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Random measures and harmonizable sequences

by

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By a *harmonizable sequence* of random variables we mean a sequence of Fourier coefficients of a random measure. A concept of prediction for strictly stationary sequences which need not have a finite variance was introduced in [19] and [21]. In particular, each stationary sequence admitting a prediction is the sum of two independent stationary sequences, one deterministic and the other completely non-deterministic. The purpose of this paper is to give a characterization of deterministic and completely non-deterministic harmonizable stationary sequences of random variables. Some modular spaces introduced by J. Musielak and W. Orlicz in [14] are used as a tool to study harmonizable sequences. They play the same role in our investigations as the L^2-spaces in the Wiener-Kolmogorov theory of the best linear least squares prediction for wide sense stationary sequences.

The first section contains a discussion of an extremal problem for Musielak-Orlicz spaces and a generalization of the famous Kolmogorov-Krein criterion for L^p-spaces. The second one contains an analogue of S. Bernstein's Theorem concerning Gaussian random variables. In the third section we study the space of all complex-valued functions which are integrable with respect to a complex-valued isotropic random measure. The main results concerning harmonizable sequences are given in the last section.

1. An extremal problem for Musielak-Orlicz spaces. Given a measure ν defined on Borel subsets of the unit interval $I = [0, 1]$, we take a real function Φ defined on $I \times R_+$, R_+ being the space of non-negative reals, satisfying the following conditions:

(i) $\Phi(t, 0) = 0$ and $\Phi(t, x) > 0$ for $x > 0$ and ν-almost all t;

(ii) $\Phi(t, x)$ is a continuous non-decreasing function of x for every $t \in I$;

(iii) $\Phi(t, x)$ is Borel measurable as a function of t for every $x \in R_+$;

(iv) $\int_I \Phi(t, 1)\nu(dt) < \infty$;

(v) (the Δ_2-condition) there exists a positive constant $\varkappa$ such that

$$\Phi(t, 2x) \leqslant \varkappa \Phi(t, x)$$

for all x and ν-almost all t.

Throughout this paper we identify functions equal ν-almost everywhere. Let f be a complex-valued Borel function on I. It is easily seen that $\Phi(t, |f(t)|)$ is also a Borel function on I. We define a modular ϱ by means of the formula

$$\varrho(f) = \int_I \Phi(t, |f(t)|)\nu(dt). \tag{1.1}$$

Let $L_\Phi(\nu)$ be the set of all complex-valued Borel functions f on I such that $\varrho(f)$ is finite. The set $L_\Phi(\nu)$ is a linear space over the complex field under usual addition and scalar multiplication. Moreover, it becomes a complete linear metric space under the non-homogeneous norm

$$\|f\| = \inf\{c : c > 0,\ \varrho(c^{-1}f) \leqslant c\}.$$

The space $L_\Phi(\nu)$ with this norm was introduced and investigated by J. Musielak and W. Orlicz in [14] and will be called a *Musielak-Orlicz space*.

A sequence $\{f_n\}$ of elements of $L_\Phi(\nu)$ is said to be *modular convergent* to an element f of $L_\Phi(\nu)$ if

$$\lim_{n\to\infty} \varrho(f_n - f) = 0.$$

From the Δ_2-condition it follows that the modular convergence is equivalent with the norm convergence in $L_\Phi(\nu)$ (see [14], theorem 1.31). Further, from (iv) it follows that all bounded Borel functions belong to $L_\Phi(\nu)$. Moreover, the set of all Borel simple functions, i.e. Borel functions assuming a finite number of values, is dense in $L_\Phi(\nu)$.

By ν_c we shall denote the absolutely continuous component of the measure ν and by $d\nu_c/dt$ a Borel measurable version of its Radon-Nikodym density function. It is clear that if the Lebesgue measure is absolutely continuous with respect to the measure ν, then

$$\frac{d\nu_c}{dt} = \left(\frac{dt}{d\nu}\right)^{-1} \tag{1.2}$$

almost everywhere in the sense of the Lebesgue measure.

We introduce auxiliary functions $\Lambda_{\varphi,\nu}$ and $\Omega_{\varphi,\nu,n}$ $(n = 1, 2, \ldots)$ by means of the formulas

$$\Lambda_{\Phi,\nu}(t, x) = \sup\left\{\frac{\log y}{\Phi(t, y)}\left(\frac{d\nu_c}{dt}\right)^{-1} : y \geqslant x\right\}, \tag{1.3}$$

$$\Omega_{\Phi,\nu,n}(t) = \inf\{x : \Lambda_{\varphi,\nu}(t, x) \leqslant n,\ x \geqslant 1\}, \tag{1.4}$$

where the infimum of an empty set is defined as ∞. It is clear that all these functions are Borel measurable and

$$1 \leqslant \Omega_{\Phi,\nu,n}(t) \leqslant \infty \qquad (n = 1, 2, \ldots).$$

The aim of this section is to prove the following theorem:

THEOREM 1.1. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space with the norm $\|\ \|$. The equation*

$$\inf\left\|1 + \sum_{k=1}^{n} a_k e^{2\pi i k t}\right\| = 0, \tag{1.5}$$

where the infimum is taken over all complex numbers $a_1, a_2, \ldots, a_n$ and $n = 1, 2, \ldots$, holds if and only if no function $\log \Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ is Lebesgue integrable over I.

This solution of an extremal problem of Szegö's type can be regarded as a generalization of the Kolmogorov-Krein criterion for L^p-spaces (see [7] and [8]). For a class of Orlicz spaces more general problem was discussed in [20]. Before proving the theorem we shall prove four lemmas.

Given an arbitrary set $\mathscr{S}$ of complex-valued Borel functions on I, we put $i(\mathscr{S}) = \inf\{\varrho(f) : f \in \mathscr{S}\}$, where ϱ is the modular in $L_\Phi(\nu)$. Let $\mathscr{P}$ be the set of all trigonometric polynomials

$$1 + \sum_{k=1}^{n} a_k e^{2\pi i k t},$$

where $a_1, a_2, \ldots, a_n$ are complex numbers and n is variable. Further, let $\mathscr{Q}$ be the set of all Borel functions on I such that $\log|f|$ is Lebesgue integrable and $\int_I \log|f(t)|\,dt \geqslant 0$.

LEMMA 1.1. $i(\mathscr{P}) = i(\mathscr{Q})$.

Proof. The inclusion $\mathscr{P} \subset \mathscr{Q}$ is a simple consequence of the Jensen inequality. Hence we get the inequality $i(\mathscr{P}) \geqslant i(\mathscr{Q})$.

To prove the converse inequality we put

$$\mathscr{Q}_{a,b} = \{f : a < |f| < b\} \cap \mathscr{Q} \qquad (a, b > 0).$$

By bounded convergence theorem we get the formula

$$i(\mathscr{Q}) = \lim_{\substack{a\to 0\\ b\to\infty}} i(\mathscr{Q}_{a,b}). \tag{1.6}$$

Consider an auxiliary modular

$$\varrho_0(f) = \varrho(f) + \int_I |f(t)|\,dt$$

on $\mathcal{Q}_{a,b}$. From condition (iv) it follows that the subset of $\mathcal{Q}_{a,b}$ consisting of trigonometric polynomials

$$\sum_{k=-m}^{m} b_k e^{2\pi ikt},$$

where $b_{-m}, b_{-m+1}, \ldots, b_m$ are complex numbers and m is variable, is dense in $\mathcal{Q}_{a,b}$ in the sense of the modular ϱ_0-convergence. Moreover, both functionals $\varrho(f)$ and $\int_I \log|f(t)|\,dt$ are continuous on $\mathcal{Q}_{a,b}$. Consequently, by (1.6), for every positive number ε there exists a trigonometric polynomial

$$g(t) = e^{-2\pi imt}\Big(\sum_{k=0}^{n} c_k e^{2\pi ikt}\Big) \qquad (m, n \geqslant 0)$$

such that

$$\varrho(g) \leqslant i(\mathcal{Q}) + \varepsilon \quad \text{and} \quad \int_I \log|g(t)|\,dt \geqslant 0. \tag{1.7}$$

Of course, we may assume that $c_0 \neq 0$. Further, we can find a trigonometric polynomial

$$h(t) = \sum_{k=0}^{n} a_k e^{2\pi ikt}$$

such that $|h(t)| = |g(t)|$ for $t \in I$ and the polynomial $\sum_{k=0}^{n} a_k z^k$ has no zero inside the unit circle. By the Jensen equation and (1.7) we have

$$\log|a_0| = \int_I \log|h(t)|\,dt \geqslant 0.$$

Hence and from (1.7) we obtain the inequality $\varrho(a_0^{-1}h) \leqslant i(\mathcal{Q}) + \varepsilon$. Since $a_0^{-1}h \in \mathcal{P}$, we infer that for every positive number ε the inequality $i(\mathcal{P}) \leqslant i(\mathcal{Q}) + \varepsilon$ holds. Thus $i(\mathcal{P}) \leqslant i(\mathcal{Q})$, which completes the proof.

LEMMA 1.2. *If the Lebesgue measure is not absolutely continuous with respect to the measure ν, then $i(\mathcal{P}) = 0$.*

Proof. Let E be a Borel subset of I such that $|E| > 0$ and $\nu(E) = 0$, where $|E|$ denotes the Lebesgue measure of E. Given $\varepsilon > 0$, we can find a positive number c such that $\varrho(c) > \varepsilon$. Taking a positive number q satisfying the condition $\log q \geqslant |E|^{-1}|\log c|$, we put $g = c + qh_E$, where h_E is the indicator of the set E. Evidently, $\varrho(g) = \varrho(c) < \varepsilon$ and $\int_I \log g(t)\,dt \geqslant 0$. Thus $i(\mathcal{Q}) = 0$ and, consequently, by Lemma 1.1, $i(\mathcal{P}) = 0$ which completes the proof.

LEMMA 1.3. *If the Lebesgue measure is absolutely continuous with respect to the measure ν and $\log \Omega_{\Phi,\nu,p}$ is Lebesgue integrable over I for an index p, then $i(\mathcal{P}) > 0$.*

Proof. Contrary to this let us assume that $i(\mathcal{P}) = 0$. Put

$$a = \int_I \log \Omega_{\Phi,\nu,p}(t)\,dt.$$

Let k be an integer satisfying the inequality

$$k > \frac{a+1}{\log 2}. \tag{1.8}$$

Further, let g be a trigonometric polynomial from $\mathcal{P}$ satisfying the inequalities

$$\varrho(g) < \frac{1}{p\varkappa^k}, \qquad \int_I \log|g(t)|\,dt \geqslant 0, \tag{1.9}$$

where $\varkappa$ is the constant appearing in the Δ_2-condition for the function Φ. Setting $h = 2^k g$ and $E = \{t: |h(t)| > \Omega_{\Phi,\nu,p}(t)\}$, we have, by virtue of (1.2), (1.3) and (1.4), the inequality

$$\log|h(t)|\frac{dt}{d\nu} \leqslant p\Phi(t, |h(t)|) \qquad (t \in E).$$

Thus

$$\int_E \log|h(t)|\,dt \leqslant p\int_E \Phi(t, |h(t)|)\nu(dt) \leqslant p\varrho(h).$$

On the other hand, by the Δ_2-condition and (1.9),

$$\varrho(h) \leqslant \varkappa^k \varrho(g) < \frac{1}{p},$$

and, consequently,

$$\int_E \log|h(t)|\,dt < 1.$$

Since

$$\int_{I\setminus E} \log|h(t)|\,dt \leqslant \int_{I\setminus E} \log \Omega_{\Phi,\nu,p}(t)\,dt \leqslant a,$$

we have

$$\int_I \log|h(t)|\,dt \leqslant a+1.$$

But, by (1.9),

$$\int_I \log|h(t)|\,dt \geqslant k\log 2$$

and, consequently, $k \leqslant (a+1)/\log 2$ which contradicts inequality (1.8). The Lemma is thus proved.

LEMMA 1.4. *If the Lebesgue measure is absolutely continuous with respect to the measure ν and no function $\log \Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ is Lebesgue integrable over I, then $i(\mathscr{P}) = 0$.*

Proof. Put

$$A_n = \{t : 1 < \Omega_{\Phi,\nu,n}(t) < \infty\} \qquad (n = 1, 2, \ldots).$$

First consider the case

$$\int_{A_n} \log \Omega_{\Phi,\nu,n}(t)\,dt = \infty \qquad (n = 1, 2, \ldots). \tag{1.10}$$

Given $\varepsilon > 0$, we take a positive number c for which

$$\varrho(c) \leqslant \frac{\varepsilon}{2}. \tag{1.11}$$

From (1.10) it follows that there exists a subset B_n of the set A_n for which

$$\int_{B_n} \log \Omega_{\Phi,\nu,n}(t)\,dt = \frac{n\varepsilon}{2} \qquad (n = 1, 2, \ldots). \tag{1.12}$$

Since, by (1.2), (1.3) and (1.4),

$$\log \Omega_{\Phi,\nu,n}(t)\frac{dt}{d\nu} = n\Phi\big(t, \Omega_{\Phi,\nu,n}(t)\big) \qquad (t \in A_n),$$

we have, by (1.12), the formula

$$\int_{B_n} \Phi\big(t, \Omega_{\Phi,\nu,n}(t)\big)\nu(dt) = \frac{\varepsilon}{2} \qquad (n = 1, 2, \ldots). \tag{1.13}$$

Put $g_n(t) = \Omega_{\Phi,\nu,n}(t)$ on B_n and $g_n(t) = c$ otherwise. By (1.11) and (1.13), we have the inequality $\varrho(g_n) \leqslant \varepsilon$. Moreover, by (1.12),

$$\int_I \log g_n(t)\,dt = \frac{n\varepsilon}{2} + |I \setminus B_n| \log c.$$

Thus $g_n \in \mathscr{Q}$ for $n \geqslant 2|\log c|/\varepsilon$ and, consequently, $i(\mathscr{Q}) \leqslant \varepsilon$ whence the formula $i(\mathscr{Q}) = 0$ follows. Now the assertion of the Lemma is a consequence of Lemma 1.1.

In the remaining case there exists an index p for which $\log \Omega_{\Phi,\nu,p}$ is Lebesgue integrable over the set A_p. Since the sequence $\Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ is monotone non-increasing, the function $\log \Omega_{\Phi,\nu,n}$ is Lebesgue integrable over the set A_n for $n \geqslant p$. Consequently, the sets

$$C_n = \{t : \Omega_{\Phi,\nu,n}(t) = \infty\} \tag{1.14}$$

have for $n \geqslant p$ positive Lebesgue measure. Given a positive number ε, we take an integer m satisfying the inequalities

$$m \geqslant p \quad \text{and} \quad m \geqslant \frac{2|\log c|}{\varepsilon}, \tag{1.15}$$

where the number c is determined by (1.11). Put

$$q = \exp\left(\frac{\varepsilon}{2} m |C_m|^{-1}\right).$$

By the definitions (1.3), (1.4) and (1.14) the function

$$h(t) = \inf\left\{x : \frac{\log x}{\Phi(t, x)}\frac{dt}{d\nu} \geqslant m,\ x \geqslant q\right\}$$

is finite on C_m and, of course, Borel measurable. Moreover,

$$\int_{C_m} \log h(t)\,dt \geqslant |C_m| \log q = \frac{m\varepsilon}{2}.$$

Consequently, there exists a Borel subset D of C_m for which

$$\int_D \log h(t)\,dt = \frac{m\varepsilon}{2}. \tag{1.16}$$

Thus

$$\int_D \Phi\big(t, h(t)\big)\nu(dt) \leqslant \frac{1}{m}\int_D \log h(t)\,dt = \frac{\varepsilon}{2}. \tag{1.17}$$

Put $g(t) = h(t)$ on D and $g(t) = c$ otherwise. By (1.11) and (1.17) we have the inequality $\varrho(g) \leqslant \varepsilon$. Moreover, by (1.15) and (1.16),

$$\int_I \log g(t)\,dt = \frac{m\varepsilon}{2} + |I \setminus D| \log c \geqslant 0.$$

Thus $i(\mathscr{Q}) \leqslant \varepsilon$ and, consequently, $i(\mathscr{Q}) = 0$. Taking into account Lemma 1.1, we get the formula $i(\mathscr{P}) = 0$ which completes the proof.

Proof of the Theorem 1.1. Since the modular convergence and the norm convergence in $L_\Phi(\nu)$ are equivalent, equation (1.5) is equivalent with the equation $i(\mathscr{P}) = 0$. Consequently, by Lemmas 1.3 and 1.4, the Theorem is true if the Lebesgue measure is absolutely continuous with respect to the measure ν. If the Lebesgue measure is not absolutely continuous with respect to the measure ν, then $d\nu_c/dt = 0$ on a set of positive Lebesgue measure and, consequently, by (1.3) and (1.4), no function $\log \Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ is Lebesgue integrable over I. In this case the Theorem is a consequence of Lemma 1.2, which completes the proof.

We conclude this section with some particular cases of the Theorem 1.1. If the function Φ does not depend upon the variable t, i.e. $\Phi(t,x) = \Phi(x)$ $(t\in I,\ x\in R_+)$, $L_\Phi(\nu)$ is called an *Orlicz space* (see [11] and [13]). In this paper we have assumed the Δ_2-condition for Φ. We say that the function Φ satisfies the Λ_a-condition for a number $a>1$ if there exists a constant $\gamma_a > 1$ such that

$$\Phi(x)\gamma_a \leqslant \Phi(ax)$$

for sufficiently large x (see [12]).

Now we shall prove the original version of the Kolmogorov-Krein criterion.

THEOREM 1.2. *Let $L_\Phi(\nu)$ be an Orlicz space and let Φ satisfy the Λ_a-condition for some constant $a>1$. Then equation* (1.5) *holds if and only if $\log(d\nu_c/dt)$ is not Lebesgue integrable over I.*

Proof. Of course, it suffices to consider the case when the Lebesgue measure is absolutely continuous with respect to the measure ν. Since, by (iv), $\nu(I)$ is finite, we have the inequality

$$\int_I \log\frac{d\nu_c}{dt}\,dt < \infty.$$

Thus to prove our statement it suffices to prove that (1.5) is equivalent to the equation

$$\int_I \log\frac{d\nu_c}{dt}\,dt = -\infty.$$

From the Δ_2-condition and the Λ_a-condition for Φ it follows that there are positive constants c_1, c_2, p and q such that

$$c_1 x^{2p} \leqslant \Phi(x) \leqslant c_2 x^q$$

for sufficiently large x (see [12]). Consequently, we can find a positive number x_0 such that

$$c_1 x^p \leqslant \frac{\Phi(x)}{\log x} \leqslant c_2 x^q \quad \text{if } x \geqslant x_0. \tag{1.18}$$

Hence, in particular, it follows that

$$\lim_{x\to\infty} \Lambda_{\Phi,\nu}(t,x) = 0$$

for almost all t in the sense of the Lebesgue measure, because $d\nu_c/dt$ is almost everywhere positive. Consequently, the functions $\Omega_{\varphi,\nu,n}$ $(n=1,2,\ldots)$ are finite almost everywhere. Put

$$F_n = \{t : \max(1,x_0) < \Omega_{\Phi,\nu,n}(t) < \infty\}.$$

Then, by the definitions (1.3) and (1.4), the function $(d\nu_c/dt)^{-1}$ is bounded on $I\setminus F_n$ almost everywhere and

$$\log\Omega_{\Phi,\nu,n}(t)\left(\frac{d\nu_c}{dt}\right)^{-1} = n\Phi\big(\Omega_{\Phi,\nu,n}(t)\big)$$

for $t\in F_n$. Hence and from (1.18) we get the inequalities

$$nc_1\,\Omega^p_{\Phi,\nu,n}(t) \leqslant \left(\frac{d\nu_c}{dt}\right)^{-1} \leqslant nc_2\,\Omega^q_{\Phi,\nu,n}(t) \qquad (t\in F_n).$$

Consequently, no function $\log\Omega_{\Phi,\nu,n}$ $(n=1,2,\ldots)$ is Lebesgue integrable over I if and only if

$$\int_I \log\frac{d\nu_c}{dt}\,dt = -\infty.$$

Our theorem is now a consequence of Theorem 1.1.

THEOREM 1.3. *If $L_\Phi(\nu)$ is an Orlicz space and*

$$\lim_{x\to\infty}\frac{\Phi(x)}{\log x} = c > 0,$$

then equation (1.5) *holds if and only if*

$$\operatorname{ess\,inf}\left\{\frac{d\nu_c}{dt} : t\in I\right\} = 0.$$

Proof. If $\operatorname{ess\,inf}\{d\nu_c/dt : t\in I\} = 0$, then the set

$$G_n = \left\{t : \frac{d\nu_c}{dt} < \frac{c}{n}\right\}$$

has for every n a positive Lebesgue measure and, by (1.3) and (1.4), $\Omega_{\Phi,\nu,n}(t) = \infty$ on G_n. Hence we infer that no function $\log\Omega_{\Phi,\nu,n}$ $(n=1,2,\ldots)$ is Lebesgue integrable over I.

Suppose now that

$$\operatorname{ess\,inf}\left\{\frac{d\nu_c}{dt} : t\in I\right\} = a > 0$$

and put

$$b = \sup\left\{\frac{\log x}{\Phi(x)} : x\geqslant 1\right\}.$$

Of course, $b<\infty$ and, by (1.3) and (1.4), $\Omega_{\Phi,\nu,m}(t) = 1$ almost everywhere for all indices m satisfying the inequality $m > a^{-1}b$. Consequently, our theorem is a simple consequence of Theorem 1.1.

In the same way we can prove the following theorems:

THEOREM 1.4. *If $L_\Phi(\nu)$ is an Orlicz space and*

$$\lim_{x\to\infty}\frac{\Phi(x)}{\log^{p+1}x}>0,$$

where p is a positive number, then equation (1.5) *holds if and only if*

$$\int_I\left(\frac{d\nu_c}{dt}\right)^{-1/p}dt=\infty.$$

THEOREM 1.5. *If $L_\Phi(\nu)$ is an Orlicz space and*

$$\lim_{x\to\infty}\frac{\Phi(x)}{\log x\log\log x}>0,$$

then equation (1.5) *holds if and only if*

$$\int_I\exp\left\{n^{-1}\left(\frac{d\nu_c}{dt}\right)^{-1}\right\}dt=\infty$$

for all positive integers n.

2. Vector-valued random measures. In this section by (x, y) and $|x|$ we shall denote the inner product and the norm respectively in R^p. Further, for any R^p-valued random variable X, $\varphi_X(t)$ $(t\in R^p)$ will denote the characteristic function of X, i.e. the expectation $\mathrm{E}e^{i(t,X)}$.

A function M defined on the σ-algebra of all Borel subsets of the unit interval I whose values are R^p-valued random variables is called an *R^p-valued random measure* or shortly a *random measure* if

(*) for every sequence $E_1, E_2, \ldots$ of disjoint Borel sets

$$M\Big(\bigcup_{n=1}^{\infty}E_n\Big)=\sum_{n=1}^{\infty}M(E_n),$$

where the series converges with probability 1,

(**) for every sequence $E_1, E_2, \ldots, E_n$ of disjont Borel sets the random variables $M(E_1), M(E_2), \ldots, M(E_n)$ are independent.

The theory of random measures was developed by A. Prékopa in [15], [16] and [17]. For further results see [6], [22] and [24].

A random measure is said to be *atomless* if $M(\{a\})=0$ with probability 1 for every one-point set $\{a\}$. In this paper we shall consider atomless random measures only. Moreover, we shall identify random variables which are equal with probability 1. Given a random measure M, we say that a Borel set E is an *M-null set* if $M(A)=0$ for all Borel subsets A of E. Relations valid except on an M-null set are said to be valid *M-almost* everywhere.

The concept of the integral of a real-valued function with respect to a real-valued random measure was introduced in [16] (the unconditional integral) and in [22]. In an analogous way the integral of an operator-valued function with respect to a vector-valued random measure was introduced in [24]. We shall quote the basic definition, which is an adaptation of the Dunford's definition of the integral with respect to a measure whose values belong to a Banach space ([4], Chapter IV).

Let M be an atomless R^p-valued random measure. If F is an operator-valued Borel simple function on I,

$$F=\sum_{j=1}^{n}C_j\chi_{E_j},$$

where E_j are Borel sets, C_j are linear operators on R^p and χ_{E_j} denotes the indicator of E_j $(j=1,2,\ldots,n)$, then the integral on every Borel set E of F with respect to M is defined by the formula

$$\int_E F(s)M(ds)=\sum_{j=1}^{n}C_jM(E_j\cap E).$$

Further, an operator-valued Borel function defined on I is said to be *M-integrable* if there exists a sequence of operator-valued Borel simple functions $\{F_n\}$ such that

1° the sequence $\{F_n\}$ converges to F M-almost everywhere on I,

2° for every Borel set E the sequence $\{\int_E F_n(s)M(ds)\}$ converges in probability.

Then, by the definition, the integral $\int_E F(s)M(ds)$ is the limit in probability of the sequence $\{\int_E F_n(s)M(ds)\}$.

A random measure M is said to be *symmetric* if for every Borel set E the random variables $M(E)$ and $-M(E)$ are identically distributed. Since the values $M(E)$ of an atomless random measure have an infinitely divisible distribution, we infer that for symmetric atomless random measures the characteristic function of the random variable $M(E)$ can be written in the Lévy-Khinchine form

$$\varphi_{M(E)}(t)=\exp\Big\{-\big(D_M(E)t,t\big)-\int_{R^p\setminus\{0\}}\big(1-\cos(t,x)\big)\frac{1+|x|^2}{|x|^2}\lambda_M(E,dx)\Big\}, \tag{2.1}$$

where $D_M(E)$ is a symmetric non-negative operator on R^p and $\lambda_M(E,\cdot)$ is a finite non-negative measure on $R^p\setminus\{0\}$. Moreover, $D_M(\cdot)$ is an operator-valued Borel measure on I and for every Borel subset A of $R^p\setminus\{0\}$

the set-function $\lambda_M(\cdot, A)$ is a non-negative Borel measure on I. In the sequel we shall use the notation

$$(2.2)\qquad \psi_X(t) = -\log\varphi_X(t)$$

for symmetric random variables X with an infinitely divisible distribution.

Given a symmetric atomless random measure M, we put

$$(2.3)\quad Q_M(u, v, E) = 2\psi_{M(E)}(u)+2\psi_{M(E)}(v)-\psi_{M(E)}(u+v)-\psi_{M(E)}(u-v).$$

By (2.1) we have

$$Q_M(u, v, E) = 2\int_{R^p\setminus\{0\}}\big(1-\cos(u, x)\big)\big(1-\cos(v, x)\big)\frac{1+|x|^2}{|x|^2}\lambda_M(E, dx).$$

Consequently, $Q_M(\cdot, \cdot, E)$ is a continuous function and $Q_M(u, v, \cdot)$ is a non-negative Borel measure on I. Let us fix an orthonormal basis $e_1, e_2, \ldots, e_p$ in R^p. Given

$$x = \sum_{j=1}^{p}\alpha_j e_j \quad\text{and}\quad y = \sum_{j=1}^{p}\beta_j e_j,$$

we put

$$x\circ y = \sum_{j=1}^{p}\alpha_j\beta_j e_j.$$

Further, for any pair F, G of M-integrable operator-valued functions we put

$$(2.4)\quad S(F, G, u, v) = 2\psi_{u\circ X}(w)+2\psi_{v\circ Y}(w)-\psi_{u\circ X+v\circ Y}(w)-\psi_{u\circ X-v\circ Y}(w),$$

where

$$w = \sum_{j=1}^{p} e_j,\qquad X = \int_I F(s)M(ds),$$

$$Y = \int_I G(s)M(ds) \quad\text{and}\quad u, v\in R^p.$$

LEMMA 2.1. *Let M be a symmetric atomless random measure and let F, G be a pair of operator-valued M-integrable functions on I. Then for every triplet a, b, r $(a < b)$ of positive numbers we have the inequality*

$$\int_{K_r}\int_{K_r} S(F, G, u, v)\,du\,dv \geqslant b^{-2p}\int_{K_{ar}}\int_{K_{ar}} Q_M\big(u, v, U_{a,b}(F)\cap U_{a,b}(G)\big)\,du\,dv,$$

where $K_c = \{x: x\in R^p, |x|\leqslant c\}$, $U_{a,b}(H) = \{s: K_a\subset H^*(s)K_1\subset K_b\}$, $H^*(s)$ being the conjugate of $H(s)$.

Proof. By the definition of M-integrable functions there are two sequences of operator-valued Borel simple functions $\{F_n\}$ and $\{G_n\}$ which converge to F and G M-almost everywhere respectively and

$$\lim_{n\to\infty}\int_I F_n(s)M(ds) = \int_I F(s)M(ds),\qquad \lim_{n\to\infty}\int_I G_n(s)M(ds) = \int_I G(s)M(ds).$$

Taking into account definitions (2.2) and (2.4) we infer that

$$\lim_{n\to\infty} S(F_n, G_n, u, v) = S(F, G, u, v)$$

uniformly on every compact subset of $R^p\times R^p$. Consequently,

$$(2.5)\qquad \lim_{n\to\infty}\int_{K_r}\int_{K_r} S(F_n, G_n, u, v)\,du\,dv = \int_{K_r}\int_{K_r} S(F, G, u, v)\,du\,dv.$$

Moreover, there exists an M-null set U such that for every positive number ε less than a we have the inclusion

$$(2.6)\qquad \liminf_{n\to\infty} U_{a-\varepsilon,b+\varepsilon}(F_n)\cap U_{a-\varepsilon,b+\varepsilon}(G_n)\supset U_{a,b}(F)\cap U_{a,b}(G)\setminus U.$$

Let n be fixed for a moment. Introducing the notation

$$F_n = \sum_{j=1}^{k} A_j\chi_{E_j},\qquad G_n = \sum_{j=1}^{k} B_j\chi_{E_j},$$

where the Borel sets $E_1, E_2, \ldots, E_k$ are disjoint, we have, by virtue of (2.3) and (2.4), the formula

$$(2.7)\qquad S(F_n, G_n, u, v) = \sum_{j=1}^{k} Q_M(A_j^* u, B_j^* v, E_j).$$

Without loss of generality we may assume that

$$(2.8)\qquad \bigcup_{j=1}^{s} E_j = U_{a-\varepsilon,b+\varepsilon}(F_n)\cap U_{a-\varepsilon,b+\varepsilon}(G_n),$$

where $s\leqslant k$ and ε is a positive number less than a. Consequently, for $j\leqslant s$ the operators A_j^* and B_j^* are invertible, $(A_j^*)^{-1}K_r\supset K_{(a-\varepsilon)r}$, $(B_j^*)^{-1}K_r\supset K_{(a-\varepsilon)r}$ and the inequalities $|\det(A_j^*)^{-1}|\geqslant(b+\varepsilon)^{-p}$, $|\det(B_j^*)^{-1}|\geqslant(b+\varepsilon)^{-p}$ hold, where the matrix representation of operators with respect to the orthonormal basis $e_1, e_2, \ldots, e_p$ is taken. Hence we get the inequality

$$\int_{K_r}\int_{K_r} Q_M(A_j^* u, B_j^* v, E_j)\,du\,dv \geqslant (b+\varepsilon)^{-2p}\int_{K_{(a-\varepsilon)r}}\int_{K_{(a-\varepsilon)r}} Q_M(u, v, E_j)\,du\,dv$$

for $j\leqslant s$. Thus, by (2.7),

$$\int_{K_r}\int_{K_r} S(F_n, G_n, u, v)\,du\,dv \geqslant (b+\varepsilon)^{-2p}\int_{K_{(a-\varepsilon)r}}\int_{K_{(a-\varepsilon)r}} Q_M\Big(u, v, \bigcup_{j=1}^{s} E_j\Big)\,du\,dv.$$

Hence, by (2.5), (2.6), (2.8) and Fatou's Lemma, we get the inequality

$$\int_{K_r}\int_{K_r} S(F, G, u, v)\,du\,dv$$
$$\geqslant (b+\varepsilon)^{-2p}\int_{K_{(a-\varepsilon)r}}\int_{K_{(a-\varepsilon)r}} Q_M\big(u, v, U_{a,b}(F)\cap U_{a,b}(G)\big)\,du\,dv.$$

Since ε is an arbitrary positive number, from the last inequality we obtain the assertion of the Lemma.

Now we shall prove a continuous analogue of the Bernstein-Darmois theorem ([1], [2]). For homogeneous random measures this problem was discussed in [9], [18] and [21].

THEOREM 2.1. *Let M be a vector-valued atomless random measure and let F and G be operator-valued M-integrable functions on I. If the integrals $\int_I F(s)M(ds)$ and $\int_I G(s)M(ds)$ are independent, then for any Borel subset E of the set $\{s: F(s)$ and $G(s)$ are invertible$\}$ the random variables $M(E)$ are Gaussian.*

Proof. By Cramér's Theorem ([10], p. 271) we may assume, without loss of generality, that the random measure M is symmetric. Moreover, it suffices to prove that the random variable $M(U)$ is Gaussian, where

$$U = \{s: F(s) \text{ and } G(s) \text{ are invertible}\}.$$

Since the integrals $\int_I F(s)M(ds)$, $\int_I G(s)M(ds)$ are independent and symmetrically distributed, we have, by definition (2.4), the equation $S(F, G, u, v) = 0$ for all $u, v \in R^p$. Consequently, by Lemma 2.1,

$$\int_{K_{ar}}\int_{K_{ar}} Q_M\big(u, v, U_{a,b}(F) \cap U_{a,b}(G)\big)\,du\,dv = 0$$

for all positive numbers a, b and r $(a < b)$. Taking into account the continuity of $Q_M(\cdot, \cdot, E)$ we infer that

$$Q_M\big(u, v, U_{a,b}(F) \cap U_{a,b}(G)\big) = 0$$

for all $u, v \in R^p$ and $b > a > 0$. Since $U = \bigcup U_{a,b}(F) \cap U_{a,b}(G)$, where the union is taken over all pairs $a < b$ of positive rational numbers, we finally get the equation

$$Q_M(u, v, U) = 0 \qquad (u, v \in R^p).$$

Consequently, by (2.3), the function $f(t) = \psi_{M(U)}(t)$ is a non-negative continuous solution of the functional equation

$$2f(u)+2f(v)-f(u+v)-f(u-v) = 0 \qquad (u, v \in R^p).$$

It is well-known (see [5]) that each non-negative continuous solution of this equation is of the form $f(t) = (Dt, t)$, where D is a non-negative symmetric operator. Hence it follows that the random variable is Gaussian which completes the proof.

3. Complex-valued isotropic random measures. In this section we identify the complex plane and the space R^2. The integral of a complex-valued function f with respect to a complex-valued random measure M is defined by means of the formula

$$\int_I f(s)M(ds) = \int_I \hat{f}(s)M(ds),$$

where $\hat{z}$ is a matrix representation of the complex number z given by the formula

$$\hat{z} = \begin{pmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{pmatrix}.$$

A complex-valued random measure M is said to be *isotropic* if for every orthogonal operator V in R^2 and every Borel subset E of I the random variables $M(E)$ and $VM(E)$ have the same probability distribution. For an atomless isotropic random measure M the characteristic function of the random variable $M(E)$, where E is a Borel subset of I, can be written in the form

$$(3.1)\qquad \varphi_{M(E)}(t) = \exp\left\{\int_0^\infty \big(J_0(x|t|)-1\big)\frac{1+x^2}{x^2}\,\mu_M(E, dx)\right\},$$

where J_0 is the Bessel function defined by

$$(3.2)\qquad J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin u)\,du$$

and $\mu_M(E, \cdot)$ is a finite non-negative Borel measure on R_+ (see [5]). Moreover, for every Borel subset A of R_+ the set-function $\mu_M(\cdot, A)$ is a non-negative Borel measure on I. Put $\nu_M(E) = \mu_M(E, R_+)$. It is obvious that $\mu_M(\cdot, A) \leqslant \nu_M$ and, consequently, all measures $\mu_M(\cdot, A)$ are absolutely continuous with respect to the measure ν_M. Put $G_M(E, x) = \mu_M\big(E, [0, x)\big)$, $x \in R_+$. By the Radon-Nikodym theorem

$$(3.3)\qquad G_M(E, x) = \int_E g_M(s, x)\,\nu_M(ds),$$

where $0 \leqslant g_M(s, x) \leqslant 1$ and the function $g_M(\cdot, x)$ is Borel measurable. Moreover, we may assume that the function $g_M(s, \cdot)$ is monotone non-decreasing and continuous to the left on R_+. In fact, we can always find a version $\tilde{g}_M(\cdot, w)$ of the Radon-Nikodym densities of $G_M(\cdot, w)$ for rational numbers w, such that $\tilde{g}_M(\cdot, w)$ is Borel measurable and monotone non-decreasing as the function of w. Setting

$$g_M(s, x) = \lim_{w\to x-}\tilde{g}_M(s, w)$$

we obtain a Radon-Nikodym density with required properties.

Put

$$\Phi_M(t, x) = \int_{1/x}^{\infty} \frac{g_M(t, u)}{u^3}\, du \qquad (t \in I,\ x \in R_+). \tag{3.4}$$

The function Φ_M and the measure ν_M corresponding to the random measure M satisfy conditions (i)-(v) from section 1. The Δ_2-condition for Φ_M is a consequence of the inequality

$$\Phi_M(t, 2x) = \int_{1/2x}^{\infty} \frac{g_M(t, u)}{u^3}\, du = 4\int_{1/x}^{\infty} \frac{g_M(t, \frac{1}{2}u)}{u^3}\, du$$

$$\leqslant 4\int_{1/x}^{\infty} \frac{g_M(t, u)}{u^3}\, du = 4\Phi_M(t, x).$$

Let $\mathscr{L}(M)$ be the set of all complex-valued M-integrable functions, where M is a complex-valued atomless isotropic random measure. We identify functions which are equal M-almost everywhere. The space $\mathscr{L}(M)$ is a complete linear metric space under usual addition and scalar multiplication with a non-homogeneous norm defined by the formula

$$\|f\|_M = \Big|\Big|\Big| \int_I f(s) M(ds) \Big|\Big|\Big|,$$

where $|||X|||$ denotes the Fréchet norm of the random variable X, i.e. the expectation $\mathrm{E}[(1+|X|)/|X|]$ (see [22] and [24]). It should be noted that the convergence of a sequence of functions in $\mathscr{L}(M)$ is equivalent to the convergence in probability of the sequence of their integrals with respect to the random measure M. Moreover, the set of all Borel simple functions on I is dense in $\mathscr{L}(M)$.

By (3.1), $\varphi_{M(E)}(t)$ depends only upon the modulus of t. Consequently, we can introduce the notation

$$\vartheta_{M(E)}(|t|) = -\log \varphi_{M(E)}(t). \tag{3.5}$$

LEMMA 3.1. *Let M be a complex-valued atomless isotropic random measure. There exists then a positive constant c_1 such that the inequality*

$$\int_0^1 \vartheta_{M(E)}(ar)\, dr \geqslant c_1 \int_E \Phi_M(s, a)\nu_M(ds) \tag{3.6}$$

holds for all non-negative numbers a and all Borel subsets E of I. Moreover, for every positive number ε there exist a positive constant c_2 and a Borel subset A of R_+ such that $\mu_M(I, A) < \varepsilon$ and the inequality

$$\vartheta_{M(E)}(a) \leqslant c_2 \int_E \Phi_M(s, a)\nu_M(ds) + 2\mu_M(E, A) \tag{3.7}$$

holds for all non-negative numbers a and all Borel subsets E of I.

Proof. Integrating by parts, from (3.3) and (3.4) we get the formula

$$\int_E \Phi_M(s, a)\nu_M(ds) = \frac{1}{2}\int_0^{\infty} \min(a^2, x^{-2})\mu_M(E, dx). \tag{3.8}$$

Taking into account the well-known inequality

$$1 - \frac{\sin y}{y} \geqslant b_1 \min(1, y^2),$$

where b_1 is a positive constant, we get, by (3.2), the inequality

$$\int_0^1 \big(1 - J_0(axr)\big)\, dr \geqslant b_2 \min(a^2x^2, 1),$$

where b_2 is a positive constant. Hence and from (3.1), (3.5) and (3.8) we obtain the inequality

$$\int_0^1 \vartheta_{M(E)}(ar)\, dr \geqslant b_2 \int_0^{\infty} \min(a^2x^2, 1)\frac{1+x^2}{x^2}\mu_M(E, dx) \geqslant 2b_2 \int_E \Phi_M(s, a)\nu_M(ds).$$

Formula (3.6) is thus proved.

Given $\varepsilon > 0$, we take a number $q \geqslant 1$ such that $\mu_M(I, A) < \varepsilon$, where $A = (q, \infty)$. It is clear that

$$\int_q^{\infty} \big(1 - J_0(ax)\big)\frac{1+x^2}{x^2}\mu_M(E, dx) \leqslant 2\mu_M(E, A) \tag{3.9}$$

for all non-negative numbers a and all Borel sets E. Further, taking into account the inequality $1 - \cos y \leqslant b_3 \min(1, y^2)$, where b_3 is a positive constant, we get, by (3.2), the inequality

$$1 - J_0(ax) \leqslant b_3 \min(a^2x^2, 1).$$

Consequently, by (3.8), for every non-negative number a and every Borel set E we have the inequality

$$\int_q^{\infty} \big(1 - J_0(ax)\big)\frac{1+x^2}{x^2}\mu_M(E, dx) \leqslant b_3 \int_0^q \min(a^2, x^{-2})(1+x^2)\mu_M(E, dx)$$

$$\leqslant c_2 \int_E \Phi_M(s, a)\nu_M(ds),$$

where $c_2 = 2b_3(1+q^2)$. Hence and from (3.1), (3.5) and (3.9) we get inequality (3.7) which completes the proof of the Lemma.

LEMMA 3.2. *Let M be a complex-valued atomless isotropic random measure. A sequence $\{f_n\}$ of Borel simple functions on I converges to 0 in $\mathscr{L}(M)$ if and only if it converges to 0 in the Musielak-Orlicz space $L_{\Phi_M}(\nu_M)$.*

Proof. Given a Borel simple function f on I, say

$$f = \sum_{j=1}^{k} a_j \chi_{E_j},$$

where the sets $E_1, E_2, \ldots, E_k$ are disjoint, we put

$$X_f = \int_I f(s) M(ds) = \sum_{j=1}^{k} a_j M(E_j).$$

Since the characteristic function $\varphi_{X_f}(t)$ depends only upon the modulus of t, we can introduce the notation $H_f(|t|) = -\log \varphi_{X_f}(t)$. Further, by (3.5), we have

$$H_f(r) = \sum_{j=1}^{k} \vartheta_{M(E_j)}(|a_j| r). \tag{3.10}$$

By ϱ_M we shall denote the modular induced by Φ_M and ν_M (see definition (1.1)). From (3.10) and Lemma 3.1, for every Borel simple function f we get the inequality

$$\int_0^1 H_f(r)\, dr \geqslant c_1 \sum_{j=1}^{k} \int_{E_j} \Phi_M(s, |a_j|) \nu_M(ds) = c_1 \varrho_M(f), \tag{3.11}$$

where c_1 is a positive constant. Moreover, for every positive number ε there exists a positive constant c_2 such that

$$H_f(r) \leqslant c_2 \varrho_M(f) + 2\varepsilon. \tag{3.12}$$

Suppose that a sequence $\{f_n\}$ of Borel simple functions converges to 0 in $\mathscr{L}(M)$. Then the sequence of random variables X_{f_n} converges to 0 in probability. Consequently,

$$\lim_{n\to\infty} H_{f_n}(r) = 0$$

uniformly in every compact interval. Hence, by (3.11), we get the formula

$$\lim_{n\to\infty} \varrho_M(f_n) = 0,$$

which shows that the sequence $\{f_n\}$ converges to 0 in $L_{\Phi_M}(\nu_M)$.

Suppose now that the sequence $\{f_n\}$ of simple functions converges to 0 in $L_{\Phi_M}(\nu_M)$, i.e.

$$\lim_{n\to\infty} \varrho_M(f_n) = 0.$$

Since the functions H_{f_n} are non-negative, we have, by (3.12), the formula

$$\lim_{n\to\infty} H_{f_n}(r) = 0$$

for every r. Consequently, the sequence $\{X_{f_n}\}$ tends to 0 in probability which implies the convergence of $\{f_n\}$ to 0 in $\mathscr{L}(M)$. The Lemma is thus proved.

In this paper two linear metric spaces $(Y, \|\ \|_1)$ and $(Y, \|\ \|_2)$ will be treated as identical if the convergence in the norm $\|\ \|_1$ is equivalent to the convergence in the norm $\|\ \|_2$. Since the set of Borel simple functions on I is dense in both spaces $\mathscr{L}(M)$ and $L_{\Phi_M}(\nu_M)$, we have, by Lemma 3.2, the following characterization of the space $\mathscr{L}(M)$ of all M-integrable functions:

THEOREM 3.1. *For each complex-valued atomless isotropic random measure M the space $\mathscr{L}(M)$ is identical with the Musielak-Orlicz space $L_{\Phi_M}(\nu_M)$.*

4. Harmonizable sequences. All random measures considered in this section are assumed to be atomless and complex-valued. The sequence

$$X_n(M) = \int_0^1 e^{2\pi n i s} M(ds) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

of the Fourier coefficients of a random measure M will be called a *harmonizable sequence* of random variables. We say that two sequences $\{X_n\}$ and $\{Y_n\}$ of random variables are *identically distributed* if for every system $n_1, n_2, \ldots, n_k$ of integers the multivariate distributions of $X_{n_1}, X_{n_2}, \ldots, X_{n_k}$ and $Y_{n_1}, Y_{n_2}, \ldots, Y_{n_k}$ are identical. Further, we say that two random measures M_1 and M_2 are *identically distributed* if for every Borel set E the random variables $M_1(E)$ and $M_2(E)$ are identically distributed. A sequence $\{X_n\}$ of random variables is called *strictly stationary*, or, shortly, *stationary*, if for every system $m, n_1, n_2, \ldots, n_k$ of integers the multivariate distribution of the random variables $X_{n_1+m}, X_{n_2+m}, \ldots, X_{n_k+m}$ is independent of m.

Let J be an arbitrary subinterval of I. Denoting by $\sigma_n(\cdot, J)$ the sequence of Fejér means of the Fourier series of the indicator $\chi_J(\cdot)$, we inferthat the functions $\sigma_n(\cdot, J)$ are bounded in common and

$$\lim_{n\to\infty} \sigma_n(s, J) = \chi_J(s)$$

except of the endpoints of J (see [25], p. 45). Consequently, by dominated convergence theorem for random integrals ([16], Theorem 2.9), we have

$$\lim_{n\to\infty} \int_I \sigma_n(s, J) M(ds) = M(J) \text{ in probability}.$$

Since the random variables $\int_I \sigma_n(s, J) M(ds)$ are linear combinations of the Fourier coefficients $X_k(M)$ $(|k| \leqslant n)$, we get, by the last formula, the following two Lemmas:

LEMMA 4.1. *The Fourier coefficients $\{X_n(M)\}$ determine the random measure M uniquely.*

LEMMA 4.2. *The random measures M_1 and M_2 are identically distributed if and only if the sequences $\{X_n(M_1)\}$ and $\{X_n(M_2)\}$ of their Fourier coefficients are identically distributed.*

Now we shall give a characterization of stationary harmonizable sequences of random variables.

THEOREM 4.1. *A sequence $\{X_n(M)\}$ is stationary if and only if the random measure M is isotropic.*

Proof. Suppose that the sequence $\{X_n(M)\}$ is stationary. Then, by Lemma 4.2, for every integer k the random measures

$$M_k(E) = \int_E e^{2\pi kis} M(ds)$$

and M are identically distributed. Further, the characteristic function $\varphi_{M(E)}(t)$ $(t \in R^2)$ can be written in the Lévy-Khinchine form

$$\varphi_{M(E)}(t) = \exp\Big\{\big(a(E), t\big) - \big(D(E)t, t\big) + \int_{R^2\setminus\{0\}} \Big(e^{i(t,x)} - 1 - \frac{i(t,x)}{1+|x|^2}\Big)\frac{1+|x|^2}{|x|^2}\mu(E, dx)\Big\}, \tag{4.1}$$

where $a(E)$ is an element of R^2, $D(E)$ is a symmetric non-negative operator on R^2 and $\mu(E,\cdot)$ is a finite non-negative Borel measure on $R^2\setminus\{0\}$. More over, $a(\cdot)$ is a vector-valued Borel measure on I and for every Borel subset A of $R^2\setminus\{0\}$ the set-function $\mu(\cdot, A)$ is a non-negative Borel measure on I. Let us denote by $\mu_1(\cdot)$ and $\mu_2(\cdot)$ the scalar variations of the measures $a(\cdot)$ and $D(\cdot)$ respectively. Put

$$\lambda(E) = \mu(E, R^2\setminus\{0\}) + \mu_1(E) + \mu_2(E).$$

Since the random measure M is atomless, we infer that the measure λ is also atomless. Moreover, all measures $a(\cdot)$, $D(\cdot)$ and $\mu(\cdot, A)$ are absolutely continuous with respect to the measure λ. Consequently, by the Radon-Nikodym theorem

$$a(E) = \int_E b(s)\lambda(ds), \quad D(E) = \int_E C(s)\lambda(ds), \tag{4.2}$$

$$\mu(E, A) = \int_E h(s, A)\lambda(ds),$$

where b is a vector-valued Borel function, C is an operator-valued Borel function, $0 \leqslant h(s, A) \leqslant 1$ and the function $h(\cdot, A)$ is Borel measurable. Moreover, we may assume that the set-function $h(s,\cdot)$ is a Borel measure on $R^2\setminus\{0\}$. In fact, we can always find a version $g(\cdot, w_1, w_2)$ of the Radon-Nikodym densities of $\mu\big(\cdot, A(w_1, w_2)\big)$, where w_1 and w_2 are rational numbers and

$$A(w_1, w_2) = \{(t_1, t_2): t_1 < w_1, t_2 < w_2, (t_1, t_2) \neq (0, 0)\},$$

such that $g(\cdot, w_1, w_2)$ is Borel measurable and monotone non-decreasing with respect to each variable w_1 and w_2. Furthermore, we may assume that

$$g(\cdot, w_1, w_2) - g(\cdot, v_1, w_2) - g(\cdot, w_1, v_2) + g(\cdot, v_1, v_2) \geqslant 0$$

whenever $v_1 \leqslant w_1$ and $v_2 \leqslant w_2$. Setting

$$h\big(s, A(u_1, u_2)\big) = \lim_{\substack{w_1\to u_1-\\ w_2\to u_2-}} g(s, w_1, w_2)$$

for every pair u_1, u_2 of real numbers we get a distribution function which uniquely determines the measure $h(s,\cdot)$.

Put

$$K(s, t) = \big(b(s), t\big) - \big(C(s)t, t\big) + \int_{R^2\setminus\{0\}} \Big(e^{i(t,x)} - 1 - \frac{i(t,x)}{1+|x|^2}\Big)\frac{1+|x|^2}{|x|^2} h(s, dx).$$

It is clear that the function $K(\cdot, t)$ is Borel measurable and the function $K(s,\cdot)$ is continuous on R^2. Moreover, by (4.1) and (4.2),

$$\varphi_{M(E)}(t) = \exp\int_E K(s, t)\lambda(ds). \tag{4.3}$$

Hence it follows that for any Borel simple function f the characteristic function of the integral $\int_E f(s)M(ds)$ is given by the expression

$$\exp\int_E K\big(s, \overline{f(s)}t\big)\lambda(ds),$$

where $\bar{z}$ denotes the complex conjugate of z. Taking a bounded sequence of Borel simple functions convergent to $e^{2\pi kis}$ everywhere on I, we infer, by the dominated convergence theorem for random integrals ([16], Theorem 2.9) that the corresponding sequence of integrals over the set E converges in probability to $M_k(E)$. Consequently, by the continuity of $K(s,\cdot)$, we have

$$\varphi_{M_k(E)}(t) = \exp\int_E K(s, e^{-2\pi kis}t)\lambda(ds).$$

Hence and from (4.3) it follows that for every integer k, $t \in R^2$ and for λ-almost all s the equation

$$K(s, e^{-2\pi kis}t) = K(s, t) \tag{4.4}$$

holds. Given $t = |t|e^{2\pi i\tau}$, we denote by B the subset of I consisting of all irrational numbers s for which equation (4.4) holds for all integers k. Since the measure is atomless, we have the formula $\lambda(I\setminus B) = 0$. Since for every $s \in B$ the sequence of multiples $s, 2s, 3s, \ldots$ (mod 1) is dense in I (see for instance [23]), we can take a sequence $\{k_n\}$ of integers for which

$$\lim_{n\to\infty} e^{2\pi k_n is} = e^{2\pi i\tau}.$$

Consequently, by (4.4) and the continuity of $K(s,\cdot)$, we have the formula $K(s,t) = K(s,|t|)$, $s \in B$. Thus, by (4.3), the characteristic function $\varphi_{M(E)}(t)$ depends only upon the modulus of t which implies that the random measure M is isotropic.

Since for isotropic random measures M the characteristic function of the integral $\int_I f(s)M(ds)$ depends only upon the absolute value of f, the converse implication is obvious. The theorem is thus proved.

Given a stationary sequence $\{X_n\}$, by $[X_n]$ and $[X_n: n\leqslant k]$ we shall denote the linear spaces closed with respect to the convergence in probability spanned by all random variables X_n and by random variables X_n with $n \leqslant k$ respectively. To each stationary sequence $\{X_n\}$ there corresponds a shift transformation $TX_n = X_{n+1}$ $(n = 0, \pm 1, \pm 2, \ldots)$, which can be extended to an invertible linear transformation T on $[X_n]$. Of course, the transformation T preserves the probability distribution.

The concept of stationary sequences admitting a prediction was introduced and discussed in [19]. We say that a stationary sequence $\{X_n\}$ *admits a prediction*, if there exists a continuous linear operator A_0 from $[X_n]$ onto $[X_n: n\leqslant 0]$ such that

(i) $A_0X = X$ whenever $X \in [X_n: n\leqslant 0]$;

(ii) if for every $Y \in [X_n: n \leqslant 0]$ the random variables X and Y are independent, then $A_0X = 0$;

(iii) for every $X \in [X_n]$ and $Y\in [X_n: n\leqslant 0]$ the random variables $X - A_0X$ and Y are independent.

The random variable A_0X can be regarded as a linear prediction of X based on the full past of the sequence $\{X_n\}$ up to the time $n = 0$. An optimality criterion is given by (iii). In what follows the operator A_0 will be called a *predictor* based on the past of the sequence $\{X_n\}$ up to the time $n = 0$.

It should be noted that Gaussian stationary sequences with zero mean always admit a prediction. This follows from the fact that in this case the concepts of independence and orthogonality are equivalent and, moreover, the square-mean convergence and the convergence in probability are equivalent. Therefore the predictor A_0 is simply the best linear least squares predictor, i.e. the orthogonal projector from $[X_n]$ onto $[X_n: n\leqslant 0]$ (see [3], Chapter XII, §1).

The predictor A_0 and the shift T induced by $\{X_n\}$ determine the predictor A_k based on the full past of $\{X_n\}$ up to the time $n = k$ by means of the formula $A_k = T^kAT^{-k}$.

A stationary sequence $\{X_n\}$ admitting a prediction is called *deterministic*, if $A_0X = X$ for every $X\in[X_n]$. Further, a stationary sequence $\{X_n\}$ admitting a prediction is called *completely non-deterministic*, if $\lim_{k\to-\infty} A_kX = 0$ for every $X\in[X_n]$.

Consider a stationary harmonizable sequence $\{X_n(M)\}$. By Theorem 4.1, the random measure M is isotropic. Further, by Theorem 3.1, $L(M)$ is a Musielak-Orlicz space. It is clear that the mapping $X_n(M)\to e^{2\pi nis}$ $(n = 0, \pm 1, \pm 2, \ldots)$ can be extended in a natural way to an isomorphism of $[X_n(M)]$ and $\mathscr{L}(M)$. Moreover,

$$[X_n(M)] = \Big\{\int_I f(s)M(ds): f\in\mathscr{L}(M)\Big\}$$

and

$$T\int_I f(s)M(ds) = \int_I e^{2\pi is}f(s)M(ds).$$

THEOREM 4.2. *Let $\{X_n(M)\}$ be a stationary harmonizable sequence admitting a prediction. There exists then a Borel subset Q of I such that $\{X_n(M_1)\}$ and $\{X_n(M_2)\}$, where $M_1(E) = M(E\cap Q)$ and $M_2(E) = M\big(E\cap(I\setminus Q)\big)$, are stationary sequences admitting a prediction. Moreover, the sequence $\{X_n(M_1)\}$ is completely non-deterministic and the sequence $\{X_n(M_2)\}$ is deterministic. Consequently, each stationary harmonizable sequence admitting a prediction is the sum of two independent stationary harmonizable sequences admitting a prediction, one completely non-deterministic and the other deterministic.*

Proof. By Theorem 1 in [19], $X_n(M) = X_n' + X_n''$, where the sequences $\{X_n'\}$ and $\{X_n''\}$ are independent, stationary and admit a prediction. Moreover, the sequence $\{X_n'\}$ is completely non-deterministic, the sequence $\{X_n''\}$ is deterministic and the space $[X_n(M)]$ is a direct sum of the subspaces $[X_n']$ and $[X_n'']$. Further, $TX_n' = X_{n+1}'$ and $TX_n'' = X_{n+1}''$, where T is the shift transformation induced by $\{X_n(M)\}$ in $[X_n(M)]$. Put $X_0' = \int_I h(s)M(ds)$. Consequently,

$$T^nX_0' = X_n' = \int_I e^{2\pi nis}h(s)M(ds) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Let us introduce the notation $Q = \{s: h(s)\neq 0\}$. It is evident that

$$[X_n'] \subset \Big\{\int_Q f(s)M(ds): f\in\mathscr{L}(M)\Big\}. \tag{4.5}$$

Since for every trigonometric polynomial w the relation

$$\int_Q w(s)h(s)(s)M(ds)\in[X_n']$$

holds and, by dominated convergence theorem for random integrals (see [16], Theorem 2,9), for every $f\in\mathscr{L}(M)$ there exists a sequence $\{w_n\}$ of trigonometric polynomials such that

$$\lim_{n\to\infty}\int_Q w_n(s)h(s)M(ds) = \int_Q f(s)M(ds)$$

in probability, we have, by (4.5), the formula

$$[X_n'] = \Big\{\int_Q f(s)M(ds): f\in\mathscr{L}(M)\Big\}. \tag{4.6}$$

Since $X_0'' = X_0(M)-X_0'$, we have

$$X_0'' = M(I\setminus Q)+\int_Q\big(1-h(s)\big)M(ds). \tag{4.7}$$

Obviously, the random variables $M(I\setminus Q)$ and $\int_Q\big(1-h(s)\big)M(ds)$ are independent. Since the sequences $\{X_n'\}$ and $\{X_n''\}$ are independent, we infer, by (4.6) and (4.7), that the random variables

$$M(I\setminus Q)+\int_Q\big(1-h(s)\big)M(ds) \quad\text{and}\quad \int_Q\big(1-h(s)\big)M(ds)$$

are independent. Hence it follows that $\int_Q\big(1-h(s)\big)M(ds)$ is a constant random variable. Finally, taking into account that the measure M is isotropic, we infer that

$$\int_Q\big(1-h(s)\big)M(ds) = 0.$$

Consequently, by (4.7), $X_0'' = M(I\setminus Q)$ and $X_0' = M(Q)$. Setting $M_1(E) = M(E\cap Q)$ and $M_2(E) = M\big(E\cap(I\setminus Q)\big)$, we have

$$X_n' = T^nX_0' = \int_Q e^{2\pi nis}M(ds) = \int_I e^{2\pi nis}M_1(ds),$$

$$X_n'' = T^nX_0'' = \int_{I\setminus Q} e^{2\pi nis}M(ds) = \int_I e^{2\pi nis}M_2(ds).$$

The theorem is thus proved.

We proceed now to a description of stationary harmonizable deterministic sequences $\{X_n(M)\}$ in terms of probabilistic characteristics of the random measure M. We remind that to every isotropic random measure M there corresponds a Borel measure ν_M on I and a function Φ_M on $I\times R_+$ (see (3.4)). Moreover, by Theorem 3.1, the space $\mathscr{L}(M)$ is identical with the Musielak-Orlicz space $L_{\Phi_M}(\nu_M)$. Further, the measure ν_M and the function Φ_M determine, by formula (1.4), a sequence of functions $\Omega_{\Phi_M,\nu_M,n}$ on I. It is evident that the sequence $\{X_n(M)\}$ is deterministic if and only if $X_0(M)\in[X_n(M): n\leqslant -1]$. Since

$$[X_n(M)] = \Big\{\int_I f(s)M(ds): f\in\mathscr{L}(M)\Big\},$$

we infer, by virtue of Theorem 3.1, that $\{X_n(M)\}$ is deterministic if and only if

$$\inf\Big\|1+\sum_{k=1}^{n}a_ke^{-2\pi kit}\Big\| = 0,$$

where the infimum is taken over all complex numbers $a_1, a_2, \ldots, a_n$ and $n = 1, 2, \ldots$, $\|\ \|$ being the norm in $L_{\Phi_M}(\nu_M)$. Since $\|f\| = \|\bar{f}\|$, we have, by Theorem 1.1, the following characterization of deterministic stationary harmonizable sequences:

THEOREM 4.3. *A stationary sequence $\{X_n(M)\}$ is deterministic if and only if no function $\log\Omega_{\Phi_M,\nu_M,n}$ $(n = 1, 2, \ldots)$ is Lebesgue integrable over I.*

We say that M is a *Poisson random measure* if there exist a probability distribution P on R^2 and a non-negative Borel measure λ on I such that for every Borel subset E of I the probability distribution of $M(E)$ is given by the expression

$$e^{-\lambda(E)}\sum_{n=0}^{\infty}\frac{\lambda^n(E)P^{*n}}{n!},$$

where the power of P is taken in the sense of convolution and P^{*0} denotes the probability measure concentrated at the origin. It is clear that the Poisson measure M is isotropic if and only if the probability measure P is isotropic. In this case the measure $\mu_M(E,\cdot)$ appearing in (3.1) is given by the formula

$$\mu_M(E, A) = \lambda(E)\int_A\frac{t^2}{1+t^2}P_0(dt),$$

where $P_0(A) = P(\{x: |x|\in A\})$. Hence, by simple computations, it follows that the function Φ_M is bounded. Consequently, by definitions (1.3) and (1.4), the functions $\Omega_{\Phi_M,\nu_M,n}$ are infinite almost everywhere. Thus from Theorem 4.3 we get the following

COROLLARY. *For every isotropic Poisson random measure M the sequence $\{X_n(M)\}$ is deterministic.*

Let M be a Gaussian isotropic random measure. We have already mentioned that for Gaussian stationary sequences the concepts of prediction presented in this paper and the best linear least squares prediction coincide. It is easy to verify that for Gaussian isotropic measures M the formula $\Phi_M(t, x) = \frac{1}{2}x^2$ is true. Consequently, $\mathscr{L}(M) = L^2(\nu_M)$. The classical characterization of completely non-deterministic wide sense stationary sequences (see [3], Chapter XII, § 4) implies the following lemma:

LEMMA 4.3. *Let M be a Gaussian isotropic random measure. The sequence $\{X_n(M)\}$ is completely non-deterministic if and only if either $M \equiv 0$ or the measure ν_M is absolutely continuous with respect to the Lebesgue measure and $\log(d\nu_M/dt)$ is Lebesgue integrable over I.*

Now we shall give a description of stationary harmonizable completely non-deterministic sequences.

THEOREM 4.4. *A stationary sequence $\{X_n(M)\}$ is completely non-deterministic if and only if either $M \equiv 0$ or M is a Gaussian random measure, ν_M is absolutely continuous with respect to the Lebesgue measure and $\log(d\nu_M/dt)$ is Lebesgue integrable over I.*

Proof. By Lemma 4.3, to prove the Theorem it suffices to prove that M is a Gaussian random measure provided $\{X_n(M)\}$ is stationary completely non-deterministic.

Let A_k be the predictor based on the full past of $\{X_n(M)\}$ up to time k. Setting

$$A_k X_0(M) = \int_I f_k(s) M(ds),$$

where $f_k \in \mathscr{L}(M)$, and

$$E_k = \{s : f_k(s) \neq 1\},$$

we have

$$A_k X_0(M) = \int_{E_k} f_k(s) M(ds) + M(I \setminus E_k).$$

Of course, the random variables $M(I \setminus E_k)$ and $\int_{E_k} f_k(s) M(ds)$ are independent and symmetrically distributed. Consequently, the relation

$$\lim_{k \to -\infty} A_k X_0(M) = 0$$

implies the relation

$$\lim_{k \to -\infty} M(I \setminus E_k) = 0. \tag{4.8}$$

By the definition of predictors, the random variables $X_0(M) - A_k X_0(M)$ and $X_k(M)$ are independent. In other words, the integrals

$$\int_I \big(1 - f_k(s)\big) M(ds) \quad \text{and} \quad \int_I e^{2\pi k i s} M(ds)$$

are independent. Since both integrands are different from 0 on E_k, we infer, by Theorem 2.1, that the random variable $M(E_k)$ is Gaussian.

Hence and from (4.8) it follows that $M(I)$, being the limit in probability of Gaussian random variables $M(E_k)$, is Gaussian too. By Cramér's Theorem ([10], p. 271), M is a Gaussian random measure which completes the proof.

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Метод эквивалентных норм в теории абстрактных почти периодических функций

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Функция $x(t)$, $-\infty < t < \infty$, со значениями в банаховом пространстве E называется *почти периодической* (п. п. функцией), если она сильно непрерывна и если для каждого $\varepsilon > 0$ можно указать такое $l = l(\varepsilon)$, что в любом интервале длины l найдется хотя бы один ε-почти период (ε-п. период) функции $x(t)$, то-есть число τ такое, что

$$\sup_t \|x(t+\tau)-x(t)\| < \varepsilon \quad (-\infty < t < \infty).$$

Для числовых п. п. функций справедлива следующая теорема об интегрировании (см. [6], стр. 29):

Теорема Боля-Бора. *Если интеграл*

$$X(t) = \int_0^t x(\eta)\,d\eta \quad (-\infty < t < \infty) \tag{1}$$

п. п. функции $x(t)$ ограничен, то он также есть п. п. функция. Более точно: для каждого $\varepsilon > 0$ существует такое $\varepsilon_1 = \varepsilon_1(x, \varepsilon)$, что каждый ε_1-п. период функции $x(t)$ есть ε-п. период функции $X(t)$.

Л. Америо [1], [2] показал, что теорема Боля-Бора распространяется на абстрактные п. п. функции, если в качестве E взять равномерно выпуклое банахово пространство. Кроме того, он привел пример п. п. функции со значениями в пространстве c всех сходящихся числовых последовательностей

$$x(t) = \{\lambda_n \cos \lambda_n t\}_{n=1}^{\infty} \quad (\lambda_n \downarrow 0),$$

интеграл от которой

$$X(t) = \{\sin \lambda_n t\}_{n=1}^{\infty}$$

есть ограниченная, но не почти периодическая, функция.

Естественно, возникает задача выделения тех пространств Банаха, которые, подобно равномерно выпуклым пространствам, допускают обобщение теоремы Боля-Бора.