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Ideals in Banach algebra extensions *

by

RICHARD ARENS (Los Angeles)

1. Let A be a commutative Banach algebra with unit, and let $c_1, \ldots, c_N$ be elements of A and suppose that the ideal $(c_1, \ldots, c_N)$ is proper. Now let B be an extension of A, and a commutative Banach algebra. It may happen that the ideal that $c_1, \ldots, c_N$ generate in B is no less than B. We present here a necessary and sufficient condition (expressed in terms of the norm in A) that this cannot happen. We prove the following. Let $C = c_1x_1 + \ldots + c_Nx_N$ and suppose there is an $\varepsilon > 0$ such that whenever $f_0, f_1, \ldots, f_{n-1}$ are homogeneous polynomials (of the degree indicated) with coefficients in A, then

$$\|\varepsilon Cf_{n-1} + \varepsilon^2 C^2 f_{n-2} + \ldots + \varepsilon^n C^n f_0\| + \|f_{n-1}\| + \ldots + \|f_1\| \geqslant \|f_0\|; \tag{1.1}$$

then there is an extension B of A in which $c_1, \ldots, c_N$ generate the unit ideal B; and conversely. The norm in (1.1) is a natural one for $A[x_1, \ldots, x_N]$.

The sufficiency of (1.1) to insure the existence of a suitable B is established by the use of a formula (2.41) below which was discovered by Dr. J.-E. Björk of Stockholm University. He very kindly told me his formula before taking the time to prove it in all generality.

2. In this section we suppose that A is a commutative ring with unit. For each N, let $A[x_1, \ldots, x_N]$ be the ring of polynomials with coefficients in A. For a monomial $x_1^{k_1} \ldots x_N^{k_N}$ we call $(k_1, \ldots, k_N)$ the *order* of that monomial, and we say

$$(k_1, \ldots, k_N) > (m_1, \ldots, m_N)$$

if the last non-zero difference $k_1 - m_1, k_2 - m_2, \ldots, k_N - m_N$ is positive. Thus each f in $A[x_1, \ldots, x_N]$ has a non-zero term of least order.

2.1. LEMMA. *Let $c_1, \ldots, c_N \in A$ and $f \in A[x_1, \ldots, x_N]$. Suppose $(c_1x_1 + \ldots + c_Nx_N)f = 0$. Then for each ν, $1 \leqslant \nu \leqslant N$, there is a p_ν such that $c_\nu^{p_\nu} b = 0$, where b is the coefficient of the non-zero term of least order in f.*

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Proof. Let $C = c_1x_1+\ldots+c_{N-1}x_{N-1}$ and write f as $g_0x_N^k+\ldots+ g_mx_N^{k+m}$, where $g_i \in A[x_1, \ldots, x_{N-1}]$ and where $g_0 \neq 0$. Then g_0 contains the coefficient b, in an obvious sense. Evidently

$$(C+c_Nx_N)(g_0+g_1x_N+\ldots+g_mx_N^m) = 0.$$

Thus $c_Ng_m = 0$, $c_Ng_{m-1}+Cg_m = 0$, $c_Ng_{m-2}+Cg_{m-1} = 0$, and so forth. Thus $c_N^2g_{m-1} = 0$, $c_N^3g_{m-2} = 0$, and so forth. This latter sequence terminates with the assertion that $c_N^{m+1}g_0 = 0$. On the other hand, we see also that $Cg_0 = 0$. Assume that 2.1 holds for N replaced $N-1$. Then some power of each c_ν ($\nu \leqslant N-1$) annihilates b. But the $(m+1)$-st power of c_N also annihilates b.

Thus the proof can be achieved by induction.

2.2. Proposition. *Let $c_1, \ldots, c_N$ be such a set of elements of A that $c_1b = \ldots = c_Nb = 0$ only for $b = 0$. Then $c_1x_1+\ldots+c_Nx_N$ is not a zero-divisor in $A[x_1, \ldots, x_N]$.*

Proof. Let b be the coefficient of the non-zero term of lowest degree of any non-zero f (if there be any) in $A[x_1, \ldots, x_N]$ for which $(c_1x_1+\ldots+c_Nx_N)f = 0$. Appealing to 2.1 we conclude that some power of each c_ν annihilates b. Hence there is an integer K so large that $c_1^{k_1}\ldots c_N^{k_N}b = 0$ whenever $k_1+\ldots+k_N = K$. Suppose that $j_1+\ldots+j_N = K-1$. Let $a = c_1^{j_1}\ldots c_N^{j_N}b$. Then $c_1a = \ldots = c_Na = 0$. Thus $a = 0$. We can now replace K by $K-1$ and repeat the argument. In the end we see that $b = 0$.

This proposition is used here only to establish the following

2.3. Proposition. *Let $c_1, \ldots, c_N$ be as in 2.2 and suppose that $(c_1x_1+\ldots+c_Nx_N-1)f$ is homogeneous (in the usual sense). Then $f = 0$.*

Proof. Denote $c_1x_1+\ldots+c_Nx_N$ by C and suppose $(C-1)f$ is homogeneous of degree n. Suppose there are terms in f of degree (in the usual sense, not in the sense of order as used in 2.1) p greater than $n-1$. Let h be the sum of those terms of f which have the highest degree. If this degree is greater than $n-1$, we must have $Ch = 0$. By virtue of 2.2, this h is 0. Hence we may assume $f = f_{n-1}+f_{n-2}+\ldots+f_m$, where these are homogeneous of the degree indicated. From $(C-1)f$ being homogeneous of degree n we conclude that $f_m = 0$, and after several such steps we obtain $f = 0$.

The following theorem establishes a formula proposed by J.-E. Björk:

2.4. Theorem. *Let $c_1, \ldots, c_N$ be such a set of elements of A that $c_1b = \ldots = c_Nb = 0$ only for $b = 0$, and let J be the ideal generated by $C-1$, where $C = c_1x_1+\ldots+c_Nx_N$, in $A[x_1, \ldots, x_N]$. Let $f \in A[x_1, \ldots, x_N]$, with a decomposition into homogeneous summands*

$$f = f_n+f_{n-1}+\ldots+f_1+f_0.$$

Then $f \in J$ only if

$$f_n = -Cf_{n-1}-C^2f_{n-2}-\ldots-C^nf_0, \tag{2.41}$$

and conversely.

Proof. Let $h = f_n+Cf_{n-1}+\ldots+C^nf_0$. Then $f-h \in J$ whence (2.41) surely implies $f \in J$. This establishes the converse. Now suppose $f \in J$. Then certainly $h \in J$. But 2.3 tells us that then h must be 0. Thus 2.4 holds.

This theorem makes it very clear that the ideal J, in the circumstances of the theorem, cannot contain any element of A itself, except 0. For if indeed $a \in J$, define $f = f_1+f_0$, where $f_1 = 0$ and $f_0 = a$. Then (2.41) asserts that $0 = Ca$ and this requires $a = 0$. Therefore $A[x_1, \ldots, x_N]$ mod J contains a subring isomorphic to A showing that A is a subring of a ring B in which there exist elements $\xi_1, \ldots, \xi_N$ such that

$$c_1\xi_1+\ldots+c_N\xi_N = 1.$$

In the next section we study an analogue of this result for Banach algebras.

3. Let B be a commutative normed algebra with unit, and suppose $c_1, \ldots, c_N$ is a set of elements of B for which there exist $b_1, \ldots, b_N$ in B such that $c_1b_1+\ldots+c_Nb_N = 1$. Then we call $\{c_1, \ldots, c_N\}$ a *regular* system. If, moreover, these b's can be selected with norms not exceeding 1, then it is a *normally regular* system. If A is a subalgebra of B and A contains a system which is regular with respect to B, we call it *subregular* with respect to A. It is clear what *normally subregular* should mean ([1]).

In our opinion, the main problem concerning normal subregularity is whether it is characterized by the (obviously necessary) condition

$$\inf_{\substack{a \in A \\ \|a\|=1}} \{\|c_1a\|+\ldots+\|c_Na\|\} \geqslant 1. \tag{3.1}$$

When $N = 1$, this condition is sufficient, but probably not in general. It is with the hope of contributing to the ultimate solution of this problem that we present a condition ((3.21) below) which is equivalent to normal subregularity, and which resembles (3.1) in spirit.

First we must define the norm in $A[x_1, \ldots, x_N]$ supposing that A is a Banach algebra. Actually, we refer the reader to the discussion in Section 2 of the paper cited in ([1]), wherein he should correct the absurdly misprinted (2.3) to read

$$\|f\| = \sum_{\xi \in S(X)} \|f(\xi)\| < \infty.$$

([1]) Compare R. Arens, *Extensions of Banach algebras*, Pac. J. Math. 10 (1960), p. 1-16.

Thus the norm of a polynomial $f \in A[x_1, \ldots, x_N]$ is the sum of the norms of the coefficients, which are of course elements of A.

Let $c_1, \ldots, c_N$ be N elements of A. As before let C be the linear polynomial $c_1x_1 + \ldots + c_Nx_N$, and let J be the ideal generated in $A[x_1, \ldots, x_N]$ by $C-1$, where 1 is the unit of A. In this notation, it is possible to express (3.1) as follows:

(3.11) *For each $a \in A$ one has $\|Ca\| \geqslant \|a\|$.*

The following is therefore a strengthened form of the observation already made:

3.2. THEOREM. *Suppose $c_1, \ldots, c_N$ is a subset of A. If $c_1, \ldots, c_N$ is a normally subregular system, then, for any set $f_0, f_1, \ldots, f_{n-1}$ of homogeneous polynomials from $A[x_1, \ldots, x_N]$ of the degrees indicated, there holds the inequality*

$$\|Cf_{n-1} + C^2 f_{n-2} + \ldots + C^n f_0\| + \|f_{n-1}\| + \ldots + \|f_1\| \geqslant \|f_0\|. \tag{3.21}$$

Here f_0 is simply an element of A, of course. To prove 3.2 suppose $A \subset B$ and $\xi_1, \ldots, \xi_N$ are elements of B of norm at most one such that

$$C(\xi_1, \ldots, \xi_N) = c_1\xi_1 + \ldots + c_N\xi_N = 1.$$

Define f_n to be $-Cf_{n-1} - C^2 f_{n-2} - \ldots - C^n f_0$. Define p to be $f_n + f_{n-1} + \ldots + f_1$. These summands are the homogeneous constituents of p, so surely

$$\|p\| = \|f_n\| + \|f_{n-1}\| + \ldots + \|f_1\|.$$

On the other hand, $p(\xi_1, \ldots, \xi_N) = -f_0$ whence

$$\|f_0\| \leqslant \|p(\xi_1, \ldots, \xi_N)\|.$$

Finally, it must be observed that norms in $A[x_1, \ldots, x_N]$ have been so defined that

$$\|p(\xi_1, \ldots, \xi_N)\| \leqslant \|p\|,$$

which completes the proof.

We now establish the converse.

3.3. THEOREM. *Suppose that whenever $f_0, f_1, \ldots, f_{n-1}$ are homogeneous polynomials of the degree indicated, then (3.21) holds. Then $c_1, \ldots, c_N$ is a normally subregular system.*

To prove this we use a principle [2], whose statement unfortunately has to have its sense restored by the insertion of "the system Σ" after "A and". In any event our problem is simply to prove that $\|p\| \geqslant \|a\|$ whenever $p - a \in J$. We now follow the suggestion of J.-E. Björk and

[2] See op. cit., 2.7.

apply 2.4. We write $p - a = f_n + f_{n-1} + \ldots + f_0$, where n is at least the degree of p. Thus

$$p = (-Cf_{n-1} - C^2 f_{n-2} - \ldots - C^n f_0) + f_{n-1} + \ldots + f_1 + (f_0 + a),$$

where the last parentheses contain the constant term of p. To save writing we merely remark that $\|p\|$ is equal to the left-hand side of (3.21) *plus* $\|f_0 + a\|$, whence

$$\|p\| \geqslant \|f_0\| + \|f_0 + a\| \geqslant \|a\|.$$

It follows from this that $B = A[x_1, \ldots, x_N] \bmod J$ contains an isometric copy of A, and $c_1, \ldots, c_N$ is obviously normally regular in B, with $\xi_i = x_i + J$. Thus 3.3 is established, and (if one likes) may be combined with 3.2 to give an "if and only if" statement.

The condition in 3.2 and 3.3 is admittedly complicated. It is interesting to notice that a neater condition implying normal subregularity is possible.

3.4. PROPOSITION. *Suppose $\|Cf\| \geqslant \|f\|$ for every homogeneous f in $A[x_1, \ldots, x_N]$. Then $c_1, \ldots, c_N$ is normally subregular.*

The proof consists in observing that this condition implies (3.21). Unfortunately, this condition $\|Cf\| \geqslant \|f\|$ is not necessary. Take any algebra A and let $c_1 = c_2 = \frac{1}{2}$ (so $N = 2$). c_1, c_2 is a normally regular system because $c_1 1 + c_2 1 = 1$. However, with $f = x_1 - x_2$ we have $\|f\| = 2$ but $\|Cf\| = 1$. As a matter of fact, C is a topological zero divisor in the algebra $A[x_1, x_2]$ because one can easily find f_n, homogeneous of degree n, such that $\|f_n\| = n+1$ and $\|Cf_n\| = 1$.

Suppose $c_1, \ldots, c_N$ and $d_1, \ldots, d_M$ are subregular systems. We might say that the former is *weaker* than the latter if whenever the former is regular in some extension B of A, then so the is latter.

This concept has an ideal-theoretic meaning which ought not be passed over.

3.5. THEOREM. *Suppose $c_1, \ldots, c_N$ is weaker than $d_1, \ldots, d_M$. Then some power of the ideal $(c_1, \ldots, c_N)$ lies in the ideal $(d_1, \ldots, d_M)$. The converse also holds.*

Proof. By multiplying the c_i by real numbers we can "make" them into a normally subregular system. According to 3.3 the system $c_\cdot, \ldots, c_N$ is regular in $B = A[x_1, \ldots, x_N] \bmod J$. Hence $d_1, \ldots, d_M$ is regular there. Hence there are polynomials $p_1, \ldots, p_M$ such that

$$d_1p_1 + \ldots + d_Mp_M - 1 \in J.$$

Thus we have a polynomial f whose coefficients lie in $(d_1, \ldots, d_M)$ such that $f - 1 \in J$. Say the degree of f is n, and let $f = f_0 + f_1 + \ldots + f_n$. By 2.4,

$$f_n + Cf_{n-1} + C^2 f_{n-2} + \ldots + C^{n-1} f_1 + C^n (f_0 - 1) = 0.$$

Thus the polynomial C^n has coefficients in $(d_1, \ldots, d_M)$. But these coefficients generate the n-th power of the ideal $(c_1, \ldots, c_N)$.

The converse also holds.

We have introduced the concept of "weaker" only in order to ask the following. Let $d_1, \ldots, d_M$ be a subregular system. Can one find a weaker system $c_1, \ldots, c_N$ such that $\|Cf\| \geqslant \|f\|$ for all homogeneous f?

Let us call a system $c_1, \ldots, c_N$ *isometric* if

$$\|a\| = \|c_1 a\| + \ldots + \|c_N a\|$$

for every element a in A.

3.6. Proposition. *An isometric system is normally subregular.*

Proof. Using the triangle inequality, we observe that the left-hand side of (3.21) is at least equal to

$$\|C^n f_0\| + \|f_{n-1}\| + \ldots + \|f_1\| - \|Cf_{n-1} + \ldots + C^{n-1} f_1\|.$$

Isometrism implies that the first term here equals $\|f_0\|$, and it also implies that

$$\|Cf_{n-1} + \ldots + C^{n-1} f_1\| \leqslant \|f_1\| + \ldots + \|f_{n-1}\|.$$

To see that one need only prove $\|Cf\| \leqslant \|f\|$ for any polynomial f. Any such f is a sum of monomials and $\|f\|$ *is the sum of their norms.* For monomial f we have $\|Cf\| = \|f\|$, thus completing the proof of 3.6. An isometric system does not make $\|Cf\| = \|f\|$ for all homogeneous f. The example given earlier is an isometric system.

It is obvious that if $c_1, \ldots, c_N$ and $d_1, \ldots, d_M$ are each isometric systems, then *the product system*

$$\{c_i d_j : 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant M\}$$

is also isometric. This brings us to the final problem. Is the product of two subregular systems itself subregular?

For sup-normed algebras A, the condition of 3.2 expressed by inequality (3.21) characterizes a simple situation, namely that for every maximal ideal m on the Shilov boundary $\partial_A \Delta(A)$ one has

$$|\hat{c}_1(m)| + \ldots + |\hat{c}_N(m)| \geqslant 1.$$

This does not follow immediately by inspection, but rather from the observations made in paper cited in (1), Section 6.

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Vecteurs cycliques et quasi-affinités

par

B. SZ.-NAGY (Szeged) et C. FOIAŞ (Bucarest)

Hommage à Monsieur W. Orlicz

1. PRÉLIMINAIRES ET THÉORÈME

1. Soit T un opérateur (linéaire, borné) dans l'espace de Hilbert (séparable ou non) $\mathfrak{H}$. Faisons la suivante

Définition. Si $\mathfrak{M}$ est un sous-espace de $\mathfrak{H}$, invariant pour T, on appelle la *T-dimension* de $\mathfrak{M}$ et on désigne par $\dim_T \mathfrak{M}$ le minimum du nombre cardinal d'un sous-ensemble $\mathfrak{E}$ de $\mathfrak{M}$ tel que

$$\mathfrak{M} = \bigvee_{n=0}^{\infty} T^n \mathfrak{E} \; (^1)$$

ou, ce qui revient au même, le minimum de la dimension d'un sous-espace $\mathfrak{M}_0$ de $\mathfrak{M}$ tel que

$$\mathfrak{M} = \bigvee_{n=0}^{\infty} T^n \mathfrak{M}_0.$$

Lorsque $\dim_T \mathfrak{M} = 1$, c'est-à-dire qu'il existe un vecteur $h \in \mathfrak{M}$ tel que $\mathfrak{M}$ est sous-tendu par $h, Th, T^2 h, \ldots$, on dira que h est un *vecteur cyclique* pour T dans $\mathfrak{M}$.

La relation suivante est évidente pour tout $h \in \mathfrak{H}$ et pour $n = 0, 1, \ldots$:

$$(1) \quad h = (I - TT^*)h + T(I - TT^*)T^*h + \ldots + T^{n-1}(I - TT^*)T^{*n-1}h + T^n T^{*n} h.$$

Dans le cas où

$$(2) \quad T^n T^{*n} h \to 0 \quad \text{pour tout } h \in \mathfrak{H} \text{ et pour } n \to \infty,$$

la relation (1) entraîne

$$(3) \quad h = \sum_{n=0}^{\infty} T^n (I - TT^*) T^{*n} h$$

(1) Pour un système quelconque $\{\mathfrak{E}_\gamma\}_{\gamma \in \Gamma}$ de sous-ensembles de l'espace de Hilbert on entend par $\bigvee_{\gamma \in \Gamma} \mathfrak{E}_\gamma$ le sous-espace sous-tendu par les ensembles $\mathfrak{E}_\gamma$.