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On the space of convolution operators in $\mathscr{K}_1'$

by

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Convolution operators in the space $\mathscr{K}_1'$ $(=\Lambda_\infty)$ of distributions of exponential growth in R^n were characterized by Hasumi [5]. The space $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ of these operators can be identified with a subspace of Schwartz's space $\mathscr{O}_c'$ of rapidly decreasing distributions. A distribution T is in $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ if and only if, for every $a\in R^n$, the product $\exp(a\cdot)T$ is bounded on R^n, i.e. belongs to the space $\mathscr{B}'$ ([7], vol. II, p. 56).

In this paper we prove that $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$, endowed with the topology induced in $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ by the space $\mathscr{L}_b(\mathscr{K}_1',\mathscr{K}_1')$ of all continuous linear mappings from $\mathscr{K}_1'$ into $\mathscr{K}_1'$ with the topology of uniform convergence on all bounded sets, is nuclear, complete and bornologic, and therefore a Montel space. We also characterize the space $\mathscr{O}_c(\mathscr{K}_1':\mathscr{K}_1')$ of C^∞-functions, which is the dual of $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$.

We denote by N, R and C the sets of natural, real and complex numbers respectively. The sets of the corresponding n-tuples are denoted by N^n, R^n and C^n; for the sum, the scalar product, etc. of points in each of these spaces we use the standard notation (see e.g. [7] or [5]).

A horizontal strip in C^n around R^n of width $b>0$ is defined as

$$V_b=\{\zeta=(\zeta_1,\ldots,\zeta_n)\in C^n\colon |\mathrm{Im}\,\zeta_j|\leqslant b,\, j=1,2,\ldots,n\}.$$

We use the function

$$\sigma_b(x)=\prod_{j=1}^{n}[\exp(bx_j)+\exp(-bx_j)],$$

where $x=(x_1,\ldots,x_n)\in R^n$ and $b\in R$. In particular, a function f on R^n is of exponential growth, if f/σ_b is bounded for some b.

The spaces $\mathscr{D}$, $\mathscr{D}'$, $\mathscr{S}'$, $\mathscr{B}'$ and $\mathscr{E}'$ are those introduced in [7]. In all spaces of distributions the scalar product of a distribution T and a test function φ (i.e. the value of T on φ) is denoted by $T\cdot\varphi$. For any $T\in\mathscr{D}'$ and $h\in R^n$, $\tau_h T$ is the translation of T by h and $\check{T}$ is obtained from T by symmetry with respect to the origin.

1. The spaces $\mathscr{K}_1$ and $\mathscr{K}_1'$. Convolution operators in $\mathscr{K}_1'$. For the sake of completeness we include a brief exposition of the main facts concerning the spaces $\mathscr{K}_1$ and $\mathscr{K}_1'$ and the convolution operators in $\mathscr{K}_1'$.

$\mathscr{K}_1$ is the space of all C^∞-functions φ on R^n such that $\exp(k|x|)D^p\varphi(x)$ is bounded in R^n for each $k \in N$ and $p \in N^n$ (D^p is the partial derivative). The topology in $\mathscr{K}_1$ is defined by the system of semi-norms

$$v_k(\varphi) = \sup_{x \in R^n,\ |p| \leqslant k} \exp(k|x|)\,|D^p\varphi(x)|, \qquad k = 1, 2, \ldots$$

Then $\mathscr{K}_1$ is a Fréchet nuclear space ([5], proposition 1).

The dual $\mathscr{K}_1'$ of $\mathscr{K}_1$ is the space of distributions containing all tempered distributions. $\mathscr{K}_1'$ is the space of "distributions of exponential growth", which are characterized by the following theorem:

THEOREM 1. (a) *A distribution $T \in \mathscr{D}'$ is in $\mathscr{K}_1'$ if and only if T can be represented in the form*

$$T = D^p[\exp(k|x|)F(x)],$$

where $p \in N^n$, $k \in R$ and F is a bounded, continuous function on R^n.

(b) *A distribution $T \in \mathscr{D}'$ is in $\mathscr{K}_1'$ if and only if each regularization $T*\alpha$, $\alpha \in \mathscr{D}$, is a continuous function of exponential growth; in that case there is a $k \in N$ such that*

$$(T*\alpha)(x) = O\big(\exp(k|x|)\big)$$

as $|x| \to \infty$, for all $\alpha \in \mathscr{D}$.

(c) *In order that a distribution T be in $\mathscr{K}_1'$ it is necessary that there exists a $k \in N$ such that the product $\frac{1}{\sigma_k}T$ is bounded in R^n (i.e. $\frac{1}{\sigma_k}T \in \mathscr{B}'$), and it is sufficient that, for each function $\varphi \in \mathscr{K}_1$, the product φT is bounded in R^n.*

(d) *In order that a distribution T be in $\mathscr{K}_1'$ it is necessary that there exists a $k \in N$ such that the set of distributions $\tau_h T/\exp(k|h|)$, $h \in R^n$, is bounded in $\mathscr{D}'$, and it is sufficient that, for any function $g(h)$ decreasing more rapidly than all powers of $\exp(-|h|)$, the set of distributions $g(h)\tau_h T$, $h \in R^n$, is bounded in $\mathscr{D}'$.*

Part (a) of the theorem was proved by Hasumi ([5], proposition 3). The proof of the remaining parts is similar to the proof of the corresponding statements for tempered distributions (see [7], vol. II, p. 96-97) and we leave it out.

The topology of $\mathscr{K}_1'$ is the strong dual topology; it makes $\mathscr{K}_1'$ into a complete, locally convex space which has all the properties proved in [7] for the space $\mathscr{S}'$ of tempered distributions. In particular, $\mathscr{K}_1'$ is a Montel space.

For a function $\varphi \in \mathscr{K}_1$, its Fourier transform

$$\hat{\varphi}(\zeta) = \int_{R^n} \exp(-2\pi i\zeta\cdot x)\varphi(x)\,dx$$

is defined for all $\zeta \in C^n$. We denote by K_1 the space of Fourier transforms of functions from $\mathscr{K}_1$. K_1 consists of all entire functions rapidly decreasing in any horizontal strip. In other words, an entire function ψ is in K_1 if and only if, for every $k \in N$,

$$w_k(\psi) = \sup_{\zeta \in V_k}(1+|\zeta|)^k|\psi(\zeta)| < \infty.$$

Fourier inversion formula also holds. The topology of K_1 is defined by the system of semi-norms w_k, $k = 1, 2, \ldots$ Then the Fourier transform is a topological isomorphism of $\mathscr{K}_1$ onto K_1.

The dual K_1' of K_1 is the space of Fourier transforms of distributions from $\mathscr{K}_1'$. For a distribution $T \in \mathscr{K}_1'$ its Fourier transform $\hat{T}$ is defined by the Parseval-Plancherell formula

$$\hat{T}\cdot\hat{\varphi} = T\cdot\check{\varphi}.$$

K_1' is provided with the strong topology. Then the Fourier transform is a topological isomorphism of $\mathscr{K}_1'$ onto K_1'.

For $S \in \mathscr{K}_1'$ and $T \in \mathscr{E}'$, the convolution $S*T$ is well defined as a distribution in $\mathscr{K}_1'$ and $T \to S*T$ is a continuous linear mapping from $\mathscr{E}'$ into $\mathscr{K}_1'$. We call S a *convolution operator* in $\mathscr{K}_1'$, if the latter mapping is continuously extendable to a mapping from $\mathscr{K}_1'$ into $\mathscr{K}_1'$. We denote by $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ the linear space of all convolution operators in $\mathscr{K}_1'$. $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ is a space of distributions, which can be characterized as follows:

THEOREM 2. *A distribution S is in $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ if and only if it satisfies one of the equivalent conditions:*

(a) *For every $a \in R^n$, the product $\exp(a\cdot)S$ is a bounded distribution, i.e. $\exp(a\cdot)S$ belongs to $\mathscr{B}'$.*

(b) *For every $k \in N$, S can be represented as a finite sum of derivatives of continuous functions F_p,*

(1) $$S = \sum_{|p| \leqslant m} D^p F_p,$$

where

(2) $$|F_p(x)| \leqslant M_p \exp(-k|x|);$$

M_p are constants.

(c) *For every $k \in N$, the set of distributions $\exp(k|h|)\tau_h S$, $h \in R^n$, is bounded in $\mathscr{D}'$.*

(d) *For every $\alpha \in \mathcal{D}$, the regularization $S * \alpha$ is in $\mathcal{K}_1$.*

Proof. It was proved by Hasumi ([5], proposition 9) that a distribution S is in $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ if and only if it satisfies condition (b). Thus for the proof of the theorem it is sufficient to show that conditions (a)-(d) are equivalent. We proceed successively.

Assume first that S satisfies condition (a). Then, for any $k \in N$, the product $\sigma_k S$ is a bounded distribution and therefore

$$\sigma_k S = \sum_{|p| \leqslant m} D^p G_p,$$

where $m \in N$ and G_p are continuous functions bounded in R^n (see [7], vol. II, p. 57). Hence

$$S = \sum_{|p| \leqslant m} \frac{1}{\sigma_k} D^p G_p,$$

where to each term we can apply the formula

$$\frac{1}{\sigma_k} D^p G_p = \sum_{0 \leqslant q \leqslant p} (-1)^{|p-q|} \binom{p}{q} D^{p-q} \left[G_p D^q \left(\frac{1}{\sigma_k} \right) \right].$$

This leads to the desired representation (1) and (2), because the functions $G_p D^q \left(\frac{1}{\sigma_k} \right)$ are continuous in R^n and

$$G_p(x) D^q \left(\frac{1}{\sigma_k(x)} \right) = O\left(\exp(-k|x|) \right)$$

as $|x| \to \infty$.

Now, for every function F_p in (1) and every compact set K in R^n, the functions

$$\exp(k|h|) F_p(x+h), \quad h \in R^n,$$

are uniformly bounded in K. Consequently

$$\exp(k|h|) \tau_h S = \sum_{|p| \leqslant m} \exp(k|h|) D^p (\tau_h F_p), \quad h \in R^n,$$

is a bounded set of distributions, and so condition (c) follows from (b).

Further, condition (c) implies (d), since for every $\alpha \in \mathcal{D}$ the C^∞-functions

$$\left(\exp(k|h|) \tau_h S \right) * \alpha = \exp(k|h|) \tau_h (S * \alpha), \quad h \in R^n,$$

are uniformly bounded in R^n.

But each distribution satisfying condition (d) is bounded (and even rapidly decreasing), by a theorem of Schwartz ([7], vol. II, p. 100). On the other hand, we have

$$\left(\exp(a \cdot) S \right) * \alpha = \exp(a \cdot) \left[S * \left(\exp(-a \cdot) \alpha \right) \right],$$

and therefore condition (d) holds also if S is replaced by any product $\exp(a \cdot) S$, $a \in R^n$. Thus from (d) we conclude that all products $\exp(a \cdot) S$, $a \in R^n$, are bounded distributions, which completes the proof of theorem 2.

Remark. Condition (d) can be replaced by the stronger condition

(d′) For every $\varphi \in \mathcal{K}_1$, the convolution $S * \varphi$ is in $\mathcal{K}_1$ and the mapping $\varphi \to S * \varphi$ of $\mathcal{K}_1$ into $\mathcal{K}_1$ is continuous. This can be easily derived from the representations (1) and (2).

We denote by $\mathcal{O}_M(K'_1 : K'_1)$ the space of all C^∞-functions extendable over C^n as entire functions slowly increasing in any horizontal strip. This means that an entire function χ is in $\mathcal{O}_M(K'_1 : K'_1)$ if and only if for each $k \in N$ there exists an $l \in N$ such that

$$\sup_{\zeta \in V_k} \frac{|\chi(\zeta)|}{(1+|\zeta|)^l} < \infty.$$

$\mathcal{O}_M(K'_1 : K'_1)$ is the space of multiplication operators in K'_1. If $\psi \in K_1$ and $\chi \in \mathcal{O}_M(K'_1 : K'_1)$, then $\psi\chi \in K_1$ and the mapping $\psi \to \psi\chi$ of K_1 into K_1 is continuous. The product χF of $\chi \in \mathcal{O}_M(K'_1 : K'_1)$ and $F \in K'_1$ is defined by equation

$$(\chi F) \cdot \psi = F \cdot (\chi \psi), \quad \psi \in K_1.$$

THEOREM 3. *The Fourier transform $S \to \hat{S}$ maps $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ onto $\mathcal{O}_M(K'_1 : K'_1)$. Moreover, for $S \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ and $T \in \mathcal{K}'_1$ we have*

$$\widehat{S * T} = \hat{S} \hat{T}. \tag{3}$$

The proof of the first part of theorem 3 is contained in [5] (proposition 8 and proposition 9). The interchange formula (3) can be easily verified.

2. The topological space $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ and its dual. We define the topology $\mathcal{T}$ of $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ to be that induced in $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ by the space $\mathcal{L}_b(\mathcal{K}'_1, \mathcal{K}'_1)$ of all continuous linear mappings from $\mathcal{K}'_1$ into $\mathcal{K}'_1$ endowed with the topology of uniform convergence on all bounded sets.

By the remark following theorem 2, $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ can also be regarded as a subspace of the space $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_1)$ of all continuous linear mappings from $\mathcal{K}_1$ into $\mathcal{K}_1$. Denote by $\mathcal{T}'$ the topology induced in $\mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ by $\mathcal{L}_b(\mathcal{K}_1, \mathcal{K}_1)$. Then we have

THEOREM 4. *The topologies $\mathcal{T}$ and $\mathcal{T}'$ in $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ coincide.*

Proof. Let $M(B, U)$ be a set in the 0-neighborhood subbase for the topology $\mathcal{T}$, i.e.

$$M(B, U) = \{S \in \mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1'):\ S*T \in U \text{ for all } T \in B\},$$

where B is a bounded set and U a 0-neighborhood in $\mathcal{K}_1'$ (1). We prove that $M(B, U)$ contains a set $M'(B', U')$ from the 0-neighborhood base for the topology $\mathcal{T}'$.

The topology of $\mathcal{K}_1'$ is the strong dual topology and also $U_1 \subset U_2$ implies $M(B, U_1) \subset M(B, U_2)$. Therefore we may assume that

$$U = U(B', \varepsilon) = \{T \in \mathcal{K}_1' : |T\cdot\varphi| < \varepsilon \text{ for all } \varphi \in B'\},$$

where B' is a bounded set in $\mathcal{K}_1$ and $\varepsilon > 0$. We also may assume that

$$B = \check{B} = \{\check{T}: T \in B\}$$

and that the same symmetry condition holds for B'.

Since $\mathcal{K}_1$ is barreled, its topology coincides with the strong topology, and so

$$U' = U'(B, \varepsilon) = \{\varphi \in \mathcal{K}_1 : |T\cdot\varphi| < \varepsilon \text{ for all } T \in B\}$$

is a 0-neighborhood in $\mathcal{K}_1$.

Now, the set

$$M'(B', U') = \{S \in \mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1'):\ S*\varphi \in U' \text{ for all } \varphi \in B'\}$$

belongs to the 0-neighborhood base for the topology $\mathcal{T}'$. Moreover, if $S \in M'(B', U')$, then

$$|T\cdot(S*\varphi)| < \varepsilon$$

for all $T \in B$ and $\varphi \in B'$. Hence, by the symmetry of B and B',

$$|(T*S)\cdot\varphi| < \varepsilon$$

for all $T \in B$ and $\varphi \in B'$. Consequently, $T*S \in U$ for each $T \in B$, and so $S \in M(B, U)$. This proves that $M'(B', U') \subset M(B, U)$.

Similarly, one can show that each set in the 0-neighborhood base for $\mathcal{T}'$ contains a set from the 0-neighborhood base for $\mathcal{T}$. Thus $\mathcal{T}$ and $\mathcal{T}'$ coincide, q.e.d.

We denote by $\mathcal{L}_s(\mathcal{K}_1, \mathcal{K}_1)$ the space $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_1)$ under the topology of simple convergence.

THEOREM 5. *The topology $\mathcal{T}'$ in $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ coincides with the topology induced in $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ by the space $\mathcal{L}_s(\mathcal{K}_1, \mathcal{K}_1)$.*

(1) The 0-neighborhood base in $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ consists of all finite intersections of sets of this form.

Let $\mathcal{T}''$ be the topology in $\mathcal{O}_M(K_1':K_1')$ corresponding to $\mathcal{T}'$ uneer the Fourier transform. Then theorem 5 can be stated equivalently as follows:

THEOREM 5'. *The topology $\mathcal{T}''$ in $\mathcal{O}_M(K_1':K_1')$ coincides with the topology induced in $\mathcal{O}_M(K_1:K_1')$ by the space $\mathcal{L}_s(K_1, K_1)$.*

Proof. Let B be a bounded set and U a 0-neighborhood in K_1. Then the set

$$N(B, U) = \{\chi \in \mathcal{O}_M(K_1':K_1') : \chi\psi \in U \text{ for all } \psi \in B\}$$

is in the 0-neighborhood base for the topology $\mathcal{T}''$. We define a finite system of functions $\psi_1, \ldots, \psi_m \in K_1$ such that the set

$$N'(\psi_1, \ldots, \psi_m; U) = \{\chi \in \mathcal{O}_M(K_1':K_1') : \chi\psi_j \in U,\ j = 1, \ldots, m\}$$

is contained in $N(B, U)$.

Suppose that U consists of all functions $\psi \in K_1$ such that

$$(1+|\zeta|)^b |\psi(\zeta)| < \varepsilon$$

for some $b \in N$, $\varepsilon > 0$, and all $\zeta \in V_b$. The function

$$\gamma(\zeta) = \sup_{\psi \in B} |\psi(\zeta)|$$

is bounded and rapidly decreasing in V_b.

Fix now a $\nu \in N$ such that

$$4n(2k+1)^{n-1}\exp[-(k\nu-1)^2+nb^2+n/4] \leqslant \frac{1}{4^k} \tag{4}$$

for $k = 1, 2, \ldots$ It is sufficient to find a ν satisfying the inequality for $k = 1$; then one can prove by induction that it holds for all k.

For each integer $k \geqslant 0$ we denote by $c^{k,l} = (c_1^{k,l}, \ldots, c_n^{k,l})$, $l = 1, \ldots, l_k$, all points $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, whose coordinates are subject to the following conditions:

(i) ξ_j/ν is integral and $|\xi_j/\nu| \leqslant k$ for all $j = 1, \ldots, n$;
(ii) $|\xi_j| = k\nu$ for at least one j.

It is easy to see that $l_0 = 1$ and that

$$l_k \leqslant 2n(2k+1)^{n-1}, \qquad k = 1, 2, \ldots \tag{5}$$

We also use the following notation:

$$I_{k,l} = \left\{\xi \in R^n : |\xi_j - c_j^{k,l}| \leqslant \frac{1}{2}, \quad j = 1, \ldots, n\right\},$$

$$\lambda_{k,l} = \sup_{\zeta \in V_b, \mathrm{Re}\,\zeta \in I_{k,l}} \gamma(\zeta), \qquad \lambda_k = \max_{1 \leqslant l \leqslant l_k} \lambda_{k,l}.$$

Note that the sequence $\{\lambda_k\}$ is rapidly decreasing, i.e. $k^\mu \lambda_k \to 0$ as $k \to \infty$ for every $\mu \in N$.

We may assume that the function γ does not change too rapidly, since otherwise we can replace B by a suitable bounded set $B^* \supset B$. Thus we assume that

$$\lambda_{0,1} \geqslant \lambda_k \exp(-k\nu), \qquad k = 1, 2, \ldots, \tag{6}$$

and that analogous conditions are satisfied for all translations of γ by the points $c^{k,l}$.

By (5) and the rapid decrease of the sequence $\{\lambda_k\}$,

$$\psi_1(\zeta) = \sum_{k=0}^{\infty} \sum_{l=1}^{l_k} 2\lambda_{k,l} \exp[-(\zeta - c^{k,l})^2 + n/4]$$

is a function in K_1, the series being convergent uniformly in every horizontal strip on multiplying each term by an arbitrary (but the same for each term) polynomial of ζ ([2]). Moreover,

$$|\psi(\zeta)| \geqslant \gamma(\zeta) \tag{7}$$

for all $\zeta \in V_b$ with

$$\operatorname{Re} \zeta \in \bigcup_{k=0}^{\infty} \bigcup_{l=1}^{l_k} I_{k,l}.$$

In fact, for $\zeta \in V_b$ with $\operatorname{Re}\zeta \in I_{0,1}$, we have

$$2\lambda_{0,1} |\exp(-\zeta^2 + n/4)| \geqslant 2\lambda_{0,1}$$

and, on the other hand,

$$\sum_{k=1}^{\infty} \sum_{l=1}^{l_k} 2\lambda_{k,l} |\exp[-(\zeta - c^{k,l})^2 + n/4]|$$

$$\leqslant \sum_{k=1}^{\infty} 4n(2k+1)^{n-1} \lambda_k \exp\left[-\left(k\nu - \frac{1}{2}\right)^2 + nb^2 + n/4\right]$$

$$\leqslant 3 \sum_{k=1}^{\infty} \frac{\lambda_k}{4^k} \exp(-k\nu) \leqslant \lambda_{0,1},$$

on account of (4), (5) and (6). Thus in this case inequality (7) is proved. For $\zeta \in V_b$ with $\operatorname{Re}\zeta \in I_{k,l}$, $k > 0$, the proof is similar.

Translating now the system of points $c^{k,l}$, $k = 0, 1, \ldots$; $l = 1, \ldots, l_k$, we construct by the same method ν^n functions $\psi_1, \ldots, \psi_{\nu^n} \in K_1$ such that

$$\max_{1 \leqslant j \leqslant \nu^n} |\psi_j(\zeta)| \geqslant \gamma(\zeta)$$

([2]) For $\zeta \in C^n$ we write ζ^2 instead of $\zeta \cdot \zeta$.

for all $\zeta \in V_b$. Hence

$$N'(\psi_1, \ldots, \psi_{\nu^n}; U) \subset N(B, U),$$

which is the desired result.

For each pair $k, l \in N$ we denote by $E_{k,l}$ the space of all functions ψ analytic in V_k and such that

$$\|\psi\|_{k,l} = \sup_{\zeta \in V_k} \frac{|\psi(\zeta)|}{(1+|\zeta|)^l} < \infty.$$

With $\| \ \|_{k,l}$ as a norm, $E_{k,l}$ is a Banach space.

For fixed k, the spaces $E_{k,l}$, $l = 1, 2, \ldots$, form an increasing sequence and the topology induced in $E_{k,l}$ by $E_{k,l+1}$ is coarser than the topology of $E_{k,l}$. We denote by E_k the inductive limit of the sequence $\{E_{k,l} : l = 1, 2, \ldots\}$.

Note that E_k is in duality with the space $\tilde{E}_k$ of analytic functions χ in V_k such that, for every $l \in N$,

$$v_{k,l}(\chi) = \sup_{\zeta \in V_k} (1+|\zeta|)^l |\chi(\zeta)| < \infty;$$

the topology in $\tilde{E}_k$ is defined by the system of semi-norms $v_{k,l}$, $l = 1, 2, \ldots$, and the canonical bilinear form of the duality is

$$(\psi, \chi) \to \langle \psi, \chi \rangle = \sup_{\zeta \in V_k} |\psi(\zeta)\chi(\zeta)|, \qquad \psi \in E_k,\ \chi \in \tilde{E}_k.$$

The sequence $\{E_k : k = 1, 2, \ldots\}$ is decreasing and we have

THEOREM 6. *$\mathcal{O}_M(K_1' : K_1')$ is the projective limit of the sequence $\{E_k : k = 1, 2, \ldots\}$.*

Proof. Let E be the projective limit of the sequence $\{E_k : k = 1, 2, \ldots\}$. It is clear that both spaces $\mathcal{O}_M(K_1' : K_1')$ and E consist of the same functions. We prove that their topologies coincide.

Suppose first that $N(B, U)$ is a 0-neighborhood in $\mathcal{O}_M(K_1' : K_1')$ for the topology $\mathcal{T}''$; B is a bounded set in K_1 formed of functions $\psi \in K_1$ such that

$$\sup_{\zeta \in V_k} (1+|\zeta|)^l |\psi(\zeta)| < M_l, \qquad l = 1, 2, \ldots,$$

and U is a 0-neighborhood in K_1,

$$U = \{\psi \in K_1 : \sup_{\zeta \in V_k} (1+|\zeta|)^l |\psi(\zeta)| < \varepsilon,\ l = 1, \ldots, l'\}.$$

We set

$$M_l^* = \max\{M_1, \ldots, M_{l+l'}\}$$

and

$$W_{k,l} = \left\{\psi \in E_{k,l} : \|\psi\|_{k,l} < \frac{\varepsilon}{M_l^*}\right\}, \quad l = 1, 2, \ldots$$

Then, the convex circled hull W_k of the union of the sequence $\{W_{k,l} : l = 1, 2, \ldots\}$ is a 0-neighborhood in E_k. Thus $W = W_k \cap E$ is a 0-neighborhood in E and it is easy to verify that $W \subset N(B, U)$.

Conversely, if $W = W_k \cap E$, where W_k is a 0-neighborhood in E_k, then the polar set W_k^0 of W_k (with respect to the duality between E_k and $\tilde{E}_k$) is bounded in $\tilde{E}_k$. Thus, by the method used in the proof of theorem 5, one can find a finite (and therefore bounded) set $B = \{\psi_1, \ldots, \psi_m\}$ in K_1 such that

$$\sup_{\psi \in W_k^0} |\psi(\zeta)| \leqslant \max_{1 \leqslant j \leqslant m} |\psi_j(\zeta)|$$

for all $\zeta \in V_k$. But, by the bipolar theorem, $W_k^{00} = W_k$. Consequently, if U is the 0-neighborhood in K_1 defined as

$$U = \{\psi \in K_1 : \sup_{\zeta \in V_k} |\psi(\zeta)| < 1\},$$

then $N(B, U)$ is a 0-neighborhood in $\mathcal{O}_M(K_1' : K_1')$ for the topology $\mathscr{T}''$ and $N(B, U) \subset W$, q.e.d.

The bounded sets in E_k are characterized by the following theorem:

THEOREM 7. *A subset B of E_k is bounded in E_k if and only if, for some $l \in N$, B is a bounded subset of $E_{k,l}$.*

Proof. Each bounded subset of any $E_{k,l}$ is obviously bounded in E_k.

Conversely, if B is bounded in E_k, then $B \cap E_{k,l}$ is bounded in $E_{k,l}$, $l = 1, 2, \ldots$ It is thus sufficient to prove that B is contained in some $E_{k,l}$. Suppose that this is not true. Then one can find functions $\psi_l \in B$ and points ${}_l\zeta \in V_k$, $l = 1, 2, \ldots$, such that

$$1 < |{}_1\zeta| < |{}_2\zeta| < \ldots < |{}_l\zeta| \to \infty,$$

and

$$|\psi_l({}_l\zeta)| > l(1 + |{}_l\zeta|)^l.$$

Now, for each $l \in N$, we set

$$U_l = \left\{\psi \in E_{k,l} : \|\psi\|_{k,l} < \frac{1}{2^{l+1}(1 + |{}_l\zeta|)^l}\right\}.$$

Then the convex circled hull U of the union of the U_l's is a 0-neighborhood in E_k and it is easy to verify that $l^{-1}\psi_l \notin U$. This is a contradiction, and so B is contained in some $E_{k,l}$, q.e.d.

COROLLARY 1. *A subset B of $\mathcal{O}_M(K_1' : K_1')$ is bounded in $\mathcal{O}_M(K_1' : K_1')$ if and only if for each $k \in N$ there is an $l \in N$ such that B is a bounded subset of $E_{k,l}$.*

This corollary can also be proved directly in an easy way.

COROLLARY 2. *A sequence $\{\psi_j\}$ converges to 0 in $\mathcal{O}_M(K_1' : K_1')$ if and only if for each $k \in N$ there is an $l \in N$ such that $\{\psi_j\}$ converges to 0 in $E_{k,l}$.*

COROLLARY 3. *A sequence $\{S_j\}$ converges to 0 in $\mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ if and only if for each $k \in N$ there exists a representation*

$$S_j = \sum_{|p| \leqslant l} D^p F_{j,p},$$

where l is independent of j and $F_{j,p}$ are continuous functions satisfying the following conditions:

(i) $\sup_{x \in R^n} \sigma_k(x) |F_{j,p}(x)| < \infty$, $|p| \leqslant l$, $j = 1, 2, \ldots$;

(ii) *for each fixed p, the sequence $\{\sigma_k F_{j,p} : j = 1, 2, \ldots\}$ converges to 0 uniformly in R^n.*

We stated in the preceding section that $\mathscr{K}_1$ is a Fréchet space, and therefore bornologic and complete. This implies that the space $\mathscr{L}_b(\mathscr{K}_1, \mathscr{K}_1)$ is complete (see [1], chap. III, § 3, exercise 18, or [2], p. 73, proposition 7). Moreover, $\mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ is a closed subspace of $\mathscr{L}_b'(\mathscr{K}_1, \mathscr{K}_1)$. In fact, if a filter $\mathscr{F}$ on $\mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ converges to S in $\mathscr{L}_b(\mathscr{K}_1, \mathscr{K}_1)$, then $\mathscr{F}$ is a base of a filter on $\mathscr{K}_1'$ converging to S. Thus $S \in \mathscr{K}_1'$ and also $S * \varphi \in \mathscr{K}_1$ for every $\varphi \in \mathscr{K}_1$. Hence $S \in \mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ by theorem 3 (d). As a closed subspace of a complete space, $\mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ is complete.

Since, in addition, $\mathscr{K}_1$ is nuclear, the space $\mathscr{L}_b(\mathscr{K}_1, \mathscr{K}_1)$ is nuclear ([4], chap. II, theorem 9, corollary 3). Consequently, $\mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ is nuclear as a subspace of $\mathscr{L}_b(\mathscr{K}_1, \mathscr{K}_1)$.

We thus proved

THEOREM 8. *$\mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ is a complete, nuclear space.*

We now proceed to prove that the space $\mathcal{O}_M(K_1' : K_1')$ is bornologic. We need a lemma:

LEMMA. *Let U be a convex circled subset of $\mathcal{O}_M(K_1' : K_1')$ that absorbs each bounded set in $\mathcal{O}_M(K_1' : K_1')$. There is a $k \in N$ and a sequence $\{M_l\}$ of positive numbers such that the sets*

$$A_{k,l} = \{\psi \in \mathcal{O}_M(K_1' : K_1') \cap E_{k,l} : \|\psi\|_{k,l} \leqslant M_l\},$$

$l = 1, 2, \ldots$, are contained in U.

Proof. Assume the converse, i.e. that there are numbers $l_k \in N$ and functions $\psi_k \in \mathcal{O}_M(K_1' : K_1') \cap E_{k,l_k}$, $k = 1, 2, \ldots$, such that

$$\|\psi_k\|_{k,l_k} < \frac{1}{k} \quad \text{and} \quad \psi_k \notin U,$$

Then the sequence $\{k\psi_k\}$ is bounded in $\mathcal{O}_M(K_1':K_1')$. But U absorbs every bounded set in $\mathcal{O}_M(K_1':K_1')$, and so there is a $\mu\in R$ such that $k\psi_k\in\mu U$, $k=1,2,\ldots$ Hence $\psi_k\in U$ for sufficient large k's, since U is circled. This is a contradiction, which proves the lemma.

THEOREM 9. *The space $\mathcal{O}_M(K_1':K_1')$ is bornologic.*

Proof. Let U be any convex circled subset of $\mathcal{O}_M(K_1':K_1')$ that absorbs all bounded sets in $\mathcal{O}_M(K_1':K_1')$. Also, let k, $\{M_l\}$ and $\{A_{k,l}\}$ be as in the preceding lemma. If the sets $A_{k,l}^*$, $l=1,2,\ldots$, are defined as

$$A_{k,l}^*=\{\psi\in E_{k,l}\colon \|\psi\|_{k,l}\leqslant M_l\},$$

then the union

$$A_k^*=\bigcup_{l=1}^{\infty}A_{k,l}^*$$

absorbs every bounded set in E_k, by virtue of theorem 7. Moreover, by the choice of k, $\{M_l\}$ and $\{A_{k,l}\}$, we have

$$A_k^*\cap\mathcal{O}_M(K_1':K_1')\subset U.$$

But, being an (LB)-space, E_k is bornologic. Therefore the convex circled hull U_k^* of the union $A_k^*\cup U$ is a 0-neighborhood in E_k. Since, by theorem 6, $\mathcal{O}_M(K_1':K_1')$ is the projective limit of the sequence $\{E_k: k=1,2,\ldots\}$ and

$$U_k^*\cap\mathcal{O}_M(K_1':K_1')=U,$$

U is a 0-neighborhood in $\mathcal{O}_M(K_1':K_1')$. The proof is thus complete.

Since the spaces $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ and $\mathcal{O}_M(K_1':K_1')$ correspond to each other under the Fourier transform, from theorems 8 and 9 we can draw the following corollary:

COROLLARY 1. *The space $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ is complete, nuclear and bornologic.*

But each quasi-complete bornologic space is barreled ([6], p. 63, corollary) and each quasi-complete barreled nuclear space is a Montel space ([6], p. 194, exercise 19b). Thus we have

COROLLARY 2. *$\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ is a Montel space.*

We now characterize the dual $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$ of $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$. First we observe that $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ is a (linear) subspace of the space $\mathcal{O}_c'(\mathcal{S}':\mathcal{S}')$ of convolution operators in $\mathcal{S}'$ (see [7], vol. II, p. 100, or [8]) and the imbedding $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')\to\mathcal{O}_c'(\mathcal{S}':\mathcal{S}')$ is continuous.

Suppose now that f is a C^∞-function such that

$$D^pf(x)=O\big(\exp(k|x|)\big) \tag{8}$$

as $|x|\to\infty$ for all $p\in N^n$ and some $k\in N$ (independent of p). Then f/σ_k is a very slowly increasing C^∞-function and therefore belongs to the dual $\mathcal{O}_c(\mathcal{S}':\mathcal{S}')$ of $\mathcal{O}_c'(\mathcal{S}':\mathcal{S}')$ (see [4], chap. II, p. 131, or [8]). Hence $f/\sigma_k\in\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$, which implies that $f\in\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$, by the continuity of the mapping $S\to\sigma_kS$ from $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ into $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$.

On the other hand, from theorem 1 it follows that the convolution $T*\varphi$ of any $T\in\mathcal{K}_1'$ and $\varphi\in\mathcal{K}_1$ is a C^∞-function satisfying condition (8), and so $T*\varphi\in\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$. Furthermore, if $S\in\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$, then

$$(S*\varphi)\cdot T=\check{S}\cdot(\varphi*\check{T})=S\cdot(\varphi*\check{T})^{\vee}; \tag{9}$$

the convolution $S*\varphi$ is in $\mathcal{K}_1$, by the remark following theorem 2.

THEOREM 10. *The dual $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$ of $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ is the space of all C^∞-functions satisfying condition* (8).

Proof. As said before, each C^∞-function satisfying condition (8) is in $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$.

Conversely, by a well known theorem ([1], chap. IV, § 2, proposition 11, or [6], p. 139, corollary 4) there exists an (algebraic) isomorphism of the tensor product $\mathcal{K}_1\otimes\mathcal{K}_1'$ onto the dual $\mathcal{L}_s'(\mathcal{K}_1,\mathcal{K}_1)$ of $\mathcal{L}_s(\mathcal{K}_1,\mathcal{K}_1)$. Under this isomorphism to each element $\sum\varphi_j\otimes T_j\in\mathcal{K}_1\otimes\mathcal{K}_1'$ there corresponds an $I\in\mathcal{L}_s'(\mathcal{K}_1,\mathcal{K}_1)$ such that

$$I(u)=\sum u(\varphi_j)\cdot T_j$$

for all $u\in\mathcal{L}(\mathcal{K}_1,\mathcal{K}_1)$.

But for $u=S\in\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ we have

$$u(\varphi_j)=S*\varphi_j.$$

Hence

$$I(S)=\sum(S*\varphi_j)\cdot T_j=\sum S\cdot(\varphi_j*\check{T}_j)^{\vee}$$

by virtue of (9). Thus the restriction of I to $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ can be identified with the finite sum $\sum(\varphi_j*\check{T}_j)^{\vee}$ of C^∞-functions satisfying condition (8). This proves the theorem.

THEOREM 11. *The space $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$ endowed with the strong topology is a complete nuclear Montel space.*

Proof. $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$ is a complete Montel space, by corollary 1 and corollary 2 from theorem 9. Furthermore, $\mathcal{O}_c'(\mathcal{K}_1':\mathcal{K}_1')$ is a closed subspace of the space $\mathcal{L}_b(\mathcal{K}_1,\mathcal{K}_1)$, which is reflexive ([4], chap. I, § 4, proposition 19, corollary 2). Hence $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$ can be identified with a quotient space of the dual $\mathcal{L}_b'(\mathcal{K}_1,\mathcal{K}_1)$ of $\mathcal{L}_b(\mathcal{K}_1,\mathcal{K}_1)$ ([3], p. 102, corollary 2). But $\mathcal{L}_b'(\mathcal{K}_1,\mathcal{K}_1)$ is nuclear ([4], chap. II, § 2, theorem 9, corollary 3), and so $\mathcal{O}_c(\mathcal{K}_1':\mathcal{K}_1')$ is also nuclear.

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On the constants of basic sequences in Banach spaces

by

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Dedicated to Professors
S. Mazur and W. Orlicz
in honour of the fortieth anniversary
of their scientific activity

1. A sequence $\{x_n\}$ in a Banach space E is called a *basic sequence* (respectively, an *unconditional basic sequence*) if it is a basis (respectively, an unconditional basis) of its closed linear span $[x_n]$ in E (see [1]). It is well known that $\{x_n\}$ is a basic sequence (respectively, an unconditional basic sequence) if and only if there exists a constant $K \geqslant 1$ (respectively, $K_u \geqslant 1$) such that

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant K \Big\|\sum_{i=1}^{n+m} a_i x_i\Big\| \tag{1}$$

for any scalars $a_1, \ldots, a_{n+m}$ (respectively, such that

$$\Big\|\sum_{i=1}^{n} \delta_i a_i x_i\Big\| \leqslant K_u \Big\|\sum_{i=1}^{n} a_i x_i\Big\| \tag{2}$$

for any scalars $a_1, \ldots, a_n, \delta_1, \ldots, \delta_n$ with $|\delta_1| \leqslant 1, \ldots, |\delta_n| \leqslant 1$); some authors call this the *K-condition*. The least such constant $C(\{x_n\}) = \min K$ (respectively, $C_u(\{x_n\}) = \min K_u$) is called the *constant* (respectively, the *unconditional constant*) of the basic sequence $\{x_n\}$; obviously we have $1 \leqslant C \leqslant C_u$. In the particular case where $C = 1$ (respectively, $C_u = 1$) $\{x_n\}$ is called a *monotone* (respectively, an *orthogonal* [5]) basic sequence.

It is well known [4] that if $\{x_n\}$ is a basis (respectively, an unconditional basis) of a Banach space E, then the sequence of coefficient functionals $\{f_n\} \subset E^*$ (i.e. for which $f_i(x_j) = \delta_{ij}$) is a basic sequence (respectively, an unconditional basic sequence) in the conjugate space E^* (but, in general, $[f_n] \neq E^*$). Therefore it is natural to ask what are the relations between the constants of $\{x_n\}$ and $\{f_n\}$, and the present note is devoted to this problem. We shall give upper and lower evaluations of $C(\{f_n\})$