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Amenability and equicontinuity

by

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§ 1. PURPOSE

In this paper I sort out, and give simple proofs for, the relationships between many conditions related to the existence of invariant means on locally compact groups.

The new idea here is the value, and the presence when needed, of equicontinuity in L_p; this enables the equivalence proofs from the discrete case, especially those in [1] and [2] to be used in the general case.

The new results here are:

(i) A simple proof that amenability is equivalent to "type (R)" of Godement [7] (which enables us to prove that approximation on finite sets is adequate to imply the uniform approximation on compact sets needed by Godement). (See Theorem 4, ((gπ) $\leftrightarrow$ (lsaπ) $\leftrightarrow$ (gu))

(ii) A simple proof that Dieudonné's, Kesten's, and Reiter's conditions are equivalent to each other and to amenability. (Theorem 4, (ldp) $\leftrightarrow$ (lkp) $\leftrightarrow$ (lsau) $\leftrightarrow$ (laπ).)

(iii) The existence of a new family of conditions (n labels in Theorems 1 and 4) intermediate between the Kesten and Godement types, which are also equivalent to the others.

Most of the history of the subject concerns discrete groups. The first paper in the field was that of von Neumann [15] in 1929, in which he accounted for the Banach-Tarski paradoxical decomposition of three-space. More references and, a survey of what was then known for semigroups can be found in my paper [1] of 1955; also see [9], § 17. Among the many contributions to the subject arranged here are those of Kesten [11] who discussed the spectrum of convolutions over countable discrete groups, Reiter [17] and [18] on strong left invariance uniform over compact sets, Hulanicki [10] on "topological" left invariance, Dieudonné [4] on norms of convolution operators in L_p, $p > 1$, and Namioka [14] and Stegman [19] proving relationships among some of the conditions above.

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§ 2. RESULTS

In the study of invariant means on locally compact groups, the problem arises as to the proper definition of "strong" amenability (see [1], § 5 for discrete groups). By analogy with the discrete case we want a net φ_n of non-negative, continuous functions vanishing outside compact sets and with integrals all 1 so that (A) such a net can be constructed converging to left invariance if a left invariant mean exists, and so that (B) proofs using this net shall go forward in close analogy to the simple proofs for the discrete case. (See [1] and especially [2].)

The critical property is so obvious in the discrete case that it is easily overlooked; every set Φ of functions in L_1 of a discrete space is a left-equicontinuous set in L_1; that is, for each $\varepsilon > 0$ there is a neighborhood W_ε of u, the identity element of G, such that $\|l_s\varphi - \varphi\|_1 < \varepsilon$ for all s in W_ε and all φ in Φ. (In the discrete case simply take all $W_\varepsilon = \{u\}$.) Here and hereafter we use $[l_s\varphi](t) = \varphi(st)$ and $[r_s\varphi](t) = \varphi(ts)$ for all t in G.

2.1. Notation. G will be a locally compact group. We will always use left Haar measure H on G, denoting it by ds or dg. (See, for example, Hewitt and Ross [9] for convolutions and Haar measure.) We will use the complex spaces $L_p = L_p(G)$ with respect to Haar measure H (see [9]). Recall that for $1 \leqslant p < \infty$ and $1/p + 1/p' = 1$, $1 < p' \leqslant \infty$, each x in $L_{p'}$ determines a conjugate linear functional $x^\vee$ in L_p^* by the usual rule

$$x^\vee(\varphi) = \int_G x(t)\overline{\varphi(t)}\,dt,$$

and that the relation of x to $x^\vee$ is a linear isometry between $L_{p'}$ and L_p^*.

We shall also need the space C_u of two-sided uniformly continuous complex functions on G; that is C_u is the set of all bounded continuous functions x on G such that for each $\varepsilon > 0$ there is a neighborhood W_ε of u such that $\|l_sx - x\|_\infty < \varepsilon$ and $\|r_sx - x\|_\infty < \varepsilon$ if s is in W_ε. Agreeing with a preference once expressed by Namioka, we call the first of these conditions *left* uniform continuity in L_∞; this does not agree with the notation of Hewitt and Ross [9].

C_{00} is the space of all complex continuous functions vanishing outside compact sets. Clearly C_{00} is a subset of C_u and of all the L_p.

On any reasonable space E of functions normed with the (essential) least upper bound of the absolute values, one defines a *mean* m on E to be an element m of E^* such that the value $m(x)$ is in the closed convex hull of the (essential) range of values of x. In the case of an AM-space ([3], p. 100), such as C_u, $C(G)$, or L_∞, this is equivalent to (i) $\|m\| = 1$, (ii) $m(e) = 1$ (where e is the function constantly 1 on G), and (iii) $m(x) \geqslant 0$ if $x \geqslant 0$.

We let $\underset{\sim}{M}$ be the set of means on L_∞ and M the set of means on C_{00} (Hewitt and Ross [9], p. 269). P is the set of non-negative φ in L_1 with

$$\int_G \varphi\,dH = 1; \qquad P_{00} \text{ is } P \cap C_{00}.$$

It is well known (and follows from properties (i) and (ii) of means and the w*-density ([3], p. 41, Theorem 4) of the image under Q of the unit ball in L_1 in the unit ball of L_∞^*) that $Q(P)$ is w*-dense in $\underset{\sim}{M}$.

We also need the fact (see [9]) that for each μ in M and each x in L_p, $1 \leqslant p < \infty$, and for L_∞, the convolution $\mu \circ x$ is defined and is in the same L_p, and $\|\mu \circ x\|_p \leqslant \|\mu\|_1 \|x\|_p$.

We say that *a net x_n of elements of a space L_p is left equicontinuous in L_p* if for each $\varepsilon > 0$ there is a neighborhood W_ε of u such that $\|l_sx_n - x_n\|_p < \varepsilon$ for all n and for all s in W_ε; this implies that $\|l_sx_n - l_tx_n\|_p < \varepsilon$ for st^{-1} in W_ε, $\|l_sx_n - l_tx_n\|_p < \varepsilon$ for all n.

For f defined on G, $f^\sim(g) = \overline{f(g^{-1})}$. For φ in L_1, $\varphi^*(x) = \overline{\varphi(x^{-1})}\Delta(x^{-1})$, where Δ is the modular function of G([9], p. 196). Then $\varphi \in P$ if and only if $\varphi^* \in P$.

For z complex and p positive define $z^p = |z|^p \operatorname{sign} z$, where $\operatorname{sign} re^{i\theta} = e^{i\theta}$ if $r \neq 0$, $= 0$ if $r = 0$.

2.2. The theorems.

THEOREM 1. *Let φ_n be a net of elements of P_{00} which are left-equicontinuous in L_1, and for any $p > 1$ define $f_n(g) = [\varphi_n(g)]^{1/p}$ for all g in G. Then the f_n are left-equicontinuous in L_p and the following conditions on the nets φ_n and f_n are equivalent:*

(LSAΠ) *For each s in G,*

$$\lim_n \|l_s\varphi_n - \varphi_n\|_1 = 0.$$

(This is derived from strong left amenability.)

(LSAU) *For each compact $K \subseteq G$, $\|l_s\varphi_n - \varphi_n\|_1$ tends to zero uniformly for s in K.*

("Uniform strong left amenability" is due to Reiter [17].)

(LSAP) *For each a in P,*

$$\lim_n \|a \circ \varphi_n - \varphi_n\|_1 = 0.$$

(LSAM) *For each μ in M,*

$$\lim_n \|\mu \circ \varphi_n - \varphi_n\|_1 = 0.$$

(Hulanicki [10] invented these two conditions and proved them equivalent to Reiter's condition.)

(LKP) *For each (or one)* $p > 1$, *for each* a *in* P,

$$\lim_n \|a \circ f_n - f_n\|_p = 0.$$

(Kesten [11] for discrete countable G and $p = 2$; Day [2] for general discrete groups. The next seven conditions (except (LKU)) are new; they help to round out the pattern from amenability to type (R) which we are constructing.)

(LKM) *Like* (LKP) *with* μ *in* M *replacing* a *in* P.

(LKΠ) *For each (or one)* $p > 1$, *for each* s *in* G,

$$\lim_n \|l_s f_n - f_n\|_p = 0.$$

(LKU) *Like* (LKΠ) *with uniform convergence on compact sets in* G.

(LNΠ) *For each (or one)* $p > 1$, *if* $F_n = f_n^{p/p'}$, *then for each* g *in* G,

$$\lim_n F_n^{\vee}(l_g f_n) = 1.$$

(LNU) *The same with uniform convergence on compact sets in* G.

(LNP) *For each (or one)* $p > 1$, *for each* a *in* P,

$$\lim_n F_n^{\vee}(a \circ f_n) = 1.$$

(LNM) *Same with* μ *in* M *replacing* a *in* P.

(LDP) *For each (or one)* $p > 1$, *for each* a *in* P,

$$\lim_n \|a \circ f_n\|_p = 1.$$

(LDM) *The same with* μ *in* M *instead of* a *in* P.

((LDP) is a formal strengthening of a condition of Dieudonné [4]; there are no Π or U analogues of these because for all G, p, n, and s, $\|l_s f_n\|_p = \|f_n\|_p$.)

(GΠ) *For* $p = 2$, *for each* g *in* G,

$$\lim_n [f_n \circ f_n^{\sim}](g) = 1.$$

(GU) *Same with uniform convergence on compact sets in* G.

(This is formally stronger than Godement's type (R); [7], p. 76. Godement showed that (R) implies that every positive definite function on G can be approximated uniformly on compact sets by functions $f \circ f^{\sim}$, f in L_2. Fell [6] showed that this is equivalent to: Every irreducible unitary representation of G is weakly contained in the regular representation, $s \to l_s^{!}(\ )$, in $L_2(G)$.)

Remark. It is true and often trivial that equicontinuity is not needed in all these implications. These easy ones are shown in Diagram A. Equicontinuity becomes vital in passing easily from, say, (LSAΠ) to (LSAU) or from (LDP) to (LKΠ), two implications which are critical in this pattern of proof.

$$\begin{tikzcd}
(LSAU) \arrow[r] \arrow[d, leftrightarrow] & (LSAM) \arrow[r] \arrow[rr, bend left] & (LSAP) & (LSA\Pi) \arrow[d, leftrightarrow] \\
(LKU) \arrow[r] \arrow[d, leftrightarrow] & (LKM) \arrow[r] \arrow[d] \arrow[rr, bend left] & (LKP) & (LK\Pi) \arrow[d, leftrightarrow] \\
(LNU) \arrow[r] \arrow[dd, leftrightarrow] & (LNM) \arrow[r] \arrow[d] \arrow[rr, bend left] & (LNP) \arrow[d] & (LN\Pi) \arrow[dd, leftrightarrow] \\
 & (LDM) \arrow[r] & (LDP) & \\
(GU) & & & (G\Pi)
\end{tikzcd}$$

Diagram A

THEOREM 2. *Let* $\zeta_n = \varphi_n^*$; *that is,*

$$\zeta_n(s) = \varphi_n(s^{-1})\Delta(s^{-1}) \quad \textit{for all } s \textit{ in } G,$$

and let $z_n = \zeta_n^{1/p}$. *If the* φ_n *are left equicontinuous in* L_1, *then the* ζ_n (*the* z_n) *are right equicontinuous in* L_1 (*in* L_p) *and all the left-handed conditions of Theorem* 1 *are equivalent to corresponding right-handed conditions on* ζ_n *and* z_n.

THEOREM 3. *Let* φ_n *be as in Theorems* 1 *and* 2, *let* $\Phi_n = \varphi_n \circ \varphi_n^*$ *and let* $F_n = \Phi_n^{1/p}$. *Then the conditions of the preceding theorems are equivalent to the corresponding two-sided conditions.* (*None here to correspond to* (GΠ) *or* (GU).)

The next problem is to consider conditions under which there exists a net φ_n or f_n with equicontinuity as well as some (hence all) of the conditions of the kinds A, K, N, D, G. The weakest sufficient conditions known to me are (lgπ), the finite-set weakening of Godement's type (R), (laπ), "pointwise" left amenability of L_∞, and (la$_0$), "pointwise" left amenability of $C_u(G)$.

THEOREM 4. *The following conditions on a locally compact group are equivalent:*

(laπ) *There exists a mean* m *on* L_∞ *such that for each* x *in* L_∞ *and each* s *in* G, $m(l_s x) = m(x)$. (This is ordinary left amenability of L_∞.)

(la$_0$) *There exists a mean* m_0 *on* C_u *such that for each* v *in* C_u *and each* s *in* G, $m_0(l_s v) = m_0(v)$.

(lap$_0$) *There exists a mean* m_0 *on* C_u *such that for each* v *in* C_u *and each* a *in* P, $m_0(a \circ v) = m_0(v)$.

(lap) *There exists a mean m on L_∞ such that for each x in L_∞ and each a in P, $m(a \circ x) = m(x)$.*

(lam) *Same with μ in M instead of a in P.*

(lsaπ) *For each finite set A in G and each $\varepsilon > 0$ there is a φ in P_{00} such that for each s in A, $\|l_s\varphi - \varphi\|_1 < \varepsilon$.*

(lsau) *Same with compact K instead of finite A.* (See Reiter [17] and [18].)

(lsap) *There exists a net φ_n in P_{00} such that for each a in P*

$$\lim_n \|a \circ \varphi_n - \varphi_n\|_1 = 0.$$

(lsam) *Same with μ in M instead of a in P.*

(lkπ) *For each (or one) $p > 1$, for each finite set A in G and each $\varepsilon > 0$, there exists f in $P^{1/p}$ such that $\|l_s f - f\|_p < \varepsilon$ for each s in A.*

(Note that it is not enough to assume that for each s and ε there is f such that $\|l_s f - f\|_p < \varepsilon$. All discrete groups have this property because all cyclic groups are amenable. But the free group on two generators is not amenable ([1], § 4, (G)), so the weakened condition is not adequate. The same comment applies to (lsaπ), (lnπ), and (gπ).)

(lku) *Same as* (lkπ) *with compact K instead of finite A.*

(lkp) *For each (or one) $p > 1$, for each a in P, left convolution by a considered as an operator from L_p to L_p, has 1 in its spectrum.* (Kesten [11] for symmetric a, countable discrete groups, and $p = 2$; Day [2] for general discrete groups.)

(lkm) *Same with μ in M replacing a in P.*

(lnπ) *For each (or one) $p > 1$, for each finite $A \subseteq G$, and each $\varepsilon > 0$, there exists f in $P^{1/p}$ such that if $F = f^{p/p'}$, for each g in G we have $|1 - F^{\vee}(l_g f)| < \varepsilon$.*

(lnu) *Same with compact K replacing finite A.*

(lnp) *For each (or one) $p > 1$, each $\varepsilon > 0$, and each a in P there exists f in L_p with $\|f\|_p = 1$ such that, with $F = f^{p/p'}$, $|1 - F^{\vee}(a \circ f)| < \varepsilon$.*

(lnm) *Same with μ in M replacing a in P.*

(ldp) *For each (or one) $p > 1$, for each a in P, left convolution by a, as an operation from L_p to L_p, has norm 1.*

(ldm) *Same with μ in M replacing a in P.*

(gu) (= *type* (R)) *For $p = 2$ there exists for each compact K in G and each $\varepsilon > 0$ an f in L_2 such that $|1 - (f \circ f^{\sim})(g)| < \varepsilon$ for all g in K.*

(gπ) *Same with finite $A \subseteq G$ replacing compact K.*

Remarks. Hulanicki [10] invented (laπ), (lam), and and their s counterparts, and proved (lsau) $\leftrightarrow$ (lap) $\leftrightarrow$ (lsap) $\leftrightarrow$ (lsam) $\leftrightarrow$ (lam) $\rightarrow$ (laπ). Dieudonné [4] studied groups with (ldp) and proved (lsau) $\rightarrow$ (lku) $\rightarrow$ (ldp) for all $p > 1$. Reiter [18] proved that (laπ) $\rightarrow$ (lsau); Stegeman [17] proved that (lsau) $\Leftrightarrow$ (lku) for all $p > 1$. The (ln-) conditions are all new, interpolated for completeness between the k and g types.

The part of this theorem which does not require equicontinuity is displayed in Diagram B.

$$\begin{tikzcd}
 & & (\mathrm{lap}_0) \arrow[d] & (\mathrm{la}_0) \arrow[l] \\
 & (\mathrm{lam}) \arrow[r] \arrow[d, leftrightarrow] & (\mathrm{lap}) & (\mathrm{la}\pi) \arrow[u] \arrow[d, leftrightarrow] \\
(\mathrm{lsau}) \arrow[r] \arrow[d, leftrightarrow] & (\mathrm{lsam}) \arrow[r] \arrow[rr, bend left] & (\mathrm{lsap}) \arrow[u] & (\mathrm{lsa}\pi) \arrow[d, leftrightarrow] \\
(\mathrm{lku}) \arrow[r] \arrow[d, leftrightarrow] & (\mathrm{lkm}) \arrow[r] \arrow[d] \arrow[rr, bend left] & (\mathrm{lkp}) & (\mathrm{lk}\pi) \arrow[d, leftrightarrow] \\
(\mathrm{lnu}) \arrow[r] \arrow[dd, leftrightarrow] & (\mathrm{lnm}) \arrow[r] \arrow[d] \arrow[rr, bend left] & (\mathrm{lnp}) \arrow[d] & (\mathrm{ln}\pi) \arrow[dd, leftrightarrow] \\
 & (\mathrm{ldm}) \arrow[r] & (\mathrm{ldp}) & \\
(\mathrm{gu}) & & & (\mathrm{g}\pi)
\end{tikzcd}$$

Diagram B

As with Theorems 2 and 3, there are the corresponding resultss Theorems 5 and 6, for right or two-sided means. It is also too obvioug to be worth mentioning that everything could have been done startin, with right Haar measure.

One of my students, Mrs. Truitt, in the paper [20] extended the results of [2] from l_p-spaces to uniformly convex Orlicz spaces [16] on discrete G. The interested reader can see how to adapt her result to give lk-like conditions for such uniformly convex Orlicz spaces on locally compact groups.

§ 3. PROOFS AND COMMENTS

We begin with the proofs for Diagram A. First we show simply that (LSAΠ) $\leftrightarrow$ (LKΠ) $\leftrightarrow$ (LNΠ) $\leftrightarrow$ (GΠ), and the same for the corresponding U conditions. The SA-to-K step proceeds as in the discrete case, beginning with a lemma whose proof carries over to the general locally compact case (See [2], Lemma 2).

LEMMA 1. *If φ and $\varrho \in P$, if $f(g) = (\varphi(g))^{1/p}$ for all g and similarly $r = \varrho^{1/p}$, then $\|f\|_p = 1 = \|r\|_p$, and $\|f - r\|_p \leqslant [\|\varphi - \varrho\|_1]^{1/p}$, and $\|\varphi - \varrho\|_1 \leqslant p \cdot 2^{p-1}\, \|f - r\|_p$.*

This shows that the homeomorphism of Mazur [13] is uniformly continuous between the positive parts of the unit spheres of L_1 and of each L_p. Noting also that $[l_s\varphi]^{1/p} = l_s[\varphi^{1/p}]$, we see that $\|l_s\varphi_n - \varphi_n\|_1$ is small if and only if $\|l_s f_n - f_n\|_p$ is small for any particular $p > 1$. This shows that (LKΠ) for one $p>1$ implies (LSAΠ) implies (LKΠ) for every $p>1$.

If $f_n \epsilon P^{1/p}$, then $F_n = f_n^{p/p'} \epsilon P^{1/p'}$, and $F_n^{\vee}$ is the linear functional in L_p^* such that $\|F_n^{\vee}\| = 1 = \|f_n\|_p = F_n^{\vee}(f_n)$. Then

$$|1-F_n^{\vee}(l_s f_n)| = |F_n^{\vee}(f_n - l_s f_n)| \leqslant \|l_s f_n - f_n\|_p,$$

so (LKΠ) implies (LNΠ).

But L_p, $p>1$, is uniformly rotund (Clarkson; see [3], p. 115) so for each $\varepsilon>0$ there is a $\delta(\varepsilon)$, depending on p also, such that $\|l_s f_n - f_n\| < \varepsilon$ if the real part of $F_n^{\vee}(l_s f_n) > 1-\delta(\varepsilon)$. ([3], p. 113, (UD) equivalent to (UR).) Hence (LNΠ) implies (LKΠ).

When $p=2$, $L_2^* = L_2$ and we see that (LNΠ) says that $\langle f_n, l_s f_n\rangle$ can be made close to 1 on any finite set. ($\langle,\rangle$ is the usual inner product in the Hilbert space L_2.) But a quick calculation shows that $[f \circ f^{\sim}](g^{-1}) = \langle f, l_g f\rangle$, so when $\langle f_n, l_s f_n\rangle$ is near 1 on a set A, $f_n \circ f_n^{\sim}$ is near 1 on A^{-1}. Hence (GΠ) is equivalent to the case $p=2$ of (LNΠ).

If uniformity holds for a compact set K in any one of those conditions, it can be passed to the others in these proofs, so we see also that (LSAU) $\leftrightarrow$ (LKU) $\leftrightarrow$ (LNU) $\leftrightarrow$ (GU).

To do the other easy cases next, we observe that any (XU) implies its (XM) because each μ can be approximated by one with compact support, and if for all t in the support of μ, the $l_{t^{-1}}x$ are near x, then so is $\mu \circ x$. Any (XM) implies its (XP) because each $a\circ$ is a $\mu\circ$. (XM) implies (XΠ) because the point measures are in M. (LNX) implies (LDX), for $X = M$ or P, because the F_n and the f_n are of norm 1.

All these simple implications were displayed in Diagram A.

We see that to complete the proof of Theorem 1 it suffices to prove

$$\text{(LSAP)} \to \text{(LSA}\Pi) \to \text{(LSAU)} \quad \text{and} \quad \text{(LDP)} \to \text{(LK}\Pi).$$

(LSAP) $\to$ (LSAΠ). Assume that (LSAΠ) fails, that is, that for some s_0 in G, some $\varepsilon>0$, and some subnet φ_{n_i}, $\|l_{s_0}\varphi_{n_i} - \varphi_{n_i}\|_1 > 2\varepsilon$ for all i. Then, by equicontinuity, there is a W_ε such that $\|l_s \varphi_n - \varphi_n\|_1 < \varepsilon$ if $s \epsilon W_\varepsilon$, so $\|l_t\varphi_n - l_{s_0}\varphi_n\|_1 < \varepsilon$ if $s_0 t^{-1} \epsilon W_\varepsilon$; that is, if $t^{-1} \epsilon s_0^{-1} W_\varepsilon$.

Then for any a in P which has support in $(s_0^{-1}W_\varepsilon)^{-1}$ we have

$$\|a\circ\varphi_{n_i} - l_{s_0}\varphi_{n_i}\|_1 = \Big\|\int_G a(t)(l_{t^{-1}}\varphi_{n_i} - l_{s_0}\varphi_{n_i})\,dt\Big\|_1$$
$$\leqslant \int_G a(t)\|l_{t^{-1}}\varphi_{n_i} - l_{s_0}\varphi_{n_i}\|_1\,dt \leqslant \int_G a(t)\,\varepsilon\,dt = \varepsilon.$$

Therefore, $\|a\circ\varphi_{n_i} - \varphi_{n_i}\|_1 > \varepsilon$ for all i; hence (LSAP) fails if (LSAΠ) does.

(LSAΠ) implies (LSAU). If (LSAU) fails then there are $\varepsilon>0$, a compact K in G, a net s_i of points of K, and a subnet φ_{n_i} of φ_n such that $\|l_{s_i}\varphi_{n_i} - \varphi_{n_i}\|_1 > \varepsilon$. Take a convergent subnet s_{i_j} with limit s_0; then by equicontinuity in L_1, $\|l_{s_{i_j}}\varphi_{n_{i_j}} - l_{s_0}\varphi_{n_{i_j}}\|_1 < \varepsilon$ as soon as j gets so large that $s_{i_j}^{-1} \epsilon s_0^{-1} W_\varepsilon$. Hence

$$\|l_{s_0}\varphi_{n_{i_j}} - \varphi_{n_{i_j}}\|_1 \geqslant \|l_{s_{i_j}}\varphi_{n_{i_j}} - \varphi_{n_{i_j}}\|_1 - \|l_{s_{i_j}}\varphi_{n_{i_j}} - l_{s_0}\varphi_{n_{i_j}}\|_1 \geqslant 2\varepsilon - \varepsilon = \varepsilon.$$

That is (LSAΠ) fails if (LSAU) fails.

(The same technique would work in the k and n lines of the diagram, but we do not need them in this pattern of proof.)

(LDP) implies (LKΠ). If (LKΠ) fails, there are $\varepsilon>0$, s_0 in G, and a subnet f_{n_i} of the net f_n such that $\|l_{s_0}f_{n_i} - f_{n_i}\|_p > 3\varepsilon$ for all i. By equicontinuity in L_p, $\|l_s f_{n_i} - l_{s_0}f_{n_i}\|_p < \varepsilon$ if $s^{-1}\epsilon s_0^{-1}W_\varepsilon$; hence if a in P has its support in $(s_0^{-1}W_\varepsilon)^{-1}$, then $\|a\circ f_{n_i} - l_{s_0}f_{n_i}\|_p < \varepsilon$ for all i. If also β in P has its support in $(W_\varepsilon)^{-1}$, then $\|\beta\circ f_{n_i} - f_{n_i}\|_p < \varepsilon$ for all i. Hence

$$\|a\circ f_{n_i} - \beta\circ f_{n_i}\|_p > \varepsilon \quad \text{for all i.}$$

By uniform rotundity of L_p ([3], p. 115), if $\gamma = (a+\beta)/2$, then

$$\|\gamma\circ f_{n_i}\|_p < 1 - \delta(\varepsilon) \quad \text{for all i.}$$

Hence (LDP) fails when (LKΠ) fails. (This is a simplification of the discrete version of this proof in [2], Lemma 4, (ii) implies (iii).)

This completes the proof of Theorem 1. Theorem 2 obviously works in the same way. To illustrate one of the harder cases of Theorem 3, note that (LDP) transforms into (PDP). For all a, β in P,

$$\lim_n \|a\circ F_n \circ \beta\|_p = 1.$$

Also (LKΠ) transforms into (ΠKΠ). For each s, t in G,

$$\lim_n \|l_s F_n r_t - F_n\|_p = 0.$$

To prove that (PDP) implies (ΠKΠ) it suffices to note that if (ΠKΠ) fails, then one of (LKΠ) or (RKΠ) fails; if the former, then take a in P as in the corresponding proof of left conditions and take β arbitrary in P to show that (PDP) fails.

Proof of Theorem 4. In cases where we can not apply the easy cases of Diagram A, we will need to work to find a net which with its other properties is left equicontinuous in the appropriate space. The easy places to achieve this equicontinuity without assuming it in advance are in conditions (lsap), (lkp), (lnp), and (ldp).

I know no direct way to get equicontinuity in (lsaπ) so I begin this proof with the drudgery of known special proofs for (lsaπ) $\to$ (laπ) $\to$ (la$_0$) $\to$ (lap$_0$) $\to$ (lap) $\to$ (lsap).

We begin by observing that each condition π or u can be converted into a condition on a net. Let n be the ordered pair (A, ε), where A is a finite (or compact) set in G and $\varepsilon > 0$; define $n \geqslant n'$ to mean $A \supseteq A'$ and $\varepsilon \leqslant \varepsilon'$. Then (lsa$\pi$), for example, becomes (LSAΠ), there is a net φ_n for which

$$\lim_n \|l_s \varphi_n - \varphi_n\|_1 = 0.$$

(lsaπ) implies (laπ). This is just like "strong amenability implies amenability" in the discrete case ([1], §5, Theorem 1); that is, if φ_n satisfies (LSAΠ), the images $Q\varphi_n$ are in the w*-compact set M, so there is a w*-convergent subnet, $Q\varphi_{n_i}$ with limit m. It is easily seen that

$$\lim_n (l_s \varphi_{n_i} - \varphi_{n_i}) = 0$$

implies $l_{s^{-1}} * m = m$, so (lsaπ) implies (laπ). (The same proof works for m or p.)

(laπ) implies (la$_0$) simply by taking m_0 to be the restriction of m to C_u.

(la$_0$) implies (lap$_0$) (see Greenleaf [8]). Given that for every y in C_u and s in G, $m_0(l_s y) = m_0(y)$, and given any α in P, take β in P_{00} within ε of α (in the L_1-metric). Then

$$\|\alpha \circ y - \beta \circ y\|_\infty \leqslant \|\alpha - \beta\|_1 \|y\|_\infty < \varepsilon \|y\|_\infty.$$

Now for each g in G,

$$[\beta \circ y](g) = \int_G \beta(t)[l_{t^{-1}} y](g)\,dt.$$

Because y is left equicontinuous in L_∞, there exist finitely many small sets E_i covering the support of β, and points τ_i in E_i such that $\|l_{t^{-1}} y - l_{\tau_i^{-1}} y\|_\infty < \varepsilon$ if $t \in E_i$.

Then

$$\Big\| \int_G \beta(t) l_{t^{-1}} y\,dt - \sum_i \int_{E_i} \beta(t) l_{\tau_i^{-1}} y\,dt \Big\|_\infty$$

$$\leqslant \sum_i \int_{E_i} \beta(t) \|l_{t^{-1}} y - l_{\tau_i^{-1}} y\|_\infty\,dt \leqslant \|\beta\|_1 \varepsilon = \varepsilon.$$

If we let

$$\beta_i = \int_{E_i} \beta(t)\,dt,$$

then we have

$$\|\alpha \circ y - \sum_i \beta_i l_{\tau_i^{-1}} y\|^\infty < 2\varepsilon.$$

But

$$m_0(\sum_i \beta_i l_{\tau_i^{-1}} y) = \sum_i \beta_i m_0(l_{\tau_i^{-1}} y) = \sum_i \beta_i m_0(y) = m_0(y).$$

Hence $|m_0(\alpha \circ y) - m_0(y)| < 2\varepsilon$ for all $\varepsilon > 0$; therefore $m_0(\alpha \circ y) = m_0(y)$ for all α in P and y in C_u.

(lap$_0$) implies (lap). (Hulanicki [10] gives Ryll-Nardzewski credit for the basic idea here.) Choose α, β in P_{00}; in particular choose $\beta = \beta^\sim$. Then for each x in L_∞, $\alpha \circ x \circ \beta$ is in C_u. Define $m(x)$ to be $m_0(\alpha \circ x \circ \beta)$. Then

(i) m is independent of α in P_{00}.

Consider $m_0(\alpha \circ x \circ \beta)$ and $m_0(\gamma \circ x \circ \beta)$ with α and γ in P_{00}: then (see Hulanicki [10]) given $\varepsilon > 0$ there exists ϱ in P_{00} such that $\|\alpha \circ \varrho - \alpha\|_1 < \varepsilon$ and $\|\gamma \circ \varrho - \gamma\|_1 < \varepsilon$, because ϱ's with small support form an approximate identity in L_1 ([9], p. 303). Then $m_0(\alpha \circ \varrho \circ x \circ \beta) = m_0(\varrho \circ x \circ \beta)$, by (lap$_0$). Therefore

$$|m_0(\alpha \circ x \circ \beta) - m_0(p \circ x \circ \beta)| = |m_0(\alpha \circ x \circ \beta) - m_0(\alpha \circ \varrho \circ x \circ \beta)|$$

$$\leqslant \varepsilon \|x \circ \beta\|_\infty \leqslant \|x\| \, \|\tilde{\beta}\|_1 \varepsilon = \varepsilon \|x\|_\infty.$$

Similarly, $|m_0(\gamma \circ x \circ \beta) - m_0(\varrho \circ x \circ \beta)| \leqslant \varepsilon \|x\|_\infty$, so

$$|m_0(\alpha \circ x \circ \beta) - m_0(\gamma \circ x \circ \beta)| \leqslant 2\varepsilon \|x\|_\infty$$

for all $\varepsilon > 0$.

Hence $m_0(\alpha \circ x \circ \beta) = m_0(\gamma \circ x \circ \beta)$ for all α, γ in P_{00}.

(b) Returning now to our original α and β fixed in P_{00}, we have for each γ in P_{00},

$$m(\gamma \circ x) = m_0(\alpha \circ \gamma \circ \beta) = m_0(\alpha \circ x \circ \beta) = m(x),$$

so (lap) holds for γ in P_{00}. But P_{00} is norm dense in the L_1-norm in P, so (lap) holds with all γ in P.

(lap) implies (lsap). As we pointed out earlier, QP_{00} is w*-dense in M, so there exists a net φ_n in P_{00} such that $\text{w*-}\lim_n Q\varphi_n = m$. Then it can be verified directly (as in [1], §5, Lemma 1) that $(\alpha\circ)^*(m) = m$ if and only if

$$w\text{-}\lim_n (\alpha \circ \varphi_n - \varphi_n) = 0.$$

But (Day [1], §5, Lemma 4) we have

LEMMA 2. *If L is a locally convex linear topological space and if (d_n) is a net of elements of L weakly convergent to an element z of L, then there is net (c_m) of finite averages of elements far out in (d_n) such that (c_m) converges to z in the original topology of L.*

Namioka greatly simplified the application of this lemma to the present problem. We apply the lemma to the net (d_n) defined in the topological product space L_1^P by $d_n = d(n, \alpha) = \alpha \circ \varphi_n - \varphi_n$. This net converges to zero in the product of the weak topologies, so there is a net c_m of averages of elements far out in d_n which coverges in the strong topology of L_1^P, but this topology is the product of the norm topology in the individual copies of L_1 so every coordinate of c_m converges in the norm topology of L_1. But if c_m is a certain average $\sum k_n d_n$ let $\theta_m = \sum k_n \varphi_n$; then $c_m = c(m, a) = a \circ \theta_m - \theta_m$. Then

$$\lim_m \|a \circ \theta_m - \theta_m\|_1 = 0$$

for all a in P. That is (lap) $\to$ (LSAP). (The same proof works for m or Π.)

To continue with the implications where equicontinuity is not important, consider ($g\pi$). As we pointed out earlier, ($g\pi$) implies that there exists a net f_n of elements of L_2 such that $f_n \circ f_n^{\sim}$ converges pointwise to 1. But $(\|f_n\|_2)^2 = f_n \circ f_n^{\sim}(u)$ which tends to 1, so we can replace f_n by $F_n = f_n/\|f_n\|_2$. If $h_n(t) = |F_n(t)|$ for all t in G, then $(\|h_n\|_2)^2 = 1$ and

$$1 \geqslant h_n \circ h_n^{\sim}(t) \geqslant |F_n \circ F_n^{\sim}(t)|$$

for all t in G. Hence h_n is a net satisfying (GΠ).

The results recently proved and the parts of Theorem 1 shown in Diagram A which do not require equicontinuity now allow us to fill in the pattern of implications shown in Diagram B.

To proceed farther we need equicontinuity. From (lsap) we had a net θ_n which satisfied (LSAP). To construct from it a net left equicontinuous in L_1 we need only take one fixed β from P_{00} and let $\varphi_n = \beta \circ \theta_n$. Then

$$l_s\varphi_n - \varphi_n = l_s(\beta \circ \theta_n) - \beta \circ \theta_n = (l_s\beta) \circ \theta_n - \beta \circ \theta_n = (l_s\beta - \beta) \circ \theta_n.$$

If for an $\varepsilon > 0$, W_ε is chosen so that $\|l_s\beta - \beta\|_1 < \varepsilon$, then the same W_ε can be used for every φ_n because

$$\|(l_s\beta - \beta) \circ \theta_n\|_1 \leqslant \|l_s\beta - \beta)\|_1 \|\theta_n\|_1.$$

We must now check that the new φ_n approximate strongly to left-P-invariance. Because $a \circ \beta$ is in P if a and β are,

$$\|a \circ \varphi_n - \varphi_n\|_1 = \|a \circ (\beta \circ \theta_n) - \beta \circ \theta_n\|_1 \leqslant \|(a \circ \beta) \circ \theta_n - \theta_n\|_1 + \|\theta_n - \beta \circ \theta_n\|_1$$

which both tend to zero as n increases.

Now that we know that (lsap) implies existence of a net φ_n satisfying (LSAP) and equicontinuity in L_1, Theorem 1 enables us to draw arrows in Diagram B leading directly to any condition but those in the upper group of four. The last gap will be closed when we prove that (ldp) implies (LDP) with equicontinuity.

Take any $p > 1$ and choose β in P_{00}. For each finite set $A = \{a_1, a_2, \ldots, a_k\} \subset P_{00}$ form $\gamma = \sum_{i \leqslant k} a_i/k$. Then $\gamma \circ \beta$ is in P, so (ldp) implies that for $\varepsilon = 1/k$, there is an f_A of norm 1 in L_p such that $\|\gamma \circ \beta \circ f_A\|_p > 1 - \varepsilon/k$. f_A may be assumed to be non-negative, because for every g in G

$$\Big|\int_G (\gamma \circ \beta)(t) f_A(t^{-1}g)\,dt\Big| \leqslant \int_G (\gamma \circ \beta)(t)|f_A(t^{-1}g)|\,dt.$$

Hence

$$1 - \varepsilon/k < \|\gamma \circ \beta \circ f_A\|_p = \Big\|\sum_i a_i \circ \beta \circ f_A\Big\|_p/k \leqslant \sum_i \|a_i \circ \beta \circ f_A\|_p/k$$
$$\leqslant (k-1)/k + \|a_i \circ \beta \circ f_A\|_p/k = 1 - 1/k + \|a_i \circ \beta \circ f_A\|_p/k.$$

Comparing ends we see that for each $i \leqslant k$,

$$\|a_i \circ \beta \circ f_A\|_p > 1 - \varepsilon.$$

Letting $F_A = (\beta \circ f_A)/\|\beta \circ f_A\|$, we have $\|F_A\|_p = 1$, but $\|\beta \circ f_A\|_p$ tends to 1 as A increases, so there is A_0 such that $\|\beta \circ f_A\|_p \geqslant 1/2$ if $A \supseteq A_0$. The functions $\beta \circ f_A$ are left equicontinuous in L_p, so the F_A, $A \supseteq A_0$, are also left equicontinuous in L_p. Hence (ldp) implies the existence of a left equicontinuous net F_A of non-negative elements of norm one in L_p such that $\|a \circ F_A\|_p$ tends to 1 for each a in P. This is (LDP) and Theorem 1 says that (ldp) implies all the conditions in the main body of the Diagram B. This completes the proof of Theorem 4.

§ 4. FALSE HOPES

For the benefit of those who may wonder whether amenability is equivalent to some other interesting properties of locally compact groups, we add the following known counterexamples.

(1) Abelian groups and compact groups are unimodular, but amenability does not imply unimodularity.

The group of real matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, $x \neq 0$, is not unimodular ([9], p. 201) but it is an extension of one abelian group by another, so it is amenable even as a discrete group ([1], §4, (E) and (H)).

(2) Abelian groups and compact groups are Type I (see Dixmier [5] for definition). But Mautner [12] displayed a very simple group, the restricted direct product of infinitely many finite non-abelian groups, which is type II$_1$. The group is also amenable ([2], §4, (I) and (F'')).

(3) Although all abelian and compact groups are of finite type, amenable groups are no better than others. Dixmier [5], p. 272, §13.10.4, reports that (a) the von Neumann algebras generated by left and right shifts in $L_2(G)$ are always semi-finite (which implies no type III part), and (b) that these algebras are of finite type if and only if there is a fundamental system of neighborhoods of the identity in G invariant under all inner automorphisms of G. Godement [7] proved that this last condition implies that the group is unimodular, and we know from (1) that amenability

does not imply that. On the other hand, all discrete groups have such invariant neighborhoods, $W = \{u\}$, so all discrete groups, whether they are amenable or not, have regular representations of finite type.

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Concerning extension of multiplicative linear functionals in Banach algebras

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A commutative complex Banach algebra A *has the ES-property* (Extension from Subalgebras) or *belongs to the class ES* (written as $A \in$ ES) if for every its (closed) subalgebra $A_0 \subset A$ and every multiplicative linear functional f defined on A_0 there exists a multiplicative linear functional F defined on A such that its restriction to A_0 equals f. In other words, $A \in$ ES if and only if every multiplicative linear functional in any subalgebra of A is extentable to such a functional defined on A. Clearly, any subalgebra of a member of ES also belongs to this class. In this paper we characterize the class ES in terms of spectra of elements of algebras in this class. Our main result reads as follows:

THEOREM 1. *A Banach algebra A belongs to the class ES if and only if for every element $x \in A$ its spectrum $\sigma(x)$ is a totally disconnected subset of the complex plane.*

To illustrate this theorem we show that for any compact group G the group algebra $L_1(G)$ belongs to the class ES. (For related results see also [1] and [3].)

Let A be a commutative complex Banach algebra with unit e. We shall write $M(A)$ for the (compact) maximal ideal space of A provided with the Gelfand topology. The spectral (semi-) norm $\|x\|_s$ is defined as

$$\|x\|_s = \sup_{f \in M(A)} |f(x)| = \sup_{M(A)} |x^\wedge(f)| = \sup |(\sigma x)| \leqslant \|x\|,$$

where $x^\wedge(f) = f(x)$ is the Gelfand transform of $x \in A$. If p is any complex polynomial in one variable, then for any $x \in A$

$$\sigma(p(x)) = p(\sigma(x)),$$

and so

$$\|p(x)\|_s = \sup_{t \in \sigma(x)} |p(t)|. \tag{1}$$