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Countably modulared spaces

by

J. ALBRYCHT and J. MUSIELAK (Poznań)

1. Let X be a linear space and let a sequence of s-convex pseudomodulars ϱ_i, $i = 1, 2, \ldots$, be defined on X. This means (see [4] and [5]) that $0 \leqslant \varrho_i(x) \leqslant +\infty$ for $x \in X$ and that: $\varrho_i(0) = 0$; $\varrho_i(-x) = \varrho_i(x)$; and $\varrho_i(\alpha x + \beta y) \leqslant \alpha^s \varrho_i(x) + \beta^s \varrho_i(y)$ for $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, where $0 < s \leqslant 1$. Moreover, let us assume that $\varrho_i(0) = 0$, for $i = 1, 2, \ldots$, implies $x = 0$. We define for $x \in X$

$$\varrho(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\varrho_i(x)}{1+\varrho_i(x)};$$

then ϱ satisfies the conditions: $\varrho(x) = 0$ if and only if $x = 0$; $\varrho(x) = \varrho(-x)$; $\varrho(x)$ is finite for all $x \in X$; $\varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y)$ for $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$. Let us remark that the last property is a generalization of that assumed in the definition of a modular in [4], i.e.

$$\varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y) \quad \text{for } \alpha, \beta \geqslant 0, \ \alpha + \beta = 1.$$

It is easily verified that the following properties of a modular given in [4] remain valid for the modular ϱ: $\varrho(\alpha x)$ is a non-decreasing function of $\alpha \geqslant 0$ for each $x \in X$;

$$\varrho\Big(\sum_{i=1}^{n} a_i x_i\Big) \leqslant \sum_{i=1}^{n} \varrho(x_i) \quad \text{for } a_i \geqslant 0, \ \sum_{i=1}^{n} a_i^s = 1, \ 0 < s \leqslant 1.$$

1.1. The linear space

$$X_\varrho = \{x : \varrho(\lambda x) \to 0 \text{ as } \lambda \to 0, \ x \in X\}$$

will be called *countably modulared*. The formula

$$\|x\|_\varrho = \inf\{\varepsilon > 0 : \varrho(x\varepsilon^{-1/s}) \leqslant \varepsilon\}$$

defines a Fréchet norm in X_ϱ (see [1]) which has the same properties as the norm defined in [4], 1.21. Let us recall that X_ϱ is said to be *strongly ϱ-complete* if there exists a constant $\lambda > 0$ such that the Cauchy condition

$\varrho(x_n - x_m) \to 0$ as $m, n \to \infty$, $x_n, x_m \in X_\varrho$, implies $\varrho[\lambda(x_n - x_0)] \to 0$ as $n \to \infty$ with an $x_0 \in X_\varrho$. As in [4] it is seen that if X_ϱ is strongly ϱ-complete, then X_ϱ is complete with respect to the norm $\|\cdot\|_\varrho$.

1.2. *Let* $X_{\varrho_i} = \{x : \varrho_i(\lambda x) \to 0$ *as* $\lambda \to 0,\ x \in X\}$; *then* $X_\varrho = \bigcap_{i=1}^{\infty} X_{\varrho_i}$ *and*

$$(*) \qquad \|x\|_{\varrho_i} = \inf\{\varepsilon > 0 : \varrho_i(x\varepsilon^{-1/s}) \leqslant 1\}$$

is an s-homogeneous pseudonorm in X_{ϱ_i} *such that* $\|x\|_{\varrho_i} = 0$ *for* $i = 1, 2, \ldots$, $x \in X_\varrho$, *implies* $x = 0$. *Moreover,*

$$\|x\|'_\varrho = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|x\|_{\varrho_i}}{1 + \|x\|_{\varrho_i}}$$

is a Fréchet norm in X_ϱ *equivalent to the norm* $\|\cdot\|_\varrho$.

This follows from the fact that $x \in X_\varrho$ if and only if $\varrho_i(\lambda x) \to 0$ as $\lambda \to 0$ for every i, separately, and each of both conditions $\|x_n\|_\varrho \to 0$, and $\|x_n\|'_\varrho \to 0$ is equivalent to the following one: $\varrho_i(\lambda x_n) \to 0$ as $n \to \infty$ for every i and every λ, separately.

1.3. Let us now define an s-convex modular in X by the formula

$$\varrho_0(x) = \sup_i \varrho_i(x).$$

Formula 1.2 (*) with $i = 0$ defines an s-homogeneous norm in the space

$$X_{\varrho_0} = \{x : \varrho_0(\lambda x) \to 0 \text{ as } \lambda \to 0,\ x \in X\},$$

which will be called the *uniformly countably modulared space*. An element $x \in X$ belongs to X_{ϱ_0} if and only if $\varrho_i(\lambda x) \to 0$ as $\lambda \to 0$ uniformly for all $i = 1, 2, \ldots$ The condition $\|x_n\|_{\varrho_0} \to 0$ as $n \to \infty$ is equivalent to the following one: for every $\lambda > 0$, $\varrho_i(\lambda x_n) \to 0$ as $n \to \infty$ uniformly for all $i = 1, 2, \ldots$ Hence and from 1.2 we conclude that $X_{\varrho_0} \subset X_\varrho$, and the imbedding of $\langle X_{\varrho_0}, \|\cdot\|_{\varrho_0}\rangle$ into $\langle X_\varrho, \|\cdot\|_\varrho\rangle$ is continuous.

1.4. *Let* $\varrho_i(x_n - x_0) \to 0$ *as* $n \to \infty$ *imply*

$$\varrho_i(x_0) \leqslant \liminf_{n\to\infty} \varrho_i(x_n)$$

for $i = 1, 2, \ldots$ *If* X_ϱ *is strongly* ϱ-*complete, then* X_{ϱ_0} *is strongly* ϱ_0-*complete. If* $\langle X_\varrho, \|\cdot\|_\varrho\rangle$ *is complete, then so is* $\langle X_{\varrho_0}, \|\cdot\|_{\varrho_0}\rangle$.

Let the sequence $\{x_n\}$, $x_n \in X_{\varrho_0}$, satisfy the Cauchy condition $\varrho_0(x_n - x_m) \to 0$ as $m, n \to \infty$. Then $\varrho_i(x_n - x_m) \to 0$ as $m, n \to \infty$, uniformly in i, and by the assumption of strong modular completeness of X_ϱ, $\varrho_i[a(x_n - x_0)] \to 0$ as $n \to \infty$ for an $x_0 \in X_\varrho$ and every i, where a is a fixed positive number independent of $\{x_n\}$ and i. Consequently, $\varrho_i\{a[(x_n - x_m) - (x_n - x_0)]\} \to 0$ as $m \to \infty$ for every i. Let us put $\lambda_0 = \min(a, 1)$. Given an $\varepsilon > 0$ there exists N independent of i such that

$$\varrho_i[\lambda_0(x_n - x_0)] \leqslant \liminf_{m\to\infty} \varrho_i[\lambda_0(x_n - x_m)] < \varepsilon \quad \text{for } n > N.$$

It remains only to prove that $x_0 \in X_\varrho$; but this follows from the inequality

$$\varrho_i(\lambda x_0) \leqslant 2^{-1/s} \varrho_i[2^{1/s}\lambda(x_n - x_0)] + 2^{-1/s} \varrho_i(2^{1/s}\lambda x_n),$$

where $\lambda \leqslant 2^{-1/s}\lambda_0$. The second part of our assertion is proved analogously.

1.5. *Let* ϱ_i *be convex (i.e.* $s = 1$*). Then* $\langle X_\varrho, \|\cdot\|'_\varrho\rangle$ *is the projective limit of spaces* $\langle X_{\varrho_i}, \|\cdot\|_{\varrho_i}\rangle$ *for* $i = 1, 2, \ldots$, *with respect to embeddings of* X_ϱ *into* X_{ϱ_i}.

This fact follows from the definition of the projective limit [6] and from 1.2.

Let us remark that the above-defined notion of spaces X_ϱ and X_{ϱ_0} is connected also with the results of [7] concerning Banach spaces.

2. Let $\varphi_i(u)$ be φ-functions such that $\varphi_i(u) = \Phi_i(u^s)$, where $0 < s \leqslant 1$ and Φ_i are convex φ-functions, $i = 1, 2, \ldots$ Moreover, let $\varphi_i(u)$ satisfy the following conditions:

1° $\varphi_i(u)$ are equicontinuous at $u = 0$;

2° for every index n there exist positive constants λ_n, β_n, v_n such that for every $u \geqslant v_n$ and $k \geqslant n$ there holds the inequality $\varphi_n(\lambda_n u) \leqslant \beta_n \varphi_k(u)$.

In the case of powers $\varphi_i(u) = |u|^{p_i}$, where $p_i > s > 0$, and the sequence $\{p_i\}$ is non-decreasing, the above conditions are satisfied always.

Let μ be a non-atomic finite measure in a σ-algebra $\mathcal{E}$ of subsets of an abstract set E. We denote by X the space of μ-measurable functions $x(t)$ defined on E and we put

$$\varrho_i(x) = \int_E \varphi(|x(t)|)\, d\mu.$$

Then X_{ϱ_i} is the Orlicz space $L^*_{\varphi_i}$ [5].

2.1. *In order that* $X_\varrho = X_{\varrho_0}$ *it is necessary and sufficient that there exist positive constants* k, c, u_0 *and an index* i_0 *such that for every* $u \geqslant u_0$ *and* $i \geqslant i_0$ *the inequality* $\varphi_i(cu) \leqslant k\varphi_{i_0}(k)$ *holds.*

Sufficiency. It is easily seen that the condition in the above assertion implies existence of $k > 0$ and an index i_0 such that for every u_0 there exists c_0 satisfying the inequality $\varphi_i(u) \leqslant k\varphi_{i_0}(c_0 u)$ for $u \geqslant u_0$ and $i \geqslant i_0$. Hence, if $x \in X_\varrho$, then

$$\varrho_i(\lambda x) \leqslant k\varrho_{i_0}(c_0\lambda x) + \mu(E)_i(u_0).$$

Given $\varepsilon > 0$ we choose u_0 so small that $\mu(E)\varphi_i(u_0) < \varepsilon/2$ for $i = 1, 2, \ldots$ Since $x \in X_{\varrho_{i_0}}$, there exists λ_0 such that $k\varrho_{i_0}(c_0\lambda x) < \varepsilon/2$ for $0 \leqslant \lambda \leqslant \lambda_0$. Hence $\varrho_i(\lambda x) < \varepsilon$ for $\lambda \leqslant \lambda_0$ and all i. Consequently, $x \in X_{\varrho_0}$.

Necessity. Let us suppose that $X_\varrho = X_{\varrho_0}$ and for every $k, c, u_0 > 0$ and any index i_0 there exist a number $u \geqslant u_0$ and an index $i \geqslant i_0$ such that $\varphi_i(cu) \geqslant k\varphi_{i_0}(u)$. Let us fix $k > 0$, and put $c = 2^{-k}$. Then there exist sequences $i_{n,m,k}$ and $u_{n,m,k}$ such that $i_{n,m,k} \geqslant n$, $u_{n,m,k} \geqslant m$, and

$$(*)\qquad \varphi_{i_{n,m,k}}(2^{-k}u_{n,m,k}) > 2^k\varphi_n(u_{n,m,k}) \quad \text{for } n, m, k = 1, 2, \ldots$$

We choose an increasing sequence of indices m_k in such a manner that $\varphi_k(m_k) \geqslant 1$ and $m_k \geqslant v_k$, and we put $u_k = u_{k,m_k,k}$. Next, we take a set $A_k \in \mathscr{E}$ for which $\mu(A_k)\varphi_k(u_k) = 2^{-k}\mu(E)$. Then $\mu(A_k) \leqslant 2^{-k}\mu(E)$ and consequently, the sets $A_1, A_2, \ldots$ may be chosen pairwise disjoint. We define $x(t) = u_k$ for $t \in A_k$, $x(t) = 0$ for $t \in E \setminus \bigcup A_k$. Then

$$\varrho_n(\lambda_n x) \leqslant \sum_{k=1}^{n-1} \mu(A_k)\varphi_n(\lambda_n u_k) + \beta_n \sum_{k=1}^{\infty} \mu(A_k)\varphi_k(u_k) < \infty,$$

and so $x \in X_\varrho$. Consequently, $x \in X_{\varrho_0}$. Hence there exists $\delta > 0$ such that $\varrho_i(\lambda x) < 1$ for $0 \leqslant \lambda \leqslant \delta$ and $i = 1, 2, \ldots$ But inequality $(*)$ implies

$$\varrho_{i_{k,m_k,k}}(2^{-k}x) \geqslant \int_{A_k} \varphi_{i_{k,m_k,k}}[2^{-k}x(t)]\,d\mu \geqslant 2^k \int_{A_k} \varphi_k[x(t)]\,d\mu = 2^k\mu(A_k)\varphi_k(u_k) = \mu(E),$$

a contradiction.

Let us remark that condition 1° was needed only in the proof of sufficiency and condition 2° only in the proof of necessity. Moreover, let us observe that in case of powers $\varphi_i(u) = |u|^{p_i}$, where $p_i \geqslant s > 0$, and the sequence $\{p_i\}$ is non-decreasing, 2.1 gives the following necessary and sufficient condition for $X_\varrho = X_{\varrho_0}$: there exists i_0 such that $p_i = p_{i_0}$ for $i \geqslant i_0$.

3. Let $\varphi_i(u)$ be a φ-function such that $\varphi_i(u) = \Phi_i(u^s)$, where $0 < s \leqslant 1$ and Φ_i are convex φ-functions, $i = 1, 2, \ldots$, and let us suppose that for every index n there exist positive constants λ_n, β_n, v_n such that for every $0 \leqslant u \leqslant v_n$ and $k \geqslant n$ there holds the inequality $\varphi_n(\lambda_n u) \leqslant \beta_n\varphi_k(u)$.

If $\varphi_i(u) = |u|^{p_i}$, where $p_i \geqslant s > 0$, and the sequence $\{p_i\}$ is non-increasing, this condition is satisfied always.

3.1. Let $\{\omega_i\}$ be a sequence of positive numbers for which

$$0 < \liminf_{k\to\infty} \omega_k < \infty, \qquad \omega_k > 0.$$

We denote by X the space of all sequences $x = \{t_i\}$, t_i—real numbers, and we put

$$\varrho_i(x) = \sum_{j=1}^{\infty} \omega_j\varphi_i(|t_j|).$$

Then X_{ϱ_i} is the space of sequences $x = \{t_i\}$ such that

$$\sum_{j=1}^{\infty} \omega_j\varphi_i(\lambda_i|t_j|) < \infty \qquad \text{for some } \lambda_i > 0.$$

3.2. *In order that $X_\varrho = X_{\varrho_0}$ it is necessary and sufficient that there exist positive constants k, c, u_0 and an index i_0 such that for every $0 \leqslant u \leqslant u_0$ and $i \geqslant i_0$ the inequality $\varphi_i(cu) \leqslant k\varphi_{i_0}(u)$ holds.*

Sufficiency. It follows from the above condition that there exist positive constants k, c, u_0 and an index i_0 such that the inequality $\varphi_i(u) \leqslant k\varphi_{i_0}(c_0 u)$ is satisfied for $0 \leqslant u \leqslant u_0$, $i \geqslant i_0$. If $x \in X_\varrho$, then $\omega_j\varphi_i(\lambda_i|t_j|) \to 0$ as $j \to \infty$ for every i and for λ_i sufficiently small. According to the assumption on $\{\omega_j\}$, this implies $t_j \to 0$ as $j \to \infty$. Hence $|\lambda t_j| \leqslant u_0$ for λ positive and sufficiently small and $j = 1, 2, \ldots$, and we conclude that $\varrho_i(\lambda x) \leqslant k\varrho_{i_0}(c_0\lambda x)$. From this inequality it follows $X_\varrho = X_{\varrho_0}$.

Necessity. If $X_\varrho = X_{\varrho_0}$ and the condition in the theorem is not satisfied, then there exist sequences $i_{n,m,k} \geqslant n$ and $0 < u_{n,m,k} \leqslant 1/m$ such that inequality 2.1 $(*)$ holds for $n, m, k = 1, 2, \ldots$ Let

$$\liminf_{k\to\infty} \omega_k = \omega.$$

We choose an increasing sequence of indices m_k in such a manner that $\omega\varphi_k(1/m_k) \leqslant 2^{-k-1}$, $1/m_k \leqslant v_k$, and we put $u_k = u_{k,m_k,k}$. Let $\{\omega_{r_j}\}$ be a subsequence of the sequence $\{\omega_j\}$ such that $\frac{1}{2}\omega < \omega_{r_j} < \frac{3}{2}\omega$. Then there exists a finite set A_k of indices j for which

$$(**)\qquad \frac{1}{2^k} \leqslant \sum_{j\in A_k} \omega_{r_j}\varphi_k(u_k) < \frac{1}{2^{k-1}},$$

and the sets $A_1, A_2, \ldots$, may be chosen pairwise disjoint. Indeed, in other case we should have

$$(\omega_{r_{j_1}} + \ldots + \omega_{r_{j_s}})\varphi_k(u_k) < \frac{1}{2^k},$$

but

$$(\omega_{r_{j_1}} + \ldots + \omega_{r_{j_s}} + \omega_{r_{j_{s+1}}})\varphi_k(u_k) \geqslant \frac{1}{2^{k-1}}$$

and consequently

$$\tfrac{3}{4}\cdot 2^{-k} \geqslant \tfrac{3}{2}\omega\varphi_k(1/m_k) \geqslant \tfrac{3}{2}\omega\varphi_k(u_k) \geqslant \omega_{r_{j_{s+1}}}\varphi_k(u_k) > 2^{-k},$$

a contradiction.

We define $t_j = u_k$ for $j \in A_k$, $t_j = 0$ if j does not belong to $\bigcup A_k$, and we put $x = \{t_j\}$. Then

$$\varrho_n(\lambda_n x) \leqslant \sum_{k=1}^{n-1} \Big(\sum_{j \in A_k} \omega_j\Big) \varphi_n(\lambda_n u_k) + \beta_n \sum_{k=n}^{\infty} \Big(\sum_{j \in A_k} \omega_j\Big) \varphi_k(u_k) < \infty,$$

and so $x \in X_\varrho$. Consequently, $x \in X_{\varrho_0}$, and we conclude that there exists $\delta > 0$ such that $\varrho_i(\lambda x) < 1$ for $0 \leqslant \lambda \leqslant \delta$ and $i = 1, 2, \ldots$ But by 2.1 (*)

$$\varrho_{i_k, m_k, k}(2^{-k} x) \geqslant \Big(\sum_{j \in A_k} \omega_j\Big) \varphi_{i_k, m_k, k}(2^{-k} u_k) \geqslant 2^k \Big(\sum_{j \in A_k} \omega_j\Big) \varphi_k(u_k) \geqslant 1,$$

a contradiction.

3.3. *Let $\{\omega_j\}$ be a sequence of positive numbers for which*

$$\liminf_{j \to \infty} \omega_j = 0 \quad \textit{and} \quad \sum_{k=1}^{\infty} \omega_k = \infty.$$

We denote by X the space of real bounded sequences $x = \{t_j\}$ and we define the modulars ϱ_i by the same formula as in 3.1. *Then the Theorem* 3.2 *remains valid.*

Since $x = \{t_j\} \in X_\varrho$ implies $\{t_j\}$ to be bounded, the sufficiency is concluded as in the proof of 3.2. In order to prove the necessity we suppose that $X_\varrho = X_{\varrho_0}$, and we define the sequences $i_{n,m,k}$ and $u_{n,m,k}$ as in 3.2. Let us choose an increasing sequence of indices m_k for which $1/m_k \leqslant v_k$ and let us put $u_k = u_{k, m_k, k}$. Given k, we define a finite subsequence of indices $A_k = \{r_{j_1}, r_{j_2}, \ldots, r_{j_s}\}$ in such a manner that inequalities 3.2 (**) hold. Applying the assumption $\liminf_{j \to \infty} \omega_j = 0$ we choose $\omega_{r_{j_1}}$ so that $\omega_{r_{j_1}} \varphi_k(u_k) < 2^{-k}$. Let us assume that the numbers $\omega_{r_{j_1}}, \ldots, \omega_{r_{j_p}}$ are chosen in such a manner that

$$(\omega_{r_{j_1}} + \ldots + \omega_{r_{j_p}}) \varphi_k(u_k) < 2^{-k}.$$

Let us write

$$\sum\nolimits_{p+1} = (\omega_{r_{j_1}} + \omega_{r_{j_2}} + \ldots + \omega_{r_{j_p}} + \omega_{r_{j_{p+1}}}) \varphi_k(u_k),$$

where $\omega_{r_{j_{p+1}}}$ is different from all $\omega_{r_{j_1}}, \ldots, \omega_{r_{j_p}}$.

We consider the following cases:

1° if $2^{-k} \leqslant \sum_{p+1} < 2^{-k+1}$ for some $\omega_{r_{j_{p+1}}}$, we put $s = p+1$;

2° if $\sum_{p+1} < 2^{-k}$ for any $\omega_{r_{j_{p+1}}}$ we choose $\omega_{r_{j_{p+1}}}$ arbitrarily.

Let us observe that 1° and 2° exhaust all possible situations. Indeed. let us suppose that $\sum_{p+1} \geqslant 2^{-k+1}$ for any $\omega_{r_{j_{p+1}}}$. Then $\omega_{r_{j_{p+1}}} \varphi_k(u_k) = \sum_{p+1} - \sum_p \geqslant 2^{-k}$ for almost all elements of the sequence $\{\omega_j\}$, a contradiction with the assumption $\liminf_{j \to \infty} \omega_j = 0$. From the divergency of the series $\sum_{k=1}^{\infty} \omega_k$ it follows that the above-defined subsequence A_k is finite

It is easily seen that our assumption makes it possible to define the sets $A_1, A_2, \ldots$ pairwise disjoint. The remaining part of the proof of necessity runs the same lines as in 3.2.

4. An example of another type is obtained if we take as X the space of infinitely differentiable functions $f(t)$ of n variables $t = (t_1, t_2, \ldots, t_n)$ and set

$$\varrho_i(f) = \int_{R^n} \varphi[|D^i f(t)|] dt,$$

where $i = (i_1, \ldots, i_n)$, $D^i = \partial^{i_1 + \ldots + i_n} / \partial t_1^{i_1} \ldots \partial t_n^{i_n}$, and $\varphi(u)$ is convex. The space X_ϱ is equal to the space $\mathscr{D}_\varphi$ (see [3]). Let us remark that $X_{\varrho_0} \neq X_\varrho$ for every $\varphi(u)$; this follows from the fact that if $f \in X_{\varrho_0}$, then either $f \equiv 0$ or the support of f is equal to R^n. Indeed, let Ω be the complement of the support of f and $\emptyset \neq \Omega \neq R^n$. We take a point $t_0 \in \partial \Omega$. Now, from $f \in X_{\varrho_0}$ it follows that

$$\int_{R^n} \varphi[\lambda |D^i f(t)|] dt \leqslant 1$$

for a $\lambda > 0$ and every i. By (4) of [2] we conclude that

$$\varphi\left[\frac{\lambda}{2^n} |D^i f(t)|\right] \leqslant \sum_{p \in P} \int_{R^n} \varphi[\lambda |D^{p+i} f(t)|] dt \leqslant 2^n,$$

where P is the set of multi-indices $p = (p_1, \ldots, p_n)$ with $p_j = 0$ or 1 for $j = 1, 2, \ldots, n$. Hence, all the derivatives $D^i f(t)$ are uniformly bounded. Consequently, f can be developped in the Taylor series in a neighbourhood of t_0. Since, $t_0 \in \partial \Omega$, $D^i f(t_0) = 0$ for every i. Hence, $f(t) = 0$ in a neighbourhood of t_0, a contradiction.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] R. Bojanic and J. Musielak, *An inequality for functions with derivation in an Orlicz space*, Proc. Amer. Math. Soc. 15 (1964), p. 902-906.

[3] J. Musielak, *On some spaces of functions and distributions (I), Spaces D_M and D'_M*, Studia Math. 21 (1962), p. 195-202.

[4] — and W. Orlicz, *On modular spaces*, ibidem 18 (1959), p. 49-65.

[5] W. Orlicz, *On spaces of Φ-integrable functions*, Proc. Int. Symp. on Linear Spaces held at the Hebrew Univ. of Jerusalem, July 5-12, 1960, p. 357-365.

[6] A. P. Robertson and W. J. Robertson, *Topological vector spaces*, Cambridge 1964.

[7] Z. Semadeni. *Limit properties of ordered families of linear metric spaces*, Studia Math. 20 (1961), p. 245-270.

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