soit $p(z) = c_0 z^n + \ldots + c_n$, $c_0 \neq 0$. Puisque $m_T(T) = O$, on a aussi $p(T) = O$, d'où il s'ensuit que, pour tout $h \in \mathfrak{H}$ et pour tout entier $m \geqslant 0$, $T^m h$ est une combinaison linéaire de $h, Th, \ldots, T^{n-1}h$. Si h est un des vecteurs cycliques pour T, les vecteurs $h, Th, \ldots, T^{n-1}h$ sous-tendent donc $\mathfrak{H}$, ce qui contredit l'hypothèse que $\mathfrak{H}$ est de dimension infinie.

Remarque 3. Toute translation unilatérale simple est unitairement équivalente à l'opérateur S dans l'espace l^2, défini par

$$S(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots).$$

Cet opérateur S est donc une transformée quasi-affine des opérateurs T envisagés dans les propositions 3 et 4.

Proposition 5. *S est une transformée quasi-affine de S^*.*

Démonstration. Soit T un opérateur autoadjoint vérifiant les hypothèses de la proposition 4, par exemple soit T l'opérateur dans l'espace $\mathfrak{H} = L^2(0, \frac{1}{2})$, défini par $Th(x) = xh(x)$. (Un vecteur cyclique est fourni par $h(x) \equiv 1$). En vertu des propositions 3 et 4, S est une transformée quasi-affine de T, c'est-à-dire qu'il existe une quasi-affinité X telle que $TX = XS$. Il en dérive $X^*T = S^*X^*$ et par conséquent

$$S^*(X^*X) = (S^*X^*)X = (X^*T) = X^*X(TX) = X^*(XS) = (X^*X)S.$$

Comme X^*X est aussi une quasi-affinité, cela prouve notre assertion.

Remarque 4. Les vecteurs cycliques de S sous-tendent l'espace l^2; il n'y a qu'à envisager les vecteurs cycliques $v_r = (1, r, r^2, \ldots)$, $|r| < 1$. Tout opérateur T qui est relié à S par la relation $TX = XS$ moyennant une quasi-affinité X, admet alors aussi des vecteurs cycliques, notamment les vecteurs Xv_r, et ces vecteurs sous-tendent linéairement l'espace de T. En particulier, il s'ensuit l'existence de vecteurs cycliques pour S^*. De plus, on déduit de la proposition 4 que *si un opérateur d'un espace $\mathfrak{H}$ de dimension infinie admet un vecteur cyclique, l'ensemble de tous les vecteurs cycliques sous-tend $\mathfrak{H}$* (5).

(5) M. L. Gehér vient d'étendre ce résultat aux opérateurs d'un espace de Banach quelconque.

Reçu par la Rédaction le 29. 1. 1968

STUDIA MATHEMATICA, T. XXXI. (1968)

On analytic functions of Smirnov-Orlicz classes

by

W. KOKILASHVILI (Tbilisi)

Introduction. The present paper deals with the class of analytic functions which is the extension of the known Smirnov E_p-class ($p > 1$). The class introduced is called the *Smirnov-Orlicz class*. Here are proved the so-called theorems on multipliers and decomposition which represent analogues of the well-known Marcinkiewicz and Littlewood-Paley theorems with regard to series of generalized Faber polynomials corresponding to the analytic functions of the Smirnov-Orlicz class. The theorems obtained are used to study the question of approximation of the analytic functions of the Smirnov-Orlicz class by polynomials in the mean on the boundary. The so-called indirect theorems of approximation have been proved. It turns out that in some cases the structural characteristics of the boundary function depend not only on the rate of approaching the best approximation to zero but also on the metric of the space in question.

For the E_p-class ($p > 1$) of analytic functions the problem of approximations by polynomials in the mean was discussed in [1] and [20]. In [5] and [6] the results of [1] and [20] were generalized and direct and indirect theorems containing the best estimate (in the sense of order) were obtained.

Definition. Suppose we are given N-function $\mu(u)$, that is, the function allowing the representation (see [7], p. 16)

$$\mu(u) = \int_0^u p(t)\,dt,$$

where $p(t) > 0$ for $t > 0$, and $p(t)$ is continuous on the right for $t \geqslant 0$, non-decreasing and satisfying the following conditions:

$$p(0) = 0, \quad p(\infty) = \lim_{t\to\infty} p(t) = \infty.$$

Let us further assume that G is a simply connected domain with a Jordan rectifiable boundary Γ. The analytic function $f(z)$ in domain G will be called an *E_μ-function* or a *function of the class E_μ if*

$$\int_{\gamma_r} \mu[|f(z)|]|dz| \leqslant C, \tag{1}$$

where γ_r is the image of the circumference $|w| = r$ with regard to a conformal mapping of the disc $|w| < 1$ onto G.

We shall call the E_μ-class the *Smirnov-Orlicz class*. If $\mu(u) = |u|^p$ $(p > 1)$, the E_μ-class coincides with the well-known E_p-Smirnov class. If the domain G represents a unit circle, the E_μ-classes turn into Hardy-Orlicz classes, whose properties were studied earlier in [8] and [16].

It is evident that any analytic function $f(z)$ belonging to the E_μ-class will also belong to the E_1-class, that is,

$$\int_{\gamma_r} |f(z)||dz| \leqslant C < +\infty$$

uniformly in r, $0 < r < 1$.

It is well known that any function of the E_1-class has summable non-tangential boundary values at almost every point of Γ. Taking this fact and the Fatou lemma into consideration, we conclude that every function of the E_μ-class has at almost all points of Γ non-tangential boundary values belonging to the L_μ-class.

Further on we shall consider the analytic functions in domain G with a smooth boundary, for which the angle $\theta(s)$ between the tangent and a certain fixed direction expressed in the form of a function of the arc length has a modulus of continuity satisfying the following condition:

$$\int_0^s \frac{\omega(\delta, \theta)}{\delta} d\delta < +\infty. \tag{2}$$

The curves satisfying condition (2) will be called *Liapunov type curves*.

Let us suppose that the function $z = \psi(w)$ conformally maps $|w| > 1$ on the domain D with the conditions $\psi(\infty) = \infty, \psi'(\infty) > 0$. D is an exterior of G. $w = \Phi(z)$ is the inverse function. Under the above-mentioned conditions the inequalities

$$0 < l_1 \leqslant |\psi'(w)| \leqslant l_2, \quad |w| \geqslant 1, \tag{3}$$

are valid [21]; l_1 and l_2 are constants depending on G. Similar conditions are satisfied by the function $z = \varphi(w)$ mapping conformally G onto $|w| < 1$. Consequently, if $f(t) \in L_\mu(\Gamma)$, then $f[\psi(w)] \in L_\mu(C)$, C is the unit circumference.

Let us introduce a norm in the Orlicz sense [11] in $L_\mu(\Gamma)$,

$$\|f(t)\|_{L_\mu(\Gamma)} = \sup \left| \int_\Gamma f(t) g(t)\, dt \right|, \tag{4}$$

where the least upper bound is taken along all $g(t) \in L_N(\Gamma)$ for which

$$\int_\Gamma N[|g(t)|]|dt| \leqslant 1$$

($N(v)$ is a function complementary to $\mu(u)$ in the sense of Young). The Banach space obtained will be denoted by $L_\mu(\Gamma)$.

The linear operator constructed on the basis of series of generalized Faber polynomials will serve as an apparatus of approximation. The generalized Faber polynomials are defined through expansion,

$$\frac{g[\psi(w)]\psi'(w)}{\psi(w) - z} = \sum_{n=0}^{\infty} \frac{B_n(z)}{w^{n+1}}, \quad |w| < 1,\ z \in G,$$

where the weight function $g(z)$ is analytic in D and positive at infinity. The properties of the generalized Faber polynomials are given in a survey article [19].

Let Γ_R be a preimage of $|w| = R > 1$ while the mapping $w = \Phi(z)$. Then the equality

$$B_n(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{g(\xi)\Phi^n(\xi)}{\xi - z} d\xi$$

is true for the z situated within Γ_R.

Hence we obtain the formula

$$B_n(z) = g(z)\Phi^n(z) + \frac{1}{2\pi i} \int_\Gamma \frac{g(\xi)\Phi^n(\xi)}{\xi - z} d\xi$$

(see [19]) for $z \in D$.

According to the main Privalov lemma (see [13], p. 182) for boundary values on Γ we infer the validity of the formula

$$B_n(t) = \frac{1}{2} g(t)\Phi^n(t) + \frac{1}{2\pi i} \int_\Gamma \frac{g(\xi)\Phi^n(\xi)}{\xi - z} d\xi. \tag{5}$$

In the sequel the assumption that the function $g(z)$ is continuous in $\bar{D}$ and differs from zero for any $z \in \Gamma$ will always be maintained.

1. On multipliers and decomposition. We shall assume that the function generating the E_μ-class satisfies an additional condition, namely that there exist α and β for which

$$1 < \alpha \leqslant \frac{up(u)}{\mu(u)} \leqslant \beta < +\infty \tag{6}$$

for sufficiently large u.

Therefore (see [7], p. 37) the functions $\mu(u)$ and $N(v)$ satisfy the Δ_2-condition.

As Orlicz has shown, this condition is equivalent to the fact that the space generated by the function $\mu(u)$ is reflexive.

THEOREM 1.1. *Let $f(z)\in E_\mu$ in a domain with a Liapunov type boundary. Suppose further that a_n are Faber coefficients, that is*

$$a_n = \frac{1}{2\pi i}\int\limits_{|w|=1}\frac{f[\psi(w)]}{g[\psi(w)]}\frac{dw}{w^{n+1}},\qquad n=0,1,2,\ldots$$

If the sequence $\{\lambda_n\}_{n=0}^{\infty}$ of complex numbers satisfies the conditions

$$(7)\qquad |\lambda_n|\leqslant A_0,\quad \sum_{\nu=2^m}^{2^{m+1}-1}|\lambda_\nu-\lambda_{\nu+1}|\leqslant A_0,$$

then there exists an analytic function $F(z)\in E_\mu$ in G for which $\{\lambda_n a_n\}_{n=0}^{\infty}$ serve as Faber coefficients

$$a_n\lambda_n = \frac{1}{2\pi i}\int\limits_{|w|=1}\frac{F[\psi(w)]}{g[\psi(w)]}\frac{dw}{w^{n+1}}$$

and for non-tangential boundary values the estimate

$$(8)\qquad \|F(t)\|_{L_\mu(\Gamma)}\leqslant A_1(\mu,\Gamma)A_0\|f(t)\|_{L_\mu(\Gamma)}$$

holds.

Proof. Let us consider the Cauchy type integral

$$(9)\qquad \frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{f[\psi(\tau)]}{g[\psi(\tau)]}\frac{d\tau}{\tau-w}$$

representing the analytic functions $h_1(w)$ in $|w|<1$ and $h_2(w)$ in $|w|>1$. According to the main Privalov lemma for non-tangential boundary values we have

$$(10)\qquad \begin{aligned} h_1^+(\tau_0) &= \frac{1}{2}\frac{f[\psi(\tau_0)]}{g[\psi(\tau_0)]}+\frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{f[\psi(\tau)]}{g[\psi(\tau)](\tau-\tau_0)}d\tau,\\ h_2^-(\tau_0) &= -\frac{1}{2}\frac{f[\psi(\tau_0)]}{g[\psi(\tau_0)]}+\frac{1}{2\pi i}\int\limits_{|\tau|=1}\frac{f[\psi(\tau)]}{g[\psi(\tau)](\tau-\tau_0)}d\tau \end{aligned}$$

almost everywhere on the boundary Γ.

Consequently, by virtue of the boundedness of a singular operator in the Orlicz space $L_\mu^*(C)$ (see [17]), we have $h_1^+(\tau_0)\in L_\mu^*(C)$ and $h_2^-(\tau_0)\in L_\mu^*(C)$. What is more, $h_1^+(\tau_0)\in L_\alpha$, $h_2^-(\tau_0)\in L_\alpha$. In fact, owing to condition (6) we have

$$\frac{up(u)}{\mu(u)}\geqslant\alpha>1$$

and hence

$$\int\limits_{u_0}^{u}\frac{p(t)}{\mu(t)}dt>\alpha\int\limits_{u_0}^{u}\frac{dt}{t}\quad\text{ or }\quad \mu(u)>\alpha\frac{\mu(u_0)}{u_0^\alpha}u^\alpha$$

for sufficiently large u.

It is known (see [2], p. 67) that if non-tangential boundary values of a Cauchy type integral are summable in power α, $\alpha>1$, then the function given by this integral is representable by a Cauchy integral, i.e. it belongs to the H_1-class. Consequently, the functions $h_1(w)$ and $h_2(w)$ belong to the H_1-class in domains $|w|<1$ and $|w|>1$ respectively.

Further, from (10) we have

$$\frac{f[\psi(\tau_0)]}{g[\psi(\tau_0)]}=h_1^+(\tau_0)-h_2^-(\tau_0).$$

Thus for Faber coefficients we can derive

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int\limits_{|w|=1}\frac{f[\psi(w)]}{g[\psi(w)]}\frac{dw}{w^{k+1}}\\ &= \frac{1}{2\pi i}\int\limits_{|w|=1}\frac{h_1^+(w)}{w^{k+1}}dw-\frac{1}{2\pi i}\int\limits_{|w|=1}\frac{h_2^-(w)}{w^{k+1}}dw. \end{aligned}$$

As the function $h_2(w)$ belongs to the H_1-class in $|w|>1$, the second integral vanishes (see [13]) and therefore a_k serve as Taylor coefficients for $h_1(w)$.

Let us consider the polynomials

$$Q_n(z)=\sum_{k=0}^{n}\lambda_k a_k B_k(z).$$

Now we shall prove that the sequence $\{Q_n(t)\}_{n=0}^{\infty}$ converges in the mean in $L_\mu^*(\Gamma)$. By (5) for any m and n the equality

$$\sum_{k=m}^{n}\lambda_k a_k B_k(t)=\frac{1}{2}\sum_{k=m}^{n}\lambda_k a_k\Phi^k(t)+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\sum_{k=m}^{n}\lambda_k a_k\Phi^k(u)g(u)}{u-t}du$$

is valid.

By the boundedness of the singular operator on contour Γ (see [17]; for $\mu(u)=|u|^p/p$, $p>1$, cf. [2]) we obtain the estimate

$$\Big\|\sum_{k=m}^{n}\lambda_k a_k B_k(t)\Big\|_{L_\mu(\Gamma)}\leqslant A_2(\mu,\Gamma)\Big\|\sum_{k=m}^{n}\lambda_k a_k w^k\Big\|_{L_\mu(C)}.$$

Taking into consideration the theorem on multipliers of Fourier series in the Orlicz space (see [12]), we conclude that

$$(11)\qquad \Big\| \sum_{k=m}^{n} \lambda_k a_k B_k(t) \Big\|_{L_\mu(\Gamma)} \leqslant A_3(\mu, \Gamma) \Big\| \sum_{k=m}^{n} a_k w^k \Big\|_{L_\mu(C)}.$$

It is known that the partial sums of the Fourier series of a function from L_μ^* converge to the given function; therefore the right side of (11) tends to zero. Let us denote the corresponding limit of $Q_n(t)$ in the mean on Γ by $F(t)$.

According to the boundedness of the partial sums of the Fourier series in $L_\mu^*(\Gamma)$, see [15], we derive the estimate

$$\Big\| \sum_{k=0}^{n} \lambda_k a_k B_k(t) \Big\|_{L_\mu(\Gamma)} \leqslant A_4(\mu, \Gamma) \|h_1^+(w)\|_{L_\mu(C)}.$$

Therefore (10) implies

$$(12)\qquad \Big\| \sum_{k=0}^{n} \lambda_k a_k B_k(t) \Big\|_{L_\mu(\Gamma)} \leqslant A_5(\mu, \Gamma) \|f(t)\|_{L_\mu(\Gamma)}.$$

Now since the sequence $\{Q_n(t)\}_{n=0}^{\infty}$ converges in the mean in $L_\mu^*(\Gamma)$, it also converges in measure, and by (12)

$$(13)\qquad \|Q_n(t)\|_{L_\mu(\Gamma)} \leqslant A_6(\mu, \Gamma).$$

Further, it is easily seen [7] that if the N-function $\mu(u)$ satisfies the Δ_2-condition, then every bounded set with respect to norm is bounde in the mean as well. So

$$\int_\Gamma \mu[|Q_n(t)|]\,|dt| \leqslant A_7(\mu, \Gamma);$$

consequently

$$\int_0^{2\pi} \mu\big[|Q_n[\varphi(e^{ix})]|\big]\,dx \leqslant A_8(\mu, \Gamma).$$

Hence

$$\int_0^{2\pi} \ln\{|Q_n[\varphi(e^{ix})]|\}\,dx \leqslant A_9(\mu, \Gamma).$$

Thus

$$\int_0^{2\pi} \ln^+\{|Q_n[\varphi(re^{ix})]|\}\,dx \leqslant A_{10}(\mu, \Gamma)$$

for any r, $0 < r < 1$.

Now the use of the Khinchin-Ostrovski theorem (see [15], p. 118) gives us the conviction that the sequence $\{Q_n[\varphi(re^{ix})]\}_{n=0}^{\infty}$ converges uniformly within the unit circle; therefore $\{Q_n(z)\}_{n=0}^{\infty}$ converges uniformly within domain G to the function $F(z)$ possessing non-tangential boundary values almost everywhere and coinciding with $F(t)$ on Γ.

Further, the function $Q_n[\varphi(re^{ix})]$ is continuous in a closed unit circle and so it is representable by the Poisson integral

$$Q_n[\varphi(re^{ix})] = \frac{1}{2\pi} \int_0^{2\pi} Q_n[\varphi(e^{iy})] \frac{1-r^2}{1-2r\cos(x-y)+r^2}\,dy.$$

By using the Jensen inequality (see [9], p. 78), integrating both sides from 0 to 2π and inverting the order of integration, we obtain

$$\int_0^{2\pi} \mu\{|Q_n[\varphi(re^{ix})]|\}\,dx \leqslant \frac{1}{2\pi} \int_0^{2\pi} M[|Q_n[\varphi(e^{ix})]|]\,dx.$$

So

$$\int_0^{2\pi} \mu\{|Q_n[\varphi(re^{ix})]|\}\,dx \leqslant A_{11}(\mu, \Gamma)$$

uniformly with respect to r, $0 < r < 1$.

Hence by virtue of the uniformly convergence of the sequence $\{Q_n(z)\}_{n=0}^{\infty}$ within G we have

$$\int_0^{2\pi} \mu\{|F[\varphi(re^{ix})]|\}\,dx \leqslant A_{12}(\mu, \Gamma),$$

or

$$\int_{\gamma_r} \mu[|F(z)|]\,|dz| \leqslant A_{13}(\mu, \Gamma)$$

uniformly with respect to r, $0 < r < 1$.

Now we shall show that

$$\lambda_n a_n = \frac{1}{2\pi i} \int_{|w|=1} \frac{F[\psi(w)]}{g[\psi(w)]} \frac{dw}{w^{n+1}}.$$

The function $\beta_n[\psi(w)] - w^n g[\psi(w)] \in H_1$ in $|w| > 1$; therefore

$$\int_{|w|=1} \frac{B_n[\psi(w)] - w^n g[\psi(w)]}{w^{m+1} g[\psi(w)]}\,dw = 0, \qquad m = 0, 1, 2, \ldots$$

That is why

$$\int\limits_{|w|=1} \frac{B_n[\psi(w)]}{g[\psi(w)]}\,\frac{dw}{w^{m+1}} = \begin{cases} 1 & \text{when } m = n, \\ 0 & \text{when } m \neq n. \end{cases}$$

Let us suppose that γ_m are Faber coefficients of $F(z)$; then for any m we have

$$\gamma_m - a_m\lambda_m = \frac{1}{2\pi i}\int\limits_{|w|=1} \frac{\{F[\psi(w)] - \sum_{k=0}^{N}\lambda_k a_k B_k[\psi(w)]\}}{w^{m+1}g[\psi(w)]}\,dw$$

with $N > m$. Thus

$$|\gamma_m - a_m\lambda_m| \leqslant A_{14}(\mu, \Gamma)\|F(t) - Q_n(t)\|_{L_\mu(\Gamma)} \to 0$$

and hence $\gamma_m = \lambda_m a_m$ for any $m = 0, 1, 2, \ldots$

In the case of $\mu(u) = |u|^p/p$, $p > 1$, from theorem 1.1 it is clear that if $\{\varepsilon_k\}_{k=0}^{\infty}$ is any sequence of numbers ± 1, then for $f(z) \in E_p$ $(p > 1)$ in domain G with a Liapunov type boundary the following inequalities are valid:

$$A_{15}(p, \Gamma)\left(\|f(t)\|_{L_p(\Gamma)}\right)^p \leqslant \Big\|\sum_{k=0}^{\infty}\varepsilon_k \Delta_k(t)\Big\|_{L_p(\Gamma)} \leqslant A_{16}(p, \Gamma)\left(\|f(t)\|_{L_p(\Gamma)}\right)^p,$$

where

$$\Delta_0(t) = a_0 B_0(t), \qquad \Delta_k(t) = \sum_{i=2^{k-1}}^{2^k-1} a_i B_i(t) \quad \text{when } k > 1.$$

Consequently, if $\{r_k(x)\}_{k=0}^{\infty}$ is a sequence of Rademacher functions and the point x is not a dyadic number, we have

$$A_{17}(p, \Gamma)\left(\Big\|\sum_{k=0}^{2^n-1} a_k B_k(t)\Big\|_{L_p(\Gamma)}\right)^p \leqslant \int\limits_{\Gamma}\Big|\sum_{k=0}^{n} r_k(x)\,\Delta_k(t)\Big|^p |dt|$$

$$\leqslant A_{18}(p, \Gamma)\left(\Big\|\sum_{k=0}^{2^n-1} a_k B_k(t)\Big\|_{L_p(\Gamma)}\right)^p.$$

If we integrate this inequality with respect to x on $(0, 1)$ and invert the order of integration taking into consideration the fact (see [20]) that for any $\sum_{k=0}^{\infty}\beta_k r_k(x)$ with $\sum_{k=0}^{\infty}|\beta_k|^2$ the inequalities

$$B_s\Big\{\sum_{k=0}^{\infty}|\beta_k|^2\Big\}^{1/2} \leqslant \Big\{\int_0^1\Big|\sum_{k=0}^{\infty}\beta_k r_k(x)\Big|^s dx\Big\}^{1/s} \leqslant C_s\Big\{\sum_{k=0}^{\infty}|\beta_k|^2\Big\}^{1/2}, \qquad s > 0,$$

are valid, then we deduce the estimate

$$A_{19}(p, \Gamma)\Big(\sum_{k=0}^{n}|\Delta_k(t)|^2\Big)^{p/2} \leqslant \int_0^1\Big|\sum_{k=0}^{n} r_k(x)\,\Delta_k(t)\Big|^p dx$$

$$\leqslant A_{20}(p, \Gamma)\Big(\sum_{k=0}^{n}|\Delta_k(t)|^2\Big)^{p/2}, \qquad t \in \Gamma,$$

and further

$$A_{21}(p, \Gamma)\left(\Big\|\sum_{k=0}^{2^n-1} a_k B_k(t)\Big\|_{L_p(\Gamma)}\right)^p \leqslant \int\limits_{\Gamma}\Big(\sum_{k=0}^{n}|\Delta_k(t)|^2\Big)^{p/2}|dt|$$

$$\leqslant A_{22}(p, \Gamma)\left(\Big\|\sum_{k=0}^{2^n-1} a_k B_k(t)\Big\|_{L_p(\Gamma)}\right)^p.$$

If we pass to the limit as $n \to \infty$, we obtain (9). Thus we have

THEOREM 1.2. *For $f(z) \in E_p$ $(p > 1)$ in domain G with a Liapunov type boundary the following inequalities hold:*

$$A_{23}(p, \Gamma)\|f(t)\|_{L_p(\Gamma)} \leqslant \Big\{\int\limits_{\Gamma}\Big(\sum_{k=0}^{n}|\Delta_k(t)|^2\Big)^{p/2}\Big\}^{1/p} \leqslant A_{24}(p, \Gamma)\|f(t)\|_{L_p(\Gamma)}. \tag{14}$$

Our purpose is to establish similar results for the E_μ-class. It is known ([7], p. 110) that if $\mu(u)$ is an arbitrary N-function and $f(t) \in L_\mu^*(\Gamma)$, then for the norm the following formula is true:

$$\|f(t)\|_{L_\mu(\Gamma)} = \inf_{k>0}\frac{1}{k}\Big(1 + \int\limits_{\Gamma}\mu[k|f(t)|]|dt|\Big).$$

Considering the above-mentioned fact and according to the Koizumi interpolation theorem (see [3]) it can be stated that if T is a linear transformation of functions defined on one space with measure into a measurable function on an other space with measure, acting as a bounded operator from L_a into L_a and from L_b into L_b, then it will be bounded as an operator from L_μ^* into L_μ^*, where the N-function $\mu(u)$ satisfies the following conditions:

$$1 < a + \varepsilon \leqslant \frac{up(u)}{\mu(u)} \leqslant b - \varepsilon < +\infty \qquad (\varepsilon > 0).$$

Consequently the theorem is valid.

THEOREM 1.3. *For $f(z) \in E_\mu$ in domain G with a Liapunov type boundary the following inequalities are true:*

$$A_{25}\|f(t)\|_{L_\mu(\Gamma)} \leqslant \Big\|\Big(\sum_{k=0}^{\infty}|\Delta_k(t)|^2\Big)^{1/2}\Big\|_{L_\mu(\Gamma)} \leqslant A_{26}(\mu, \Gamma)\|f(t)\|_{L_\mu(\Gamma)}. \tag{15}$$

The obtained result is also new for E_p, $p > 1$.

Now we shall assume that $g(z) \equiv 1$ and the corresponding polynomials are denoted by $\Phi_n(z)$.

THEOREM 1.4. *For analytic functions whose derivative belongs to the E_μ-class in domain G with a Liapunov type boundary the inequalities*

$$(16)\qquad A_{27}\|f'(t)\|_{L_\mu(\Gamma)} \leqslant \Big\|\Big(\sum_{k=0}^{\infty}|\Delta_k'(t)|^2\Big)^{2}\Big\|_{L_\mu(\Gamma)} \leqslant A_{28}(\mu,\Gamma)\|f'(t)\|_{L_\mu(\Gamma)}$$

are correct.

The proof is similar to the previous one. Let us outline the proof with $r = 1$.

At first we prove that if $f'(z) \in E_\mu$ in domain G with a Liapunov type boundary and the sequence $\{\lambda_n\}_{n=0}^{\infty}$ of complex numbers satisfies conditions (7), then the series

$$(17)\qquad \sum_{k=0}^{\infty}\lambda_k a_k \Phi_k'(z)$$

converges uniformly into G to the analytic function $F(z) \in E_\mu$, its non-tangential boundary values $F(t)$ almost everywhere on Γ coincide with the sum in the mean of (17) and the estimate

$$\|F(t)\|_{L_\mu(\Gamma)} \leqslant A_{29}(\mu,\Gamma)\cdot A_0\|f'(t)\|_{L_\mu(\Gamma)}$$

holds true.

Let us consider integral (9). Since the function $f'(z) \in E_\mu$ (and consequently $\in E_a$), its primitive is continuous in $\bar{G}$ and absolutely continuous on the boundary as a function of arc length ([13], p. 208); in addition, the derivative $f'(t)$ of boundary values almost everywhere coincides on Γ with non-tangential boundary values of $f'(z)$.

In view of the fact for a Liapunov type curve the inequality

$$\left|\frac{S(t)-S(t_0)}{t-t_0}\right| \leqslant M \quad \text{for any } t, t_0 \in \Gamma,$$

is valid.

$S(t)$ being the arc abscissa corresponding to the point $t \in \Gamma$, the function $f(t)$ is absolutely continuous with respect to t and therefore $f[\psi(\tau)] = f_0(\tau)$ is absolutely continuous on $|\tau| = 1$.

Consequently,

$$h_1'(w) = \frac{1}{2\pi i}\int_{|\tau|=1}\frac{f[\psi(\tau)]}{(\tau-w)^2}\,d\tau = \frac{1}{2\pi i}\int_{|\tau|=1}\frac{f_0'(\tau)}{\tau-w}\,d\tau.$$

Further $f_0'(\tau) = f_\psi[\psi(\tau)]\cdot\psi'(\tau)$. Thus $f_0'(\tau) \in L_\mu^*(C)$.

The arguments of the proof of theorem 1.1 lead us to the conclusion that $h_1'(w) \in H_1$ and the non-tangential boundary values

$$[h_1'(w)]^+ = \frac{1}{2}f_0'(w) + \frac{1}{2\pi i}\int_{|\tau|=1}\frac{f_0'(\tau)}{\tau-w}\,d\tau$$

belong to L_μ^* on $|w| = 1$. And from the formula

$$\Phi_k'(z) = \frac{1}{2\pi i}\int_\Gamma\frac{k[\Phi(t)]^{k-1}\Phi'(t)}{t-z}\,dt$$

for the boundary values on Γ we obtain

$$\Phi_k'(t) = \frac{1}{2}k[\Phi(t)]^{k-1}\Phi'(t) + \frac{1}{2\pi i}\int_\Gamma\frac{k[\Phi(\tau)]^{k-1}\Phi'(\tau)}{\tau-t}\,d\tau, \qquad t \in \Gamma.$$

Let us note that $k\cdot a_k$ are Taylor coefficients for $h_1'(w) \in H_1$ whose non-tangential boundary values belong to $L_\mu^*(C)$.

Now the proof of theorem 1.4 can be completed in the manner of theorems 1.1 and 1.3.

2. On approximation in the mean of E_μ-class functions. Here we shall discuss a question about the order of approximation of analytic functions by polynomials in the mean on the boundary.

For $f(z) \in E_\mu$ we introduce the following quantities: the modulus of smoothness of non-tangential boundary values

$$\omega_k^{(\mu)}(\delta, f) = \sup_{|h|\leqslant\delta}\|\Delta_k^h f_0(\theta)\|_{L_\mu(\Gamma)},$$

where

$$\Delta_k^h f_0(\theta) = \sum_{\nu=1}^{n}(-1)^{k-\nu}f_0(\theta+\nu h), \qquad f_0(\theta) = f[\psi(e^{i\theta})],$$

and the best approximations by polynomials in the mean on the boundary

$$\varrho_n^{(\mu)}(f,\Gamma) = \inf\|f(t) - P_k(t)\|_{L_\mu(\Gamma)},$$

where the greatest lower bound is taken over all the possible polynomials $P_k(t)$ of degree $\leqslant n$.

THEOREM 2.1. *Let us assume that the function $\Psi(u) = \mu(u^{1/a})$ is convex for some a, $1 < a \leqslant 2$. Then for $f(z) \in E_\mu$ in domain G with a Liapunov type boundary the estimate*

$$\omega_1^{(\mu)}\left(\frac{1}{n}, f\right) \leqslant \frac{A_{30}(\mu,\Gamma,k)}{n}\left\{\sum_{\nu=1}^{n}\nu^{a-1}[\varrho_{\nu-1}^{(\mu)}(f,\Gamma)]^a\right\}^{1/a}$$

is right.

In the general case we have

THEOREM 2.2. *For the functions $f(z)\in E_\mu$ in domain G with a Liapunov type boundary we have the following inequality:*

$$\omega_1^{(\mu)}\left(\frac{1}{n}, f\right) \leqslant \frac{A_{31}(\mu, \Gamma, k)}{n} \sum_{\nu=1}^{n} \varrho_{\nu-1}^{(\mu)}(f, \Gamma).$$

We shall need to consider two lemmas to arrive at the proof of the theorems.

LEMMA 1. *For $f(z)\in E_\mu$ in a domain with a Liapunov type boundary we have*

$$\|f(t)-S_n(f, t)\|_{L_\mu(\Gamma)} \leqslant A_{32}(\mu, \Gamma)\,\varrho_n^{(\mu)}(f, \Gamma),$$

where

$$S_n(f, t) = \sum_{\nu=0}^{n} \alpha_\nu \Phi_\nu(t)$$

and the constant $A_{32}(\mu, \Gamma)$ depends only on the boundary and the space.

Proof. For Faber polynomials we have

$$\Phi_n(t) = \frac{1}{2}[\Phi(t)]^n + \frac{1}{2\pi i}\int_\Gamma \frac{[\Phi(v)]^n}{v-t}\,dv, \qquad t\in\Gamma.$$

Consequently

$$S_n(f, t) = \frac{1}{2}\sum_{\nu=0}^{n} \alpha_\nu [\Phi(t)]^\nu + \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{\nu=0}^{n}\alpha_\nu[\Phi(v)]^\nu}{v-t}\,dv. \tag{18}$$

As we have a bounded character of the singular operator and condition (3), we obtain

$$\|S_n(f, t)\|_{L_\mu(\Gamma)} \leqslant A_{33}(\mu, \Gamma)\left\|\sum_{\nu=0}^{n}\alpha_\nu w^\nu\right\|_{L_\mu(c)}. \tag{19}$$

As was mentioned above, $\{\alpha_\nu\}_{\nu=0}^{\infty}$ are Taylor coefficients for $h_1(w)$ whose non-tangential boundary values belong to $L_\mu^*(\Gamma)$. Therefore, according to the boundedness of the partial sums of the Fourier series in the reflexive Orlicz space [15], (19) implies

$$\|S_n(f, t)\|_{L_\mu(\Gamma)} \leqslant A_{34}(\mu, \Gamma)\cdot\|h_1^+(w)\|_{L_\mu(c)}.$$

Then by (10)

$$h_1^+(w) = \frac{1}{2}f[\psi(w)] + \frac{1}{2\pi i}\int_{|\tau|=1}\frac{f[\psi(\tau)]}{\tau-\tau_0}\,d\tau.$$

Hence

$$\|S_n(f, t)\|_{L_\mu(\Gamma)} \leqslant A_{35}(\mu, \Gamma)\|f(t)\|_{L_\mu(\Gamma)}.$$

Let us suppose $Q_n(t)$ is the polynomial of the best approximations of degree n for the function $f(z)$ in the mean on the boundary

$$\varrho_n^{(\mu)}(f, \Gamma) = \|f(t)-Q_n(t)\|_{L_\mu(\Gamma)}.$$

Thus we have got

$$\|f(t)-S_n(f, t)\|_{L_\mu(\Gamma)} \leqslant \|f(t)-Q_n(t)\|_{L_\mu(\Gamma)} + \|S_n(Q_n-f, t)\|_{L_\mu(\Gamma)}$$
$$\leqslant A_{36}(\mu, \Gamma)\,\varrho_n^{(\mu)}(f, \Gamma).$$

The lemma is proved.

LEMMA 2. *If $P_n(z)$ is a polynomial of degree n, then for a Liapunov type boundary the following inequality holds:*

$$\|P_n'(t)\|_{L_\mu(\Gamma)} \leqslant A_{37}(\mu, \Gamma)\,n\,\|P_n(t)\|_{L_\mu(\Gamma)}.$$

This lemma can be proved quite like theorem 3 of [1]; it is sufficient to use the inequality for the derivative of trigonometrical polynomials in L_μ^* deduced in [14].

The set $L_\mu(\Gamma)$ with convex $\mu(u)$ can be turned into a Banach space with the help of other norms.

Let (see [7], p. 95)

$$\|f(t)\|_{L_\mu(\Gamma)}^* = \inf_{\tau>0}\left\{\int_\Gamma \mu\left[\frac{|f(t)|}{\tau}\right]|dt| \leqslant 1\right\}. \tag{20}$$

It is known that the norms (4) and (20) are equivalent:

$$\|f(t)\|_{L_\mu(\Gamma)}^* \leqslant \|f(t)\|_{L_\mu(\Gamma)} \leqslant 2\,\|f(t)\|_{L_\mu(\Gamma)}^*.$$

Proof of theorem 2.1. Let us suppose that $1/2^{m+1} \leqslant h < 1/2^m$, $n = [1/h]$. Then we shall have

$$\|f[\psi(w)]-f[\psi(we^{ih})]\|_{L_\mu(C)} \tag{21}$$
$$\leqslant \|\Delta_h\{f[\psi(w)]-S_{2^{m+1}-1}[\psi(w)]\|_{L_\mu(C)} + \|\Delta_h S_{2^{m+1}-1}[\psi(w)]\|_{L_\mu(C)}.$$

By lemma 1 one can conclude that

$$\|\Delta_h\{f[\psi(w)]-S_{2^{m+1}-1}[\psi(w)]\}\|_{L_\mu(C)} \leqslant A_{37}(\mu, \Gamma)\,\varrho_n^{(\mu)}(f, \Gamma). \tag{22}$$

Let us write

$$q(x) = S_{2^{m+1}-1}[\psi(e^{ix})].$$

It is easy to see that

$$\|q(x+h)-q(x)\| = \Big\|\int_0^h q'(x+u)\,du\Big\| \leqslant \int_0^h \|q'(x+u)\|\,du.$$

Consequently

$$\|q(x+h)-q(x)\| \leqslant A_{38}(\mu,\Gamma)\,h\,\|S'_{2^{m+1}-1}(t)\|_{L_\mu(\Gamma)}.$$

According to theorem 1.4 we receive

$$\|S'_{2^{m+1}-1}(t)\| \leqslant A_{39}(\mu,\Gamma)\,h\,\Big\|\Big(\sum_{r=0}^{m}\Big|\sum_{\nu=2^r}^{2^{r+1}-1} a_\nu \Phi'_\nu(t)\Big|^2\Big)^{1/2}\Big\|_{L_\mu(\Gamma)}$$

or

$$\|S'_{2^{m+1}-1}(t)\| \leqslant A_{39}(\mu,\Gamma)\,h\,\Big\|\Big[\sum_{r=0}^{m}\delta_r^2(t)\Big]^{1/2}\Big\|_{L_\mu(\Gamma)},$$

where

$$\delta_r(t) = \Big|\sum_{\nu=2^r}^{2^{r+1}-1} a_\nu \Phi'_\nu(t)\Big|.$$

Then (21) implies

$$\|S'_{2^{m+1}-1}(t)\| \leqslant A_{40}(\mu,\Gamma)\inf_{\tau>0}\left\{\int_\Gamma \mu\left[\frac{\Big(\sum_{r=1}^{m}\delta_r^2(t)\Big)^{1/2}}{\tau}\right]|dt| \leqslant 1\right\}$$

$$= A_{40}(\mu,\Gamma)\inf_{\tau>0}\left\{\int_\Gamma \Psi\left[\frac{\Big(\sum_{r=1}^{m}\delta_r^2(t)\Big)^{a/2}}{\tau^a}\right]|dt| \leqslant 1\right\}.$$

From the condition $1 < a \leqslant 2$ it is clear that

$$\|S'_{2^{m+1}-1}(t)\| \leqslant A_{40}(\mu,\Gamma)\inf_{\tau>0}\left\{\int_\Gamma \Psi\left[\frac{\sum_{r=\mu}^{m}\delta_1^a(t)}{\tau^a}\right]|dt| \leqslant 1\right\}$$

or

$$\|S'_{2^{m+1}-1}(t)\|_{L_\mu(\Gamma)} \leqslant A_{40}(\mu,\Gamma)\Big(\Big\|\sum_{r=1}^{m}\delta_r^a(t)\Big\|^*_{L_\Psi(\Gamma)}\Big)^{1/a}. \tag{23}$$

In view of lemma 2 we deduce

(24)

$$\|S'_{2^{m+1}-1}(t)\| \leqslant A_{41}(\mu,\Gamma)\Big(\sum_{r=1}^{m}\|\delta_r^a(t)\|_{L_\Psi(\Gamma)}\Big)^{1/a} \leqslant A_{41}(\mu,\Gamma)\Big(\sum_{r=1}^{m}\|\delta_r(t)\|^a_{L_\mu(\Gamma)}\Big)^{1/a}.$$

Now by applying lemma 1 it is easily seen that

$$\|\Delta_h S_{2^{m+1}-1}[\psi(w)]\|_{L_\mu(C)} \leqslant A_{42}(\mu,\Gamma)\cdot h\Big\{\sum_{\nu=0}^{m} 2^{(1+\nu)a}[\varrho^{(\mu)}_{2^\nu}(f,\Gamma)]^a\Big\}^{1/a}. \tag{25}$$

Since the sequence of the best approximations converges monotonically to zero, (21), (22) and (25) imply the desired estimate.

Proof of theorem 2.2. Above we had the inequality

$$\|f[\psi(w)]-f[\psi(we^{ih})]\| \leqslant A_{43}(\mu,\Gamma)\{h\|S'_{2^{m+1}-1}(t)\|+\varrho^{(\mu)}_n(f,\Gamma)\}. \tag{26}$$

Thus

$$S'_{2^{m+1}-1}(t) = S'_0(t)+\sum_{r=1}^{m+1}[S'_{2^r-1}(t)-S'_{2^{r-1}-1}(t)];$$

therefore

$$\|S'_{2^{m+1}-1}(t)\| \leqslant \sum_{r=1}^{m+1}\|S'_{2^r-1}(t)-S_{2^{r-1}-1}(t)\|.$$

By lemma 1

$$\|S_{2^{r+1}-1}(t)-S_{2^r-1}(t)\| \leqslant A_{44}(\mu,\Gamma)\cdot\varrho^{(\mu)}_{2^{r-1}-1}(f,\Gamma).$$

Consequently

$$\|S'_{2^{m+1}-1}(t)\| \leqslant A_{45}(\mu,\Gamma)\sum_{r=1}^{m+1}2^r\varrho^{(\mu)}_{2^{r-1}-1}(f,\Gamma). \tag{27}$$

Now the desired estimate is received from (26) and (27).

The method of the proof of the above-mentioned theorems ensures the correctness of the following statements:

THEOREM 2.3. *Let the function $\Psi(u) = \mu(u^{1/a})$ be convex for some a, $1 < a \leqslant 2$, and let the series*

$$\sum_{\nu=1}^{\infty}\nu^{a-1}[\varrho^{(\mu)}_\nu(f,\Gamma)]^a$$

converge.

Then $f'(z)\in E_\mu$ and the following inequality is valid:

$$\omega_1^{(\mu)}\Big(\frac{1}{n},f'\Big) \leqslant A_{46}(\mu,\Gamma,r,k)\left\{\frac{1}{n}\Big(\sum_{\nu=1}^{n}\nu^{a(1+k)-1}[\varrho^{(\mu)}_{\nu-1}(f,\Gamma)]^a\Big)^{1/a}+\right.$$

$$\left.+\Big(\sum_{\nu=n+1}^{\infty}\nu^{a-1}[\varrho^{(\mu)}_\nu(f,\Gamma)]^a\Big)^{1/a}\right\}.$$

In the general case we have

THEOREM 2.4. *Let $f(z)\in E_\mu$ in a domain with a Liapunov type boundary and let the series*

$$\sum_{\nu=0}^{\infty} \varrho_\nu^{(\mu)}(f,\Gamma)$$

converge; then $f'(z)\in E_\mu$ and the estimate

$$\omega_1^{(\mu)}\left(\frac{1}{n},f'\right)\leqslant A_{47}(\mu,\Gamma,r,k)\left\{\frac{1}{n}\left(\sum_{\nu=1}^{n}\varrho_{\nu-1}^{(\mu)}(f,\Gamma)+\left(\sum_{\nu=n+1}^{\infty}\varrho_\nu^{(\mu)}(f,\Gamma)\right)\right\}\right.$$

holds.

References

[1] С. Я. Альпер, *О приближении в среднем аналитических функций класса E_p. Исследования по современным проблемам теории функций комплексного переменного*, Москва 1960.

[2] Б. В. Хведелидзе, *Линейные разрывные граничные задачи теории функций, сингулярные интегральные уравнения и некоторые их приложения*, Труды Мат. инст. АН Груз. ССР 23 (1956), стр. 3-158.

[3] S. Koizumi, *On the singular integrals*, Proc. Japan Academy 34 (1958), p. 193-198.

[4] V. Kokilashvili, *On extension of some Marcinkiewicz and Littlewood-Paley theorems*, Bull. Acad. Pol. Sci., Série sci. math., astr. et phys., 15 (1967), p. 239-243.

[5] В. М. Кокилашвили, *О приближении аналитических функций класса E_p*, Сообщения АН Груз. ССР 18, № 1 (1967), p. 3-7.

[6] — *О приближении аналитических функций класса E_p*, Доклады АН СССР 177, № 2 (1967), p. 261-264.

[7] М. А. Красносельский и Я. Б. Рутицкий, *Выпуклые функци и пространства Орлича*, Москва 1958.

[8] R. Leśniewicz, *On Hardy-Orlicz spaces I*, Bull. Acad. Pol. Sci., Série sci. math., astr. et phys., 14 (1966), p. 145-150.

[9] J. Littlewood and R. Paley, *Theorems on Fourier series and power series*, Proc. London Math. Soc. 42 (1937), p. 52-89.

[10] J. Marcinkiewicz, *Sur les multiplicateurs des séries de Fourier*, Studia Math. 8 (1939), p. 78-91.

[11] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Intern. Acad. Pol. Sci., Série A, Cracovie 1936.

[12] В. Г. Пономаренко, *О приближении периодических функций в пространствах Орлича*, Сибирский мат. ж. 7 (1966), стр. 1337-1346.

[13] И. И. Привалов, *Граничные свойства аналитических функций*, Москва-Ленинград 1950.

[14] E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. J. 3 (1937), p. 41-50.

[15] R. Ryan, *On the conjugate functions in Orlicz space*, Pacific J. of Math. 13 (1963), p. 1371-1377.

[16] Г. П. Сафронова, *О некоторых граничных свойствах аналитических функций*, Вестник Ленинградского унив. 13, серия математики, механики и астрон., вып. 3 (1959), p. 52-58.

[17] И. Б. Симоненко, *Краевая задача Римана для пар функций с измеримыми коэффициентами ...*, Известия АН СССР, серия матем., 28 (1964), p. 277-306.

[18] V. Smirnov, *Sur les formules de Cauchy et de Green et quelques problèmes qui s'y rattachent*, ibidem 7 (1932), p. 337-372.

[19] П. К. Суетин, *Основные свойства полиномов Фабера*, Успехи матем. наук 19, вып. 4 (II 8) (1964), p. 125-154.

[20] J. L. Walsh and H. G. Russel, *Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions*, Trans. Amer. Math. Soc. 92 (1959), p. 355-369,

[21] S. Warschawski, *Über das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung*, Math. Z. 35 (1932), p. 321-456.

[22] A. Zygmund, *Trigonometric series II*, Cambridge 1959.

Reçu par la Rédaction le 7. 2. 1968