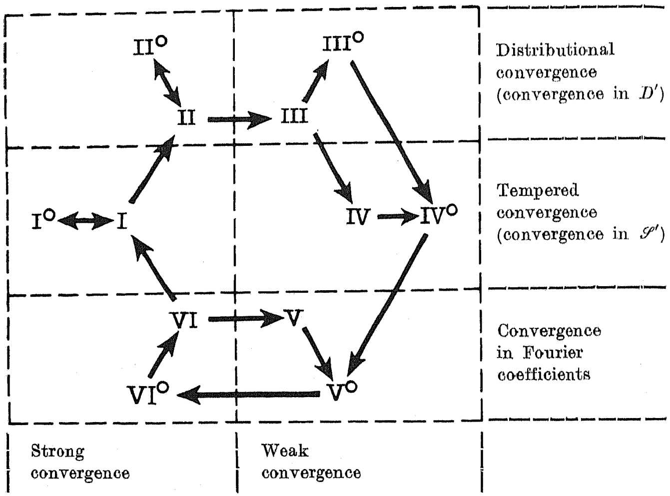
$h_n^{(e_j)}$. Thus f_n is I-convergent. Since j may be chosen arbitrary, it follows by induction that all sequences belonging to some (r) are I-convergent. Thus, every VI-convergent sequence is I-convergent.

10. The following diagram shows which implications between the considered kinds of convergence have been stated, so far:



From this diagram we can immediately read that all possible implications hold between the 12 kinds of convergence, i.e., that all the 12 kinds of convergence are, for sequences of periodic distributions, equivalent.

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On symplectic mappings of contraction operators*

by

R. S. PHILLIPS (Stanford)

Dedicated to
Stanisław Mazur and Władysław Orlicz

One of the more familiar theorems in function theory states that every conformal mapping of the unit disk onto itself is a fractional linear transformation. In 1943, Siegel [3] proved that this result holds as well for symmetric complex matrices. Our purpose is to generalize this theorem still further and show that it holds both for contraction operators and for symmetric (as distinguished from Hermitian symmetric) contraction operators.

More precisely, let $\mathscr{J}_1$ denote the set of all strictly contractive linear operators on a Hilbert space H,

$$\mathscr{J}_1 = [J;\, |J| < 1],$$

and let $\mathscr{Z}_1$ denote the set of all strictly contractive symmetric linear operators on H,

$$\mathscr{Z}_1 = [Z;\, |Z| < 1 \text{ and } Z = Z'],$$

where for a given conjugation $\mathscr{C}$,

$$Z' = \mathscr{C}Z^*\mathscr{C}.$$

We shall consider the group $\mathscr{G}$ $[\mathscr{S}]$ of one-to-one bianalytic mappings φ of $\mathscr{J}_1$ $[\mathscr{Z}_1]$ onto itself with the metric

$$|\varphi_1 - \varphi_2| = \sup |\varphi_1(J) - \varphi_2(J)| \text{ over } \mathscr{J}_1 \ [\text{or } \mathscr{Z}_1].$$

Let $\mathscr{G}_1$ $[\mathscr{S}_1]$ denote the principal component of $\mathscr{G}$ $[\mathscr{S}]$. It will turn out that $\mathscr{S}_1 = \mathscr{S}$. The analogous assertion does not hold for $\mathscr{G}$ even in the case of matrices; for example $\varphi(J) = J'$ belongs to $\mathscr{G}$ but not to $\mathscr{G}_1$.

The transformation

$$J \to (AJ+B)(CJ+D)^{-1} \tag{1}$$

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with A, B, C, D linear bounded operators on H is called *general symplectic* if

$$(2)\qquad \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$$

and *symplectic* if in addition $D = \bar{A}$ and $C = \bar{B}$; here we denote $\mathscr{C}D\mathscr{C}$ by $\bar{D}$.

In terms of these concepts our result becomes

THEOREM 1. *Every mapping in* $\mathscr{G}_1$ $[\mathscr{S}]$ *is general symplectic* [*symplectic*].

The proof begins with a few remarks on the symplectic group which permit us to restrict our considerations to the analytic mappings φ for which $\varphi(0) = 0$. Then by suitably generalizing Siegel's proof we show that such a φ is a linear isometry. Finally we make use of results due to Kadison [2] and Wigner [5] to show that φ is actually symplectic.

In the finite-dimensional case it is known that a one-to-one analytic map is necessarily bianalytic. We do not know whether this is true in the more general case treated in this paper.

1. The symplectic group. We note that J belongs to $\mathscr{J}_1$ if and only if $J^*J - I < cI$ for some negative c and that this is equivalent by (2) to

$$(AJ+B)^*(AJ+B)-(CJ+D)^*(CJ+D) < c'I$$

for some negative c'; it follows that the general symplectic transformation takes $\mathscr{J}_1$ onto $\mathscr{J}_1$. The additional conditions imposed on the symplectic transformation are just enough to show that it takes $\mathscr{Z}_1$ onto $\mathscr{Z}_1$. It is also clear from (1) and (2) that the general symplectic and the symplectic transformations form transformation groups. Further condition (2) shows that transformation (1) is general symplectic if and only if

$$(1.1)\qquad A^*A - C^*C = I = D^*D - B^*B \quad\text{and}\quad A^*B = C^*D.$$

and

$$(1.2)\qquad AA^* - BB^* = I = DD^* - CC^* \quad\text{and}\quad AC^* = BD^*.$$

Hence in the symplectic case necessary and sufficient conditions on A and B are

$$(1.3)\qquad A^*A - B'\bar{B} = AA^* - BB^* = I,\ A^*B = B'\bar{A} \quad\text{and}\quad AB' = BA'.$$

Finally we remark that (1.1) and (1.2) imply that A and D are necessarily regular and the general symplectic transformation is analytic since $(CJ+D)^{-1}$ can be expanded in powers of J.

LEMMA 1.1. *The general symplectic group and the symplectic group are transitive.*

Proof. To prove the first assertion of the lemma it suffices to show that 0 goes into any given J_0 in $\mathscr{J}_1$ under some general symplectic transformation. For such a transformation we see by (1) that $J_0 = BD^{-1}$ so that (1.1) requires that $D^*D - D^*J_0^*J_0D = I$, or equivalently that $(DD^*)^{-1} = I - J_0^*J_0 > 0$. This suggests that we define

$$(1.4)\qquad D = (I - J_0^*J_0)^{-1/2} \quad\text{and}\quad B = J_0D,$$

where we take the positive square root. Also by (1.1), $A^*J_0D = A^*B = C^*D$, which requires

$$(1.5)\qquad C = J_0^*A;$$

and inserting this in the first relation of (1.1) we see that an appropriate choice for A is

$$(1.6)\qquad A = (I - J_0J_0^*)^{-1/2},$$

where once again we take the positive square root. It is clear that the operators A, B, C, D defined by (1.4)-(1.6) satisfy relations (1.1) and hence define a general symplectic transformation taking 0 into J_0. Finally, if we replace J_0 by Z_0 belonging to $\mathscr{Z}_1$ in the above-mentioned formulae, then the resulting operators also satisfy $D = \bar{A}$, $C = \bar{B}$ and (1.3); and therefore define a symplectic transformation taking 0 into Z_0.

LEMMA 1.2. *The general symplectic group and the symplectic group are each connected.*

Proof. Suppose the given general symplectic transformation takes 0 into J_0. We then construct a one-parameter family of general symplectic transformation as in Lemma 1.1 for the operators $[tJ_0; 0 \leqslant t \leqslant 1]$. The resulting operators A_t, B_t, C_t, D_t are continuous in t as are the coefficients for the inverse transformations, namely, $A_t^*, -C_t^*, -B^*, D_t^*$. Composing these inverse transformations with the given transformation, we see that the given transformation is connected within the class of general symplectic transformations with a general symplectic transformation taking 0 into 0 and hence of the form

$$(1.7)\qquad K = AJD^{-1},$$

where by (1.1) A and D are each unitary operators. Using the spectral representations for A and D it is easy to see that they can be connected to the identity by a one-parameter family of unitary operators, say $[A_t, D_t]$. In the symplectic case we proceed in the same fashion. In this case $D = \bar{A}$ in (1.7); we can therefore choose $D_t = \bar{A}_t$ and stay within the class of symplectic transformations.

2. The modified φ is a linear isometry. Given a one-to-one bianalytic map φ of $\mathscr{J}_1$ $[\mathscr{Z}_1]$ onto itself, we now compose it with a general symplectic [symplectic] transformation taking $\varphi(0)$ into 0 so that the resulting mapping takes 0 into 0. We call this resulting map the modified φ and continue to use the same symbol for it. We consider the $\mathscr{G}$ and $\mathscr{S}$ cases in turn.

LEMMA 2.1. *Any one-to-one bianalytic map of $\mathscr{J}_1$ onto $\mathscr{J}_1$ which takes 0 into 0 is a linear isometry.*

The proof of this lemma will be broken up into three steps.

Step 1. Applying Schwarz's lemma (see [1], Theorem 3.13.4) in turn to $\varphi(\zeta J)$ and $\varphi^{-1}(\zeta J)$, $|\zeta|<|J|^{-1}$, yields

$$|\varphi(J)| = |J|. \tag{2.1}$$

Next we express $\varphi(J)$ in a Taylor series (see [1], Chapter 26),

$$\varphi(J) = \sum_{n=1}^{\infty} P_n(J), \tag{2.2}$$

where P_n is a continuous homogeneous polynomial of degree n on $\mathscr{J}_1$ to $\mathscr{J}_1$; the series converges absolutely and uniformly for all J of norm $\leqslant r<1$. Thus for J in $\mathscr{J}_1$ and $|\zeta|\leqslant 1$ we have

$$\big||\varphi(\zeta J)| - |\zeta|\,|P_1(J)|\big| = O(|\zeta|^2)$$

and making use of (2.1) we obtain

$$\big||J| - |P_1(J)|\big| = O(|\zeta|).$$

This implies

$$|P_1(J)| = |J|. \tag{2.3}$$

The analogous assertion holds for

$$\varphi^{-1}(J) = \sum_{n=1}^{\infty} Q_n(J).$$

Now

$$J = \varphi^{-1}\big(\varphi(J)\big) = \sum_{k,n\geqslant 1} Q_k\big(P_n(J)\big);$$

and comparing the first order terms on each side of this relation shows that

$$Q_1\big(P_1(J)\big) = J. \tag{2.4}$$

Likewise

$$P_1\big(Q_1(J)\big) = J \tag{2.4$'$}$$

so that P_1 and Q_1 are linear isometries of $\mathscr{J}_1$ onto $\mathscr{J}_1$. Finally, we note that

$$0 \leqslant \frac{1}{2\pi}\int_0^{2\pi} [I-\varphi(e^{i\theta}J)^*\varphi(e^{i\theta}J)]\,d\theta = I - \sum_{n=1}^{\infty} P_n^*(J)P_n(J). \tag{2.5}$$

Step 2. We set

$$\theta(J) = \varphi\big(Q_1(J)\big) = \sum_{n=1}^{\infty} P_n\big(Q_1(J)\big) \equiv J + \sum_{n\geqslant 2} R_n(J).$$

The results of step 1 also hold for θ. Suppose next that U is a partial isometry and let x lie in the range of $E\equiv U^*U$. Applying (2.5) to ζU, $|\zeta|<1$, we see that $R_n(\zeta U)x = 0$ for all $n\geqslant 2$ and hence that $\theta(\zeta U)x = \zeta Ux$. We would like to show that this holds as well for all x in H. If $E = I$, there is nothing more to prove. If $E\neq I$ but $UU^* = I$, we set

$$(x, y)_\zeta = (x, y) - |\zeta|^{-2}\big(\theta(\zeta U)x, \theta(\zeta U)y\big)$$

for any ζ of absolute value <1. By (2.1) this is a positive semi-definite form and since $(x, x)_\zeta = 0$ for x in the range of E, it follows that $(x, y)_\zeta = 0$ for such x and any y in H. In particular, for y orthogonal to the range of E we have

$$0 = \big(\theta(\zeta U)x, \theta(\zeta U)y\big) = \big(\zeta Ux, \theta(\zeta U)y\big)$$

and since the range of U is all of H in this case, we conclude that $\theta(\zeta U)y = 0$. As this holds for all $|\zeta|<1$, it follows that $R_n(U)y = 0$ for all n. Thus

$$R_n(U) = 0, \quad n\geqslant 2,$$

when U is a partial isometry and either UU^* or U^*U equals I.

Step 3. We now represent an arbitrary J in $\mathscr{J}_1$ in polar form: $J = US$, where U is a partial isometry with either $UU^* = I$ or $U^*U = I$ and S is a positive selfadjoint contraction. Approximating S by a finite linear combination of projection operators leads to an approximation of J by a simple operator of the form

$$J''(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i U_i,$$

where the U_i are mutually orthogonal partial isometries (i.e. $U_i^*U_j = 0 = U_jU_i^*$ for $i\neq j$) and either $\sum U_i^*U_i = I$ or $\sum U_iU_i^* = I$. In either case step 2 shows that

$$R_n(J'') = 0, \quad n\geqslant 2,$$

whenever $|\lambda_i| = 1$ for all of the i. Since $R_n(J''(\lambda_1, \lambda_2, \ldots, \lambda_n))$ is analytic in the polycylinder

$$\mathscr{D} = [|\lambda_i| \leqslant 1,\ i = 1, 2, \ldots, n],$$

it follows from Cauchy's formula for $\mathscr{D}$ that $R_n(J'') = 0$ for all $\{\lambda_i\}$ in $\mathscr{D}$. By continuity, $R_n(J) = P_n(Q_1(J)) = 0$ for $n \geqslant 2$ and all J in $\mathscr{J}_1$ and, since the range of Q_1 is again $\mathscr{J}_1$, we can assert that $P_n(J) = 0$ for $n \geqslant 2$ and all J in $\mathscr{J}_1$; in other words, $\varphi(J) = P_1(J)$, as desired.

Before proving the analogous result for $\mathscr{Z}_1$ we need two preparatory lemmas.

LEMMA 2.2. *Each selfadjoint operator S is unitarily equivalent to a "real" selfadjoint operator Q, that is a Q satisfying $Q = Q^* = \bar{Q}$.*

Proof. By the usual procedure employed in multiplicity theory (see [4], Chapter 7) we can represent H as a direct sum of $L_2(R, m)$ spaces (m is a measure on the Borel subsets of the real numbers),

$$H' = \Sigma \oplus L_2(R, m_a) \tag{2.6}$$

on which the action of S is multiplication by the real variable λ. We define an auxiliary conjugation $\mathscr{C}'$ on H' as:

$$[\mathscr{C}' f_a](\lambda) = \overline{f_a(\lambda)}.$$

It is obvious that $S = \mathscr{C}' S \mathscr{C}'$. Now the given conjugation $\mathscr{C}$ splits H into "real" and "imaginary" parts:

$$H = H_r \oplus iH_r; \quad H_r = [x; \mathscr{C}x = x].$$

A similar decomposition holds for $\mathscr{C}'$:

$$H = H_r' \oplus iH_r'; \quad H_r' = \text{real-valued function in (2.6)}.$$

Let V be any unitary map of H_r onto H_r' extended to H by $V(x+iy) = Vx + iVy$ for all x, y in H_r. Since $V\mathscr{C} = \mathscr{C}'V$ and $\mathscr{C}V^* = V^*\mathscr{C}'$, it is clear that $Q = V^*SV$ satisfies all of the assertions of the lemma.

LEMMA 2.3. *If $Z = Z'$, then there exists a unitary operator U and a "real" positive operator P ($P = P^* = \bar{P} \geqslant 0$) such that*

$$Z = U'PU. \tag{2.7}$$

Proof. Let S denote the positive square root of $ZZ^* = Z\bar{Z}$. According to Lemma 2.2 there exists a selfadjoint operator Q with $Q = \bar{Q}$ and a unitary operator V such that $S = V^*QV$; obviously $S^2 = V^*Q^2V$. Following Siegel [3] we set

$$F = VZV'. \tag{2.8}$$

Then $F = F'$; moreover, F is normal. In fact,

$$FF^* = VZZ^*V^* = Q^2,$$

$$F^*F = \bar{V}Z^*ZV' = [VZ\bar{Z}V^*]^- = \bar{Q}^2 = Q^2.$$

The W^*-algebra generated by F and F^* is therefore Abelian and consists only of (′)-symmetric operators. The resolution of the identity $\{E_\lambda\}$ for F is contained in this algebra and therefore

$$F = \int_{C_1} \lambda dE_\lambda,$$

where $E_\lambda' = E_\lambda$. We write $\lambda = |\lambda| \exp(i\theta(\lambda))$ and set

$$P = \int |\lambda| dE_\lambda \quad \text{and} \quad W = \int \exp(i\theta(\lambda)/2) dE_\lambda.$$

Then $P \geqslant 0$, $P = P'$; W is unitary, $W = W'$; and

$$F = W'PW.$$

Setting $U = W\bar{V}$ we obtain (2.7).

LEMMA 2.4. *Any one-to-one bianalytic map of $\mathscr{Z}_1$ onto $\mathscr{Z}_1$ which takes 0 into 0 is a linear isometry.*

Proof. The proof of this lemma follows that of Lemma 2.1. In fact, Step 1 carries over as stated while the argument in Step 2 holds in particular for all unitary U such that $U' = U$. Using the result of Lemma 2.3 and noting that the resolution of the identity for P is (′)-symmetric, we see that any Z in $\mathscr{Z}_1$ can be approximated [1] in norm by a simple operator of the form

$$Z''(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i U_i,$$

where the U_i are mutually orthogonal partial isometries such that $U_i' = U_i$ and $\sum U_i U_i^* = I = \sum U_i^* U_i$. Hence Step 3 with J'' replaced by Z'' can be used to complete the proof of Lemma 2.4.

We note that the above linear isometries can obviously be extended to the set of all bounded linear operators $\mathscr{J}$ and the set of all bounded linear (′)-symmetric operators $\mathscr{Z}$, respectively.

3. A characterization of linear isometries. After normalizing φ so that it takes I into I, we shall show that it preserves the Jordan structure of $\mathscr{J}$ [and $\mathscr{Z}$]. In essence the proofs of these facts can be found in Kadison [2]; however, since Kadison's paper assumes a ring structure whereas $\mathscr{Z}$ has only a Jordan structure, some of Kadison's arguments will have to be slightly modified. For notational convenience we shall in this section denote the *closed* unit balls in $\mathscr{J}$ and $\mathscr{Z}$ by $\mathscr{J}_1$ and $\mathscr{Z}_1$, respectively.

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[1] As pointed out by Ebbe Thue Poulsen, this approximation can also be obtained by making use of a generalized spectral representation due to A. Ghika, Revue Math. Pures et Appl. 2 (1957), p. 61-109, in which $Z = \int_0^\infty \lambda dU_\lambda$, where U_λ is an increasing family of partial isometries; that $U_\lambda = U_\lambda'$ follows from the uniqueness of this representation.

LEMMA 3.1 (Kadison [2], Lemma 2). *The identity is an extreme point of both* $\mathcal{J}_1$ *and* $\mathcal{Z}_1$.

LEMMA 3.2 (Kadison [2], Theorem 1). *The extreme points of* $\mathcal{J}_1$ *consist of the maximal partial isometries, that is partial isometries* U *such that either* $UU^* = I$ *or* $U^*U = I$.

LEMMA 3.3. *The extreme points of* $\mathcal{Z}_1$ *consist of the* $(')$*-symmetric unitary operators.*

Proof. Let U be a unitary operator in $\mathcal{Z}_1$ and suppose that there exist operators A, B in $\mathcal{Z}_1$ and positive numbers a, b with $a+b = 1$ such that

$$U = aA+bB.$$

The W^*-algebra generated by U and U^* is contained in $\mathcal{Z}$ and contains a unitary square root of U^*, which we denote by $U^{-1/2}$. Hence

$$I = aA_1+bB_1,$$

where

$$A_1 = U^{-1/2}AU^{-1/2} \quad \text{and} \quad B_1 = U^{-1/2}BU^{-1/2}$$

both belong to $\mathcal{Z}_1$. According to Lemma 3.1, $A_1 = I = B_1$, from which it follows that $A = U = B$, so that U is an extreme point of $\mathcal{Z}_1$.

Suppose next that W is an extreme point of $\mathcal{Z}_1$. As in the proof of Lemma 2.3 we can find a unitary operator V such that $F = VWV'$ is $(')$-symmetric and normal. Obviously F and W will be extreme points of $\mathcal{Z}_1$ together. However, since F is normal, it is clear from the function space representation of F that it can be extreme only if it is unitary in which case $W = V^*F\overline{V}$ is also unitary.

A linear isometry will of course take extreme points into extreme points. In particular, the modified φ acting on $\mathcal{Z}$ will then take I into a unitary operator. Kadison [2], Theorem 7, has shown that this is also true of φ acting on $\mathcal{J}$.

We now set

$$\eta(J) = \varphi(I)^*\varphi(J), \quad J \text{ in } \mathcal{J}, \tag{3.1}$$

and

$$\nu(Z) = \varphi(I)^{-1/2}\varphi(Z)\varphi(I)^{-1/2}, \quad Z \text{ in } \mathcal{Z}; \tag{3.2}$$

here, as in the proof of Lemma 3.3, $\varphi(I)^{-1/2}$ is a square root of $\varphi(I)^*$ belonging to $\mathcal{Z}$. Then η $[\nu]$ is a linear isometry of $\mathcal{J}$ onto $\mathcal{J}$ $[\mathcal{Z}$ onto $\mathcal{Z}]$ taking I into I.

Kadison [2] has proved that η preserves the Jordan structure of $\mathcal{J}$. Actually his arguments apply equally well to ν since they use only the Jordan structure of $\mathcal{Z}$. Thus his Lemma 8 shows that $\nu(Z^*) = \nu(Z)^*$ and it follows from this (see Kadison's proof of [2], Theorem 7) that ν is order preserving. If we denote the positive operators in $\mathcal{Z}_1$ by $\mathcal{Z}_1^+$, then the argument used by Kadison in [2], Theorem 4, shows that the extreme points of $\mathcal{Z}_1^+$ are the set of orthogonal projections in $\mathcal{Z}$. As a consequence, ν maps the projections in onto themselves. Finally, Kadison's alternative ending to [2], Theorem 7, shows that ν preserves the Jordan structure of $\mathcal{Z}$. We shall also need the following

LEMMA 3.4 (Kadison [2], Lemma 6). *If* ϱ *preserves the Jordan structure, then*

$$\varrho(BAB) = \varrho(B)\varrho(A)\varrho(B). \tag{3.3}$$

The pure states of both $\mathcal{J}$ and $\mathcal{Z}$ are 1-dimensional projections of the form

$$Ex = (x, f)f, \tag{3.4}$$

where $|f| = 1$. If $E' = E$ as when E lies in $\mathcal{Z}$, then since $E^* = E$, we also have $\bar{E} = E$, in other words,

$$Ex = (x, \mathscr{C}f)\mathscr{C}f. \tag{3.5}$$

Now f is determined only up to a factor a of absolute value 1 and since (3.4) and (3.5) imply $\mathscr{C}f = cf$ for some $|c| = 1$, there are two ways and only two ways of choosing a so that $\mathscr{C}(af) = af$, namely $a = \pm c^{1/2}$.

If ϱ is a linear isometry of $\mathcal{J}$ onto $\mathcal{J}$ $[\mathcal{Z}$ onto $\mathcal{Z}]$ taking I into I, then–as we have seen above–it maps the projections onto the projections and is order preserving. It follows that it maps the pure states onto the pure states. We can therefore write

$$\varrho(E)x = (x, g)g, \tag{3.6}$$

where $|g| = 1$. In this way we can correspond to f a vector ag, where $|a| = 1$, and in the case of $\mathcal{Z}$ the choice of a can be limited to ± 1 by requiring that $\mathscr{C}f = f$ and $\mathscr{C}g = g$.

Finally, we note for E of the form (3.4) and $E_1x = (x, f_1)f_1$ that

$$E_1EE_1 = |(f, f_1)|^2E_1.$$

Applying Lemma 3.4 with $\varrho(E)$ of the form (3.6) and $\varrho(E_1)x = (x, g_1)g_1$, we see that

$$|(g, g_1)|^2 = |(f, f_1)|^2; \tag{3.7}$$

that is ϱ preserves the *transition probabilities* between states.

LEMMA 3.5 (Wigner [5], p. 233-236). *If* ϱ *is a mapping of the pure states of* $\mathcal{J}$ $[\mathcal{Z}]$ *onto themselves which preserves the transition probabilities between states, then there exists either a unitary or an anti-unitary operator* U *such that*

$$\varrho(E) = UEU^* \tag{3.8}$$

for all of the pure states in $\mathscr{J}$ $[\mathscr{Z}]$. In the case of $\mathscr{Z}$ it is possible to choose U to be unitary with $\overline{U} = U$.

Strictly speaking, Wigner considers only the case $\mathscr{J}$. We now sketch his argument and indicate how it can be modified so as to yield the $\mathscr{Z}$ assertion. First choose a complete orthonormal set $\{f_a\}$ for H which in the case of $\mathscr{Z}$ consists entirely of vectors in H_r. As shown above, ϱ determines a correspondence between vectors up to a factor of absolute value one,

$$f_a \to g_a,$$

where in the $\mathscr{Z}$ case we choose the g's in H_r. We single out a particular member of this set, say f_0, and fix g_0, leaving the other g's arbitrary to within a factor of absolute value 1 in the case of $\mathscr{J}$ and to within a factor ± 1 in the case of $\mathscr{Z}$. The set $\{g_a\}$ is a complete orthonormal set by (3.7) and the onto property of the mapping. Now we also have the correspondence

$$f_0 + f_a \to g_{0a}$$

and it is easy to see that g_{0a} is of the form

$$g_{0a} = a_{0a}(g_0 + a_a g_a).$$

We choose $a_{0a} = 1$ and adjust g_a so that

$$g_{0a} = g_0 + g_a.$$

Finally, any vector $x = \sum c_a f_a$, $c_0 \neq 0$, will correspond in this way to

$$x \to y = a\left(\sum c'_a g_a\right),$$

where $|a| = 1$ and $|c'_a| = |c_a|$; in the case of $\mathscr{Z}$ we limit the c's to be real. We now determine a so that $ac'_0 = c_0$, in which case $|c_0 + c_a| = |c_0 + c'_a|$. In particular, $f_0 + if_a \to g_0 + \varepsilon_a g_a$, where $\varepsilon_a = \pm i$. This imposes a second condition on the c's, namely, $|c_0 - ic_a| = |c_0 + \bar{\varepsilon}_a c'_a|$. As Wigner shows, there are only two possibilities: if $\varepsilon_a = i$, then $c'_a = c_a$ for all vectors, whereas if $\varepsilon_a = -i$, then $c'_a = \bar{c}_a c_0/\bar{c}_0$ for all vectors (assuming that $c_0 \neq 0$). In the case of $\mathscr{Z}$, where both x and y lie in H_r, the c_a and the c'_a are necessarily real so that $c'_a = c_a$. It is clear by continuity that the above-mentioned correspondence also holds for states for which $c_0 = 0$. Finally we note that $\varepsilon_a = \varepsilon_\beta$ since otherwise we would have both $f_a + f_\beta \to g_a + g_\beta$ and $i(f_a + f_\beta) \to \varepsilon_a(g_a - g_\beta)$ which does not preserve transition probabilities. We now define the operator

$$U: \sum c_a f_a \to \sum c'_a g_a,$$

where $c'_a = c_a$ or $\bar{c}_a$ for $\mathscr{J}$, and $c'_a = c_a$ for $\mathscr{Z}$. Thus U may be anti-unitary for $\mathscr{J}$, but because we need only consider real vectors in describing the pure states of $\mathscr{Z}$, U can be chosen to be unitary on $\mathscr{Z}$; note that in the case of $\mathscr{Z}$ the resulting operator satisfies $\overline{U} = U$. Finally, we see that in either case (3.8) is satisfied.

Proof of Theorem 1 for $\mathscr{Z}$. Given ψ in $\mathscr{S}$, we can find a symplectic transformation S by Lemma 1.1 such that $\varphi = S \circ \psi$ belongs to $\mathscr{S}$ and takes 0 into 0. According to Lemma 2.4, φ is a linear isometry and as shown above

$$\nu(Z) \equiv \varphi(I)^{-1/2}\varphi(Z)\varphi(I)^{-1/2}$$

preserves the Jordan structure of $\mathscr{Z}$. Further, Lemma 3.5 shows that one can find a unitary operator U with $\overline{U} = U$ such that

$$\nu(E) = UEU'$$

for all of the pure states in $\mathscr{Z}$. Applying Lemma 3.4 we see that for arbitrary Z in $\mathscr{Z}$

$$\nu(EZE) = \nu(E)\nu(Z)\nu(E).$$

Taking E of the form (3.4), $EZE = (Zf, f)E$ and hence by linearity

$$(Zf, f)\,UEU' = UEU'\nu(Z)\,UEU';$$

cancelling out the extreme U's gives

$$(Zf, f)E = E[U'\nu(Z)\,U]E = \big(U'\nu(Z)\,Uf, f\big)E,$$

so that we finally obtain

$$(3.9)\qquad (Zf, f) = \big(U'\nu(Z)\,Uf, f\big)$$

for all f in H. If Z is selfadjoint and (′)-symmetric, then so is $\nu(Z)$ (and hence $U'\nu(Z)\,U$) and in this case (3.9) implies by polarization that $Z = U'\nu(Z)\,U$, that is

$$(3.10)\qquad \nu(Z) = UZU'.$$

Since any Z can be written as a linear combination of selfadjoint operators in $\mathscr{Z}$, it follows that (3.10) holds for all Z in $\mathscr{Z}$. Thus

$$\varphi(Z) = [\varphi(I)^{1/2}\,U]Z[\varphi(I)^{1/2}\,U]'$$

and since $\varphi(I)^{1/2}$ is (′)-symmetric, we see that φ is symplectic and therefore so is $\psi = S^{-1}\circ\varphi$.

Proof of Theorem 1 for $\mathscr{J}$. Let $\mathscr{G}_0$ denote the general symplectic group. We wish to prove that $\mathscr{G}_0$ is the principal component of $\mathscr{G}$. Since $\mathscr{G}_0$ is connected by Lemma 1.2, it suffices to prove that there exists an $\varepsilon > 0$ such that the ε-neighborhood of $\mathscr{G}_0$ in $\mathscr{G}$ is just $\mathscr{G}_0$.

Suppose for a given ψ in $\mathscr{G}$ and G in $\mathscr{G}_0$ that

$$(3.11)\qquad |\psi - G| < \varepsilon.$$

Setting $\omega = \psi \circ G^{-1}$ we see that

(3.12) $$|\omega - \iota| < \varepsilon;$$

here ι denotes the identity transformation. Next let $J_0 = \omega(0)$; then $|J_0| < \varepsilon$ and if we construct the general symplectic transformation G_1 taking 0 into J_0 as described in the proof of Lemma 1.1, it is easy to see that $|I-A| < \varepsilon^2$, $|B| < \varepsilon$, $|C| < \varepsilon$ and $|I-D| < \varepsilon^2$. As before

$$G_1^{-1} \sim \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}$$

and a crude estimate shows that

(3.13) $$|G_1^{-1} - \iota| < 3\varepsilon$$

for ε sufficiently small. Finally, we set $\varphi = G_1^{-1} \circ \omega$ and conclude from (3.12) and (3.13)

(3.14) $$|\varphi - \iota| < 4\varepsilon.$$

Since φ takes 0 into 0, it follows from Lemma 2.1 that φ is a linear isometry and from previous material in this section that

$$\eta(J) = \varphi(I)^* \varphi(J)$$

preserves the Jordan structure of $\mathscr{J}$. According to Lemma 3.5 there exists an operator U either unitary or anti-unitary such that

$$\eta(E) = UEU^*$$

for all pure states E. Combining this with (3.14) we get

$$|\varphi(I)UEU^* - E| < 4\varepsilon$$

and setting $V = \varphi(I)U$ and $W = U^*$ we finally obtain

(3.15) $$|VEW - E| < 4\varepsilon$$

for all pure states E in $\mathscr{J}$.

Next suppose that U is anti-unitary; then so are V and W. Taking E of the form (3.4) with $x = f$, inequality (3.15) implies

(3.16) $$|(f, Wf)Vf - f| < 4\varepsilon.$$

Since W is anti-unitary, we can adjust f by a factor of absolute value 1 so that $(f, Wf) \geqslant 0$. We can likewise choose g with $|g| = 1$ orthogonal to f so that $(g, Wg) \geqslant 0$. We then obtain from (3.16)

$$|f - Vf| < 8\varepsilon \quad \text{and} \quad |g - Vg| < 8\varepsilon,$$

and consequently, if $h = (f + ig)/\sqrt{2}$, we get

$$|h - (Vf + iVg)/\sqrt{2}| < 16\varepsilon.$$

On the other hand, (3.16) applied to h gives

$$|h - (h, Wh)Vh| < 4\varepsilon.$$

However, this is impossible if $\varepsilon < 1/32$, since then h is a distance of less than 1/2 from two orthogonal vectors, namely $(Vf + iVg)/\sqrt{2}$ and $(h, Wh)(Vf - iVg)/\sqrt{2}$, one of which is of unit length.

It follows that with $\varepsilon < 1/32$, U must be unitary. By an argument analogous to that used in the proof of Theorem 1 for $\mathscr{L}$ we can show that

$$\eta(J) = UJU^*$$

for all J in $\mathscr{J}$. Hence

$$\varphi(J) = \varphi(I)\eta(J) = [\varphi(I)U]JU^*$$

is a general symplectic transformation and so is $\psi = G_1^{-1} \circ \varphi \circ G$. This concludes the proof of Theorem 1 for $\mathscr{J}$.

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