (e^{int}), in the complex case) are bases. They are represented by the matrix $[n^{p-1}]$, which is of type (D_1).

According to Ogrodzka [6], spaces $C^\infty(M)$ of all infinitely differentiable functions on a compact finite-dimensional manifold M are represented, in general, by the matrix $[n^{p-1}]$, and therefore are isomorphic to the space $C^\infty(T)$.

We shall conclude the preliminaries with two propositions concerning Kolmogorov diameters.

1.9. *If* (x_n) *is a basis in* X, a_n, b_n *are positive numbers such that* $b_1/a_1 \geqslant b_2/a_2 \geqslant b_3/a_3 \geqslant \ldots$ *and* A, B *denote the cubes*

$$\Big\{x = \sum_{n=1}^{\infty} t_n x_n\colon\ |t_n| \leqslant a_n\Big\}, \qquad \Big\{x = \sum_{n=1}^{\infty} t_n x_n\colon\ |t_n| \leqslant b_n\Big\},$$

then $d_n(B, A) = b_{n+1}/a_{n+1}$ *for* $n = 0, 1, 2, \ldots$

This proposition is intuitively clear; a precise proof of it follows, for instance, from the Pietsch's argument ([7], Sect. 9.1).

By the *diametral dimension* and the *inverse diametral dimension* of the space X we mean the sets of scalar sequences:

$$\delta(X) = \{(t_n)\colon\ \underset{U}{\exists}\,\underset{V}{\forall}\ \lim_n t_n/d_{n-1}(V, U) = 0\}$$

and

$$\delta'(X) = \{(t_n)\colon\ \underset{U}{\forall}\,\underset{V}{\exists}\ \lim_n t_n\cdot d_{n-1}(V, U) = 0\},$$

where V and U run over all convex symmetric neighbourhoods of zero in X.

1.10. *The sets* $\delta(X)$ *and* $\delta'(X)$ *are isomorphic invariants of spaces* X, *moreover, if* X *and* Y *are nuclear Fréchet spaces and* Y *is isomorphic to a subspace of* X, *then*

(3) $$\delta(Y) \subset \delta(X) \quad \textit{and} \quad \delta'(Y) \supset \delta'(X).$$

If X *has a basis* (x_n) *represented by a matrix* $[a_{pn}]$ *of type* (D_i), *then*

(4)
$$\delta(X) = \Big\{(t_n)\colon\ \underset{p}{\exists}\,\underset{q}{\forall}\ \lim_n t_n\frac{a_{qn}}{a_{pn}} = 0\Big\}, \qquad \delta'(X) = \Big\{(t_n)\colon\ \underset{p}{\forall}\,\underset{q}{\exists}\ \lim_n t_n\frac{a_{pn}}{a_{qn}} = 0\Big\},$$

(5) $$\delta(X) = \{(t_n)\colon\ \underset{p}{\forall}\ \lim_n t_n a_{pn} = 0\} = \Big\{(t_n)\colon\ \sum_n t_n x_n \textit{ is convergent}\Big\}, \quad \textit{if } i = 1;$$

(6) $$\delta'(X) = \{(t_n)\colon\ \underset{p}{\forall}\ \lim_n t_n a_{pn} = 0\} = \Big\{(t_n)\colon\ \sum_n t_n x_n \textit{ is convergent}\Big\}, \quad \textit{if } i = 2.$$

The invariance of sets $\delta(X)$ and $\delta'(X)$ immediately follows from their definitions. For the proof of (3) see Mitiagin [5], p. 81.

Proof of (4). Write

(7) $$|x|_p = \sup_n |f_n(x)\,a_{pn}|, \quad p = 1, 2, \ldots$$

If $(\|\cdot\|_p)$ is any admissible (x_n)-normal system of seminorms on X such that $\|x_n\|_p = a_{pn}$, then, by 1.5, for every p there is a q such that

$$c_p = \sum_{n=1}^{\infty} \|x_n\|_p/\|x_n\|_q < \infty,$$

and therefore

$$\|x\|_p \leqslant \sum_{n=1}^{\infty} |f_n(x)|\cdot\|x_n\|_p \leqslant c_p\|x\|_q,$$

and since the seminorms $(\|\cdot\|_p)$ are (x_n)-normal, we obtain

$$|x|_p \leqslant \|x\|_p \leqslant c_p\|x\|_q \quad \text{for all } x \text{ in } X,$$

i.e. system (7) is also admissible. Using this fact, and also

(8) $$d_n(V', U') \subset d_n(V, U) \quad \text{for } V' \subset V,\ U \subset U';$$

$$d_n(aV, bU) = \frac{a}{b}\cdot d_n(V, U) \quad \text{for } a, b > 0,$$

we conclude that

$$\delta(X) = \{(t_n)\colon\ \underset{p}{\exists}\,\underset{q}{\forall}\lim_n t_n/d_n(K_q, K_p) = 0\}$$

and

$$\delta'(X) = \{(t_n)\colon\ \underset{n}{\forall}\,\underset{q}{\exists}\lim_n t_n\cdot d_n(K_q, K_p) = 0\},$$

where $K_j = \{x\colon\ |x|_j \leqslant 1\} = \{x = \sum_{n=1}^{\infty} t_n x_n\colon\ |t_n| \leqslant 1/a_{jn}\}$ for $j = 1, 2, \ldots$

Now, from condition (D_0) it follows that $d_n(K_q, K_p)$ can be computed by means of 1.9. This gives the required formula (4).

Proof of (5) and (6). From conditions (D_1) and (D_2) it follows that in the first formula of (4) we may replace a_{pn} by $a_{1n} = 1$ and in the second formula we may replace a_{qn} by $\lim_q a_{qn} = 1$, which gives the middle expressions in (5) and (6). By 1.5, the condition $\underset{p}{\forall}\ \lim_n t_n|x_n|_p = 0$ implies that $\sum_{n=1}^{\infty} |t_n x_n|_p < \infty$ for all p which in turn implies that the series $\sum_n t_n x_n$ is convergent. Hence we get the (second) equalities of (5) and (6).

2. Dragilev theory. All the results of this section are consequences of the following:

2.0. Crucial Lemma. *Suppose that* X *is a nuclear Fréchet space with a basis* (x_n), (f_n) *is the sequence of coefficient functionals of the basis, and* (y_n) *is a CBS in* X. *Then there exist positive integers* k_n *with* $\lim_n k_n = \infty$, *such that $f_{k_n}(y_n) \neq 0$ for all n and such that for any (x_n)-normal seminorm $|\cdot|_0$ there is a seminorm $\|\cdot\|$ satisfying the condition*

$$a_n |x_{k_n}|_0 \leqslant |y_n|_0 \leqslant a_n \|x_{k_n}\|, \tag{9}$$

where

$$a_n = |f_{k_n}(y_n)| \quad \textit{for } n = 1, 2, \ldots \tag{10}$$

Proof. Reordering the basis (x_n), if necessary, we may assume that the assertion of 1.7 is satisfied. Let (g_n) be linear functionals biorthogonal to (y_n) and satisfying condition (1) of 1.3. Since,

$$1 = g_n(y_n) = \sum_{k=1}^{\infty} g_n(x_k) f_k(y_n),$$

we get

$$\sum_{k=1}^{\infty} |g_n(x_k) f_k(y_n)| \geqslant 1.$$

Comparing the last series with $\sum_k k^{-2}$, we conclude that for every n there is a k_n with $|g_n(x_{k_n}) f_{k_n}(y_n)| \geqslant \sigma^{-1} \cdot k_n^{-2}$, where $\sigma = \sum_{n=1}^{\infty} n^{-2}$. Hence, by (10),

$$|g_n(x_{k_n})| \geqslant \sigma^{-1} \cdot k_n^{-2} \cdot a_n^{-1} \quad \text{for all } n. \tag{11}$$

By condition (1) of 1.3 and by 1.1, $|x| = \sup_n |g_n(x) y_n|_0$ is a seminorm on X. Hence, by 1.7 and 1.5, there is a seminorm $\|\cdot\|$ such that

$$|x_n| \leqslant \sigma^{-1} n^{-2} \|x_n\| \quad \text{for all } n \tag{12}$$

and

$$\sum_{n=1}^{\infty} |y_n|_0 / \|y_n\| < \infty. \tag{13}$$

Then

$$|g_n(x_{k_n}) y_n|_0 \leqslant \sup_j |g_j(x_{k_n}) y_j|_0 = |x_{k_n}| \leqslant \sigma^{-1} k_n^{-2} \|x_{k_n}\|,$$

and by (11),

$$|g_n(x_{k_n}) y_n|_0 = |g_n(x_{k_n})| \, |y_n|_0 \geqslant a_n \sigma^{-1} k_n^{-2} |y_n|_0,$$

i.e

$$|y_n|_0 \leqslant a_n \|x_{k_n}\|.$$

The left-hand inequality (9) follows directly from the fact that $|\cdot|_0$ is (x_n)-normal and from (10). To prove that $k_n \to \infty$, observe that by (9)

$$|y_n|_0 / \|y_n\| \geqslant a_n |x_{k_n}|_0 / (a_n \|x_{k_n}\|) = |x_{k_n}|_0 / \|x_{k_n}\|,$$

and, by (13),

$$\lim_n |x_{k_n}|_0 / \|x_{k_n}\| = 0.$$

Hence, for any j for which $|x_j|_0 \neq 0$, the set $\{n : k_n = j\}$ must be finite. But since for any given j we can find an (x_n)-normal seminorm $|\cdot|_0$ for which $|x_j|_0 \neq 0$, all the sets $\{n : k_n = j\}$ are finite, i.e. $\lim_n k_n = \infty$.

2.1. Corollary. *Let X, (x_n), (y_n), (a_n), (k_n) be such as in 2.0, and let $(|\cdot|_p)$ be an admissible system of (x_n)-normal seminorms on X. Then for every index p there exist a $q(p)$ and a constant $C_p > 0$ such that*

$$a_n |x_{k_n}|_p \leqslant |y_n|_p \leqslant C_p a_n |x_{k_n}|_{q(p)} \quad \textit{for all } n.$$

The proof immediately follows from 2.0 and 1.0.

2.2. Theorem. *Suppose that X is a nuclear Fréchet space with a basis (x_n), $(x_n) \sim [b_{pn}]$, and (y_n) is a CBS in X. Then there are positive integers k_n, with $\lim_n k_n = \infty$ and positive real numbers a_n such that $(a_n^{-1} y_n) \sim [b_{pk_n}]$.*

Proof. Let $(|\cdot|_p)$ be an admissible system of seminorms on X such that $|x_n|_p = b_{pn}$, and let (a_n), (k_n) be such as in 2.0. Further, we let $Y = \operatorname{span}(y_n)$ and $g_n \in Y^*$ be the coefficient functionals of (y_n).

Using 1.0 and 1.5 we easily conclude that the system of seminorms

$$|y|_p^0 = \sup_n |g_n(y)| \, |y_n|_p, \quad y \in Y, \ p = 1, 2, \ldots,$$

is admissible for Y. Hence, by Corollary 2.1, also the system

$$\|y\|_p = \sup a_n |g_n(y)| \, |x_{k_n}|_p^0, \quad p = 1, 2, \ldots,$$

is admissible for Y. We have

$$\|a_n^{-1} y_n\|_p = |x_{k_n}|_p = b_{pk_n} \quad \text{for all } n \text{ and } p,$$

which completes the proof.

2.3. Corollary. *Let X be a nuclear Fréchet space. If X admits a basis (x_n) of a type (D_i), then every CBS (y_n) in X is quasisimilar to a CBS of the same type (D_i).*

Proof. Applying Corollary 2.2 and reordering the sequence (y_n), if necessary, we get $(a_n y_n) \sim [b_{pk_n}]$ and the matrix $[b_{pk_n}]$ is, by 1.8, of type (D_i).

2.4. Corollary (Theorem of Dragilev). *If X is a nuclear Fréchet space having a basis (x_n) which is either of type (D_1) or is of type (D_2), then all the bases in X are quasi-similar.*

Proof. Let (w_n) be an arbitrary basis in X. By Corollary 2.3, there is a basis (y_n) of type (D_i) which is quasisimilar to (w_n). Hence, applying formula $(4+i)$ of 1.10 first to the basis (x_n) and next to (y_n), we conclude that these bases are similar.

2.5. COROLLARY. *If X is one of the spaces $A(\mathscr{C}^r)$, $A(\mathscr{D})$, $C^\infty(T)$, listed in the preliminaries, then all the bases in X are quasisimilar.*

PROBLEM 1. Is it true that in every nuclear Fréchet space with a basis all the bases are quasisimilar? How about the cartesian product $A(\mathscr{C}^r)\times A(\mathscr{D})$ and the tensor product $A(\mathscr{C}^r)\hat{\otimes} A(\mathscr{D})$?

This question is related to a problem on isomorphic classification of spaces of analytic functions (Rolewicz [8], p. 142).

Conditions (D_1) and (D_2) concerned the bases in a given nuclear Fréchet space X. Now we are going to discuss briefly similar conditions stated in terms of the space X itself; they are

$$(\mathrm{d}_1)\qquad \underset{U\,W\,V}{\exists\forall\exists}\ \lim_n d_n(V, W)/d_n(W, U) = 0,$$

and

$$(\mathrm{d}_2)\qquad \underset{U\,W\,V}{\forall\exists\forall}\ \lim_n d_n(W, U)/d_n(V, W) = 0,$$

where U, V, W denote convex symmetric neighbourhoods of zero in X.

These conditions (stated in an equivalent but more complicated way) have been introduced by Dragilev [3]. The following is obvious:

2.6. PROPOSITION. *The properties* (d_1) *and* (d_2) *are isomorphically invariant. A space satisfying* (d_1) *cannot satisfy* (d_2).

The next proposition can be found in Dragilev [3] in an implicit form:

2.7. PROPOSITION. *Let i be either* 1 *or* 2. *Let X be a nuclear Fréchet space with a basis (x_n) of type (D_0). Then the following are equivalent:*

(i) *(x_n) is semisimilar with a basis of type (D_i);*

(ii) *every basis in X is quasisimilar with a basis of type (D_i);*

(iii) *X satisfies the condition* (d_i).

From the last two propositions it follows

2.8. COROLLARY. *There is no nuclear Fréchet space having bases of both types: (D_1) and (D_2).*

Now, by 2.3, we get

2.9. COROLLARY. *Let i and j be any positive integers with $i+j=3$. Then there is no nuclear Fréchet space having a basis of the type (D_i) and a CBS of type (D_j).*

In the last statement one can not replace "CBS" by "basic sequence": *The space $A(\mathscr{C}_0)$, of all analytic functions on the disk, which can be represented by a matrix of type (D_2) contains a subspace isomorphic to $A(\mathscr{C})$* ([8], Theorem 2.4) *and therefore there are basic sequences in $A(\mathscr{C}_0)$ of type (D_1).*

3. Complemented subspaces of X.

Here X denotes a nuclear Fréchet space with a basis (x_n) represented by a matrix $[a_{pn}]$. Since by the Dynin-Mitiagin theorem (mentioned in the preliminaries) (x_n) is an absolute basis, every subsequence (x_{j_n}) is a CBS. Hence, if

(14) *Y is a space representable by a matrix $[a_{pj_n}]$, where $j_1<j_2<\dots$ then Y is isomorphic to a complemented subspace of X.*

By Corollary 2.2, condition (14) is "almost" necessary: we have only to replace the increasing sequence (j_n) by a non-decreasing one. This suggests the following:

3.0. CONJECTURE. Suppose that X and Y are nuclear Fréchet spaces with bases and X is representable by a matrix $[a_{pn}]$. Then Y is isomorphic to a complemented subspace of X if and only if Y is representable by $[a_{pj_n}]$ for some strictly increasing sequence (j_n).

We shall discuss the situation in the case of a special class of nuclear Fréchet spaces: *power series spaces of infinite type* ([7], 6.1.5), i.e. such spaces X which are representable by a matrix the p-th row of which is the $(p-1)$-power of the second row (such matrices are obviously of type (D_1)). In particular, we will show that Conjecture 1 is true for $X=A(\mathscr{C})$ and for $X=C^\infty(T)$.

From 1.10 it follows

3.1. LEMMA. *Suppose that X and Y are nuclear Fréchet spaces, X is representable by a matrix $[a_n^{p-1}]$ and Y is representable by $[b_n^{p-1}]$, where (a_n) and (b_n) are non-decreasing sequences. Then $\delta(Y)\subset\delta(X)$ if and only if*

$$(15)\qquad \underset{p}{\exists}\lim a_n/b_n^p = 0.$$

Given non-decreasing sequences of positive numbers: (a_n) and (b_n), we shall write $(b_n)\prec(a_n)$ if condition (15) holds, and $(a_n)\approx(b_n)$ if both $(a_n)\prec(b_n)$ and $(b_n)\prec(a_n)$.

3.2. PROPOSITION. *If X is representable by $[a_n^{p-1}]$ and the sequence (a_n) has the following property:*

(*) *if $k_1\leqslant k_2\leqslant\dots$ and $(a_{k_n})\prec(a_n)$, then there is a strictly increasing sequence (j_n) of indices such that $(a_{k_n})\approx(a_{j_n})$,*

then every complemented subspace of X having a basis must fulfil (14).

This proposition follows directly from Lemma 3.1 and Corollary 2.2.

3.3. COROLLARY. *Let Y be a nuclear Fréchet space with a basis. Then Y is isomorphic to a complemented subspace of $A(\mathscr{C})$ if and only if Y is representable by a matrix $[(e^{j_n})^{p-1}]$ and Y is isomorphic to a complemented subspace of $C^\infty(T)$ if and only if Y is representable by a matrix $[j_n^{p-1}]$, for some strictly increasing sequence (j_n) of indices.*

Proof. We check the condition (*). Let $(a_{k_n})\prec(a_n)$. If $a_n=e^n$, then $(a_{k_n})=(e^{k_n})\approx(e^ne^{k_n})=(a_{j_n})$, with $j_n=k_n+n$. If $a_n=n$, then $(k_n)\approx(nk_n)$, i.e. $(a_{k_n})\approx(a_{j_n})$ for $j_n=nk_n$.

3.4. COROLLARY. *Let Y be a nuclear Fréchet space with a basis. Then Y is isomorphic to a complemented subspace of the space $C^\infty(T)$ if and only if Y is a power series space of infinite type.*

Proof. By 3.3 the proof can be reduced to that of the following elementary fact: *If* (a_n) *is a non-decreasing sequence of positive numbers such that* $\sum_n a_n < \infty$, *then there are integers* $j_1 < j_2 < j_3 \ldots$ *with* $(j_n) \approx (a_n)$.

Remark. The method based on the invariance of diametral dimension and Lemma 3.1 is insufficient to settle conjecture 1 in the case of power series spaces of infinite type. In fact, if

$$a_n = \exp\big(\exp(\exp n)\big) \quad \text{and} \quad (b_n) = (a_2, a_2, a_4, a_4, a_6, a_6, \ldots),$$

then $(b_n) \prec (a_n)$, but $(b_n) \approx$ to no subsequence of (a_n).

Added in proof. A. Using the method of section 3 one can show that Conjecture 3.0 is true for all the spaces listed in Preliminaries, p. 427.

B. An interesting Dragilev-type theorem concerning the spaces $l^2[a_n^{-1/p}]$ (not necessarily nuclear) is given in [11].

References

[1] C. Bessaga and A. Pełczyński, *On embedding of nuclear spaces into the space of all infinitely differentiable functions on the real line* (Russian), Dokl. Akad. Nauk SSSR 134 (1960), p. 745-748.

[2] M. M. Dragilev, *On the canonical form of bases in the spaces of analytic functions* (Russian), Usp. Mat. Nauk 15 (2) (1960), p. 181-188.

[3] — *On regular bases in nuclear spaces* (Russian), Matem. Sbornik 68 (1965), p. 153-173.

[4] — A. S. Dynin and B. S. Mitiagin, *Criterion for nuclearity in terms of approximative dimension*, Bull. Acad. Polon. Sci. (Sér. sci. math.) 8 (1960), p. 535-540.

[5] B. S. Mitiagin, *The approximative dimension and bases in nuclear spaces* (Russian), Usp. Mat. Nauk 16 (4) (1961), p. 63-132.

[6] Z. Ogrodzka, *On simultaneous extensions of infinitely differentiable functions*, Studia Math. 28 (1967), p. 193-207.

[7] A. Pietsch, *Nukleare lokalkonvexe Räume*, Berlin 1965.

[8] S. Rolewicz, *On spaces of holomorphic functions*, Studia Math. 21 (1961), p. 135-160.

[9] H. G. Tillman, *Randverteilungen analytischer Funktionen und Distributionen*, Math. Zeitschrift 59 (1953), p. 61-83.

[10] V. P. Zaharynta, *On bases and isomorphism of the spaces of analytic functions on convex domains of many variables* (Russian), Теория функций и функциональный анализ 5 (1967), p. 5-12.

[11] — *On quasi-equivalence of bases in finite centers of Hilbertian scales*, Dokl. Akad. Nauk SSSR 130, No 4 (1968), p. 786-788.

Reçu par la Rédaction le 23. 1. 1968

STUDIA MATHEMATICA, T. XXXI. (1968)

A general maximum principle for optimization problems

by

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Halkin and Neustadt [10] have presented a very general maximum principle for a mathematical programming problem over an arbitrary set. This principle includes and extends all the most important necessary conditions for optimization problems. Two basic features are characteristic for the Halkin-Neustadt principle. The first one is the "dual form" of the necessary criterion. Roughly speaking, by the "dual form" we mean some relation (equality or inequality) in a proper conjugate space. A classical example is Lusternik's theorem ([11], p. 339): The problem is to find an $\hat{x}$ such that $F(\hat{x}) = \min\{F(x) | P(x) = 0\}$, $F\colon X \to R^1$, $P\colon X \to Y$, X, Y being Banach spaces. The necessary condition in the "dual form" is $F'(\hat{x}) = \overline{P'(\hat{x})}l$ for some $l \in Y$, where $\overline{P'(\hat{x})}$ denotes the adjoint of the Fréchet derivative $P'(\hat{x})$ of P at $\hat{x}$. However, the essential role is played by the necessary condition in the "primary form" $\max\{F'(\hat{x})y | P'(\hat{x})y = 0, \|y\| = 1\} = 0$, which is more suitable for constructing computational algorithms (see [3]-[5]). This observation seems to be of general character. The second characteristic feature of the Halkin-Neustadt principle is that the assumptions as well as the additional equality constraints are closely connected with the method of proof based on Brouwer's fixed-point theorem. This method may be considered as a further development of the method of Canon, Cullum and Polak [6], and Halkin [9].

The purpose of this paper is to present a more general and stronger maximum principle. In comparison with the Halkin-Neustadt principle it has the following properties. The necessary condition has a "primary form" (which implies the "dual form") and is stronger than the Halkin-Neustadt necessary condition. The additional equality constraints are of a much more general nature. Finally, the abstract framework for the optimization process under consideration is much more general and does not depend on the application of Brouwer's fixed-point theorem. Moreover, it is impossible to apply Brouwer's fixed-point theorem under such general hypotheses. Besides, it should be emphasized that no continuity assumptions are made in the general case.