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A bounded orthonormal system of polygonals

by

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1. Introduction. There is an old problem to characterize the Lipschitz functions in terms of the coefficients of their classical Fourier expansions. In this note we solve the corresponding problem in the case of an orthonormal and uniformly bounded system of polygonals. The new orthonormal set has some of the properties of the Walsh and trigonometric systems. The Walsh functions are not suitable for our purpose because they are discontinuous. The relation between our new set and the Franklin set is the same as between Walsh and Haar systems. Since we know how to characterize the Lipschitz functions in terms of the coefficients of their Fourier-Franklin expansions (cf. [1] and [2]), we are able to do the same for the new orthonormal set of polygonals.

2. Construction of the orthonormal system. We assume that all functions considered below are defined on the closed interval $\langle 0, 1\rangle$.

Let $r_m(t)$ denote the m-th Rademacher function, i.e. $r_m(t) = \operatorname{sign}\sin 2^m\pi t$. The Walsh functions are defined as follows: $w_1(t) = 1$ and $w_n(t) = r_{n_1+1}(t)\ldots r_{n_p+1}(t)$ whenever $n = 1+2^{n_1}+\ldots+2^{n_p}$ with $0\leqslant n_1 < \ldots < n_p$. The Haar orthonormal set $\{\chi_n, n = 1, \ldots\}$ is indexed as in [1].

The definition of Walsh functions gives immediately

$$w_{2^m+k}(t) = r_{m+1}(t)w_k(t), \qquad m\geqslant 0,\ 1\leqslant k\leqslant 2^m. \tag{1}$$

This and the definition of Haar functions imply

$$(w_{2^m+k}, \chi_{2^n+l}) = \begin{cases} 0 & \text{for } m\neq n,\\ 2^{-m/2}w_k\left(\dfrac{2l-1}{2^{n+1}}\right) & \text{for } m = n,\end{cases} \tag{2}$$

whenever $1\leqslant k\leqslant 2^m$, $1\leqslant l\leqslant 2^n$, $m\geqslant 0$ and $n\geqslant 0$. It should be remembered that

$$w_k\left(\frac{2l-1}{2^{m+1}}\right) = w_k(t) \quad \text{for } t\in\left(\frac{l-1}{2^m}, \frac{l}{2^m}\right) \text{ and } 1\leqslant k\leqslant 2^m.$$

For given $m \geqslant 0$ we define

$$A_{kl}^{(m)} = (w_{2^m+k}, \chi_{2^m+l}) \quad \text{for } 1 \leqslant k \leqslant 2^m \text{ and } 1 \leqslant l \leqslant 2^m.$$

It is clear that $(A_{kl}^{(m)})$, $1 \leqslant k \leqslant 2^m$, $1 \leqslant l \leqslant 2^m$, is an orthogonal matrix. We are also going to show that it is symmetric. According to (2) it is sufficient to prove

$$w_k\left(\frac{2l-1}{2^{m+1}}\right) = w_l\left(\frac{2k-1}{2^{m+1}}\right), \quad 1 \leqslant k \leqslant 2^m,\ 1 \leqslant l \leqslant 2^m. \tag{3}$$

Since $w_1(t) = 1$ for all t and $w_k(t) = 1$ for $t \epsilon (0, 1/2^m)$ and for $1 \leqslant k \leqslant 2^m$, it follows that (3) is satisfied if either k or l is equal to 1. In the case of $k > 1$ and $l > 1$ we proceed as follows. Let $k = 1+2^{k_1}+\ldots+2^{k_r}$, $0 \leqslant k_1 < \ldots < k_r$, $l = 1+2^{l_1}+\ldots+2^{l_s}$, $0 \leqslant l_1 < \ldots < l_s$ and let

$$t = \sum_{i=1}^{\infty} \frac{\varepsilon_i(t)}{2^i}$$

be the diadic expansion of $t \epsilon \langle 0, 1\rangle$. We know that $r_i(t) = 1-2\varepsilon_i(t)$ for almost all t. Since

$$\frac{2l-1}{2^{m+1}} = \sum_{i=1}^{s} \frac{1}{2^{m-l_i}} + \frac{1}{2^{m+1}} = \sum_{i \epsilon B_l} \frac{1}{2^i},$$

where $B_l = \{m-l_s, \ldots, m-l_1, m+1\}$, we have for $p \leqslant m$

$$r_p\left(\frac{2l-1}{2^{m+1}}\right) = \begin{cases} -1 & \text{for } p \epsilon B_l, \\ 1 & \text{for } p \notin B_l. \end{cases}$$

On the other hand, if we denote by A_k the set $\{1+k_1, \ldots, 1+k_r\}$, then

$$w_k(t) = \prod_{i \epsilon A_k} r_i(t).$$

Since $1 + k_r \leqslant m$, we obtain

$$w_k\left(\frac{2l-1}{2^{m+1}}\right) = (-1)^{N_{kl}},$$

where N_{kl} is the cardinality of $A_k \cap B_l$. If $p \epsilon A_k \cap B_l$, then $p = k_i+1 = m-l_j$ for some i and j, whence $l_j+1 = m-k_i$ and therefore $q = l_j+1 = m-k_i$ is in $A_l \cap B_k$. Since q is uniquely determined by p, it follows that $N_{kl} \leqslant N_{lk}$ and by symmetry $N_{kl} = N_{lk}$.

As a simple corollary of (2) and (3) we obtain

$$(w_n, \chi_m) = (w_m, \chi_n) \quad \text{for } n \geqslant 1 \text{ and } m \geqslant 1. \tag{4}$$

Now, the well known relation between Haar and Walsh functions (cf. [3]) can be stated as follows ($1 \leqslant k \leqslant 2^m$, $1 \leqslant l \leqslant 2^m$, $m \geqslant 0$):

$$w_{2^m+k}(t) = \sum_{l=1}^{2^m} A_{kl}^{(m)} \chi_{2^m+l}(t),$$

$$\chi_{2^m+l}(t) = \sum_{k=1}^{2^m} A_{kl}^{(m)} w_{2^m+k}(t).$$

These formulas can be used to define one of the systems $\{w_n\}$, $\{\chi_n\}$ in terms of the other one. In general, to every orthonormal system we can construct in this way a new orthonormal set. In particular, let us start with the Franklin orthonormal set $\{f_n, n = 0, 1, \ldots\}$ (see [1]). The new orthonormal set is defined as follows:

$$c_0(t) = f_0(t) = 1, \quad c_1(t) = f_1(t) = t,$$

$$c_{2^m+k}(t) = \sum_{l=1}^{2^m} A_{kl}^{(m)} f_{2^m+l}(t), \tag{5}$$

where $m \geqslant 0$, $1 \leqslant k \leqslant 2^m$, $1 \leqslant l \leqslant 2^m$. Clearly,

$$f_{2^m+k}(t) = \sum_{l=1}^{2^m} A_{kl}^{(m)} c_{2^m+l}(t). \tag{6}$$

Thus $\{c_n, n = 0, 1, \ldots\}$ is related to the Franklin set in the same way as $\{w_n\}$ is related to $\{\chi_n\}$.

THEOREM 1. *$\{c_n, n = 0, 1, \ldots\}$ is a bounded, orthonormal and complete system in $L^2\langle 0, 1\rangle$ such that*

$$(w_n, \chi_m) = (c_n, f_m) \quad \textit{for } n \geqslant 0 \textit{ and } m \geqslant 0.$$

It is clear that the boundedness only requires a proof. However, (5) and (2) give

$$|c_{2^m+k}(t)| \leqslant 2^{-m/2} \sum_{l=1}^{2^m} |f_{2^m+l}(t)|,$$

whence by Theorem 5 of [1]

$$|c_{2^m+k}(t)| \leqslant 2^5 \cdot 3^{1/2}, \quad 1 \leqslant k \leqslant 2^m,\ m \geqslant 0. \tag{7}$$

COROLLARY 1. *There exist absolute constants M_1 and M_2 such that*

$$0 < M_1 n \leqslant \operatorname*{var}_{\langle 0,1\rangle} c_n \leqslant M_2 n, \quad n > 0.$$

Indeed, formula (2) and Theorem 1 give

$$2^{m/2} = \sum_{l=1}^{2^m} |(c_{2^m+k}, f_{2^m+l})|, \quad 1 \leqslant k \leqslant 2^m,$$

whence by Theorem 16 of [2]

$$\operatorname*{var}_{\langle 0,1\rangle} c_{2^m+k} \geqslant M_1(2^m+k).$$

On the other hand, c_{2^m+k} is a polynomial of the degree 2^{m+1} and therefore, by Theorem 2′ of [1],

$$\operatorname*{var}_{\langle 0,1\rangle} c_{2^m+k} = \int_0^1 |c'_{2^m+k}(t)|\,dt \leqslant 8\cdot 2^m \int_0^1 |c_{2^m+k}(t)|\,dt.$$

This and (7) give the second inequality of Corollary 1.

It may be worth to indicate that (5) and (34) of [2] imply

$$0 < M_1 \leqslant \|c_n\|_p \leqslant M_2, \qquad n \geqslant 0,\ 1 \leqslant p \leqslant \infty,$$

where M_1 and M_2 are absolute constants.

3. Lebesgue functions and Fourier coefficients. Let us define

$$L_n(t) = \int_0^1 \Big|\sum_{i=1}^n c_i(t)c_i(s)\Big|\,ds$$

and let

$$L_n = \max_{\langle 0,1\rangle} L_n(t).$$

THEOREM 2. *The Lebesgue constants of the orthonormal set $\{c_n\}$ satisfy the following inequalities*:

$$L_{2^m} = O(1) \quad \textit{and} \quad L_n = O(\log n).$$

Since the matrices $(A_{kl}^{(m)})$ are orthogonal, it follows that

$$\sum_{k=1}^{2^m} c_{2^m+k}(t)\,c_{2^m+k}(s) = \sum_{l=1}^{2^m} f_{2^m+l}(t)f_{2^m+l}(s).$$

Thus, $L_{2^m}(t)$ is identical with the corresponding Lebesgue function for the Franklin system and therefore the first part of the theorem follows.

The proof of the second part is based on the first part and on the estimates of the Lebesgue constants for the Walsh system. It is known that for the Walsh system the Lebesgue functions are bounded from above by $O(\log n)$ (see [3], appendix, p. 455). It remains to show that uniformly in $t\in\langle 0,1\rangle$ and p, $1\leqslant p\leqslant 2^m$,

$$\int_0^1 \Big|\sum_{i=1}^p c_{2^m+i}(t)c_{2^m+i}(s)\Big|\,ds = O(m).$$

This is shown as follows. According to (34) of [2] there is a constant M such that

$$\begin{aligned}
&\int_0^1 \Big|\sum_{i=1}^p\sum_{k=1}^{2^m}\sum_{l=1}^{2^m} A_{ik}^{(m)}A_{il}^{(m)} f_{2^m+k}(t)f_{2^m+l}(s)\Big|\,ds\\
&\leqslant M 2^{-m/2}\sum_{l=1}^{2^m}\Big|\sum_{k=1}^{2^m}\Big(\sum_{i=1}^p A_{ik}^{(m)}A_{il}^{(m)}\Big)f_{2^m+k}(t)\Big|\\
&= M2^{-m/2}2^{-m}\sum_{l=1}^{2^m}\Big|\sum_{k=1}^{2^m}\Big(\sum_{i=1}^p w_i\Big(\frac{2k-1}{2^{m+1}}\Big)w_i\Big(\frac{2l-1}{2^{m+1}}\Big)\Big)f_{2^m+k}(t)\Big|\\
&= M\,2^{-m/2}\int_0^1\Big|\sum_{k=1}^{2^m}\Big(\sum_{i=1}^p w_i\Big(\frac{2k-1}{2^{m+1}}\Big)w_i(s)\Big)f_{2^m+k}(t)\Big|\,ds\\
&\leqslant M2^{-m/2}\sum_{k=1}^{2^m}|f_{2^m+k}(t)|\int_0^1\Big|\sum_{i=1}^p w_i\Big(\frac{2k-1}{2^{m+1}}\Big)w_i(s)\Big|\,ds\\
&\leqslant M^2\max_{1\leqslant k\leqslant 2^m}\int_0^1\Big|\sum_{i=1}^p w_i\Big(\frac{2k-1}{2^{m+1}}\Big)w_i(s)\Big|\,ds\\
&= O(\log p) = O(m).
\end{aligned}$$

In what follows we shall employ the following notation. For $x\in L_1\langle 0,1\rangle$ we write

$$S_n(x;t) = \sum_{k=0}^n (x,f_k)f_k(t),$$

$$C_n(x;t) = \sum_{k=0}^n (x,c_k)c_k(t).$$

The symbols $\|\ \|_p$, $\|\ \|$, $\omega_1^{(p)}(\delta;x)$, $\omega_1(\delta;x)$ and $E_n^{(p)}(x)$ and $E_n(x)$ have the same meaning as in [2].

THEOREM 3. *There exists an absolute constant M such that for $x\in L_p\langle 0,1\rangle$, $1\leqslant p<\infty$,*

$$\|x-C_n(x)\|_p \leqslant M[1+\log(n+1)]E_n^{(p)}(x), \qquad n\geqslant 0,$$

and for $x\in C\langle 0,1\rangle$

$$\|x-C_n(x)\| \leqslant M[1+\log(n+1)]E_n(x), \qquad n\geqslant 0.$$

The proof is standard and it is based on Theorem 2.

COROLLARY 2. *For* $x \in L_p\langle 0,1\rangle$, $1 \leqslant p < \infty$,

$$\|x - C_n(x)\|_p \leqslant M_1[1+\log(n+1)]\,\omega_1^{(p)}\left(\frac{1}{n+1}; x\right),$$

and for $x \in C\langle 0,1\rangle$

$$\|x - C_n(x)\| \leqslant M_1[1+\log(n+1)]\,\omega_1\left(\frac{1}{n+1}; x\right),$$

where M_1 *is a constant.*

This immediately follows from Theorem 3, Theorem 8 of [2] and Theorem 3 of [1].

THEOREM 4. *If* $x \in L_1\langle 0,1\rangle$, *then there exists an absolute constant* M *such that*

$$|(x, c_n)| \leqslant M\omega_1^{(1)}\left(\frac{1}{n+1}; x\right), \qquad n \geqslant 0.$$

We observe that formula (5) implies

$$|(x, c_{2^m+l})| \leqslant 2^{-m/2}\sum_{k=1}^{2^m}|(x, f_{2^m+k})|, \qquad 1 \leqslant l \leqslant 2^m.$$

This and (34) of [2] give

$$2^{-m/2}\sum_{k=1}^{2^m}|(x, f_{2^m+k})| \leqslant 2^5\cdot 3^{1/2}\|S_{2^{m+1}}(x) - S_{2^m}(x)\|_1,$$

whence by Theorem 8 of [2] the required result follows.

COROLLARY 3. *If* x *is of bounded variation on* $\langle 0,1\rangle$, *then*

$$|(x, c_n)| = O\left(\frac{1}{n}\right).$$

4. Characterization of the Lipschitz functions.

THEOREM 5. *Let* $0 < a < 1$ *and let* $1 \leqslant p < \infty$. *Then*

$$\sum_{n=0}^{\infty} b_n c_n(t) \tag{8}$$

is a Fourier series, convergent in the $L_p\langle 0,1\rangle$*-norm, of a function* $x \in L_p\langle 0,1\rangle$ *satisfying the Lipschitz condition* $\omega_1^{(p)}(\delta; x) = O(\delta^a)$ *if and only if*

$$\begin{gathered} b_{2^m+k} = 2^{-m(1/2+a)}\sum_{l=1}^{2^m} A_{kl}^{(m)} a_{2^m+l}, \qquad 1 \leqslant k \leqslant 2^m, \\ \left(2^{-m}\sum_{l=1}^{2^m}|a_{2^m+l}|^p\right)^{1/p} = O(1). \end{gathered} \tag{9}$$

Remark. Theorem 5 can be extended in the usual way to the case of infinite p.

To prove the first part of Theorem 5 we assume that $x \in L_p\langle 0,1\rangle$ and $\omega_1^{(p)}(\delta; x) = O(\delta^a)$. Corollary 2 gives the $L_p\langle 0,1\rangle$-convergence of (8). Since $\{f_n\}$ is a basis in $L_p\langle 0,1\rangle$, we have

$$x = \sum_{n=0}^{\infty} d_n f_n, \qquad d_n = (x, f_n). \tag{10}$$

The sums $C_{2^m}(x)$ and $S_{2^m}(x)$ are identical and therefore

$$\sum_{k=1}^{2^m} b_{2^m+k} c_{2^m+k}(t) = \sum_{l=1}^{2^m} d_{2^m+l} f_{2^m+l}(t),$$

hence

$$b_{2^m+k} = \sum_{l=1}^{2^m} A_{kl}^{(m)} d_{2^m+l}.$$

Now, Theorem 11 of [2] gives

$$2^{m(1/2+a)}\left(2^{-m}\sum_{l=1}^{2^m}|d_{2^m+l}|^p\right)^{1/p} = O(1).$$

Thus, $a_{2^m+1} = 2^{m(1/2+a)} d_{2^m+1}$ satisfies (9).

The converse can be proved as follows. We assume that (9) is satisfied and that a_n, b_n and d_n are related as above and therefore

$$2^{m(1/2-1/p)}\left(\sum_{k=1}^{2^m}|d_{2^m+k}|^p\right)^{1/p} = 2^{-am}\left(2^{-m}\sum_{k=1}^{2^m}|a_{2^m+k}|^p\right)^{1/p} = O(2^{-am}).$$

Applying Theorem 11 of [2] we find that x given by (10) is in $L_p\langle 0,1\rangle$ and $\omega_1^{(p)}(\delta; x) = O(\delta^a)$. This and Corollary 2 imply that (8) is the Fourier series of x convergent in the $L_p\langle 0,1\rangle$-norm.

Remark. Theorem 5 remains true if all the capital O's are replaced by little o's.

Suppose that there is given a sequence $\{n_k\}$ of positive integers such that $n_{k+1}/n_k > q > 1$. The series in (8) is said to be *lacunary* if $b_n = 0$ for $n \neq n_k$, $k = 1, 2, \ldots$

The following result has its analogue in the trigonometric case:

THEOREM 6. *Let* $0 < a < 1$ *and let* (8) *be lacunary. Then* (8) *is a Fourier series of an* $x \in C\langle 0,1\rangle$ *with* $\omega_1(\delta; x) = O(\delta^a)$ *if and only if* $b_n = O(n^{-a})$.

In one direction the theorem immediately follows from Theorem 4. On the other hand, $b_n = O(n^{-a})$ and

$$a_{2^m+l} = 2^{m(1/2+a)}\sum_{k=1}^{2^m} A_{kl}^{(m)} b_{2^m+k},$$

whence

$$|a_{2^m+l}| \leqslant 2^{am} \sum_{k=1}^{2^m} |b_{2^m+k}| = 2^{am} \sum_{2^m<n_k<2^{m+1}} |b_{n_k}| = O(1),$$

but this means that the b's and a's satisfy (9) with $p = \infty$. Applying Theorem 5 adapted to this case we complete the proof.

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О спектре сингулярных интегральных операторов в пространствах L_p

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Рассмотрим сперва самый простой класс одномерных сингулярных интегральных операторов — дискретные операторы Винера-Хопфа.

Пусть T_a — линейный ограниченный оператор, определенный в пространстве l_2 бесконечной матрицей $\|a_{j-k}\|_{j,k=0}^{\infty}$, где a_j — коэффициенты Фурье некоторой ограниченной функции $a(\zeta)$ $(|\zeta| = 1)$.

Если функция $a(\zeta)(|\zeta| = 1)$ непрерывна, то *спектр оператора T_a состоит из всех точек кривой $a(\zeta)$ $(|\zeta| = 1)$ и всех комплексных чисел λ, не лежащих на этой кривой, для которых величина*

$$\operatorname{ind}(a-\lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi}\left[\arg\left(a(e^{i\theta})-\lambda\right)\right]_{\theta=0}^{\theta=2\pi} \neq 0.$$

Это предложение сохраняет силу (см. [7]), если пространство l_2 заменить многими другими банаховыми пространствами. В частности, пространство l_2 можно заменить любым пространством h_p $(1 < p < \infty)$ последовательностей коэффициентов Фурье функций из соответствующего пространства Харди H_p (см. [2]). Положение усложняется, если функция $a(\zeta)$ не является непрерывной.

В случае, когда функция $a(\zeta)$ $(|\zeta| = 1)$ непрерывна слева и имеет конечное число точек разрыва $\zeta_1, \zeta_2, \ldots, \zeta_n$ *спектр оператора T_a в пространстве l_2* (см. [3]) *состоит из всех точек кривой $V(a)$, полученной добавлением к множеству всех значений функции $a(\zeta)$ отрезков $\mu a(\zeta_k)+(1-\mu)a(\zeta_k+0)$ $(0 \leqslant \mu \leqslant 1)$, а также из точек $\lambda \notin V(a)$, для которых*

$$\operatorname{ind}(a-\lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi} \oint_{V(a)} d_t \arg(t-\lambda) \neq 0.$$

Этот результат перестает быть верным в пространствах h_p $(p \neq 2, 1 < p < \infty)$: *в случае кусочно-непрерывной функции $a(\zeta)$ спектр оператора T_a в h_p меняется с изменением p.*