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Smooth operators and commutators*

by

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1. Introduction. In a previous paper [4] we introduced, in connection with the problem of wave operators in scattering theory, the notion of *T-smooth operators* when T is a linear operator of a certain type in a Hilbert space $\mathfrak{H}$. In the present paper we propose to study further properties of T-smooth operators when T is selfadjoint and bounded. In particular, we shall determine all T-smooth operators when $\mathfrak{H}$ is separable. Roughly speaking, A is T-smooth if and only if A^*A is an integral operator with L^∞-kernel in a representation in which T is diagonal.

The results are applied to the *commutators* of two bounded selfadjoint operators H, K. In particular, we are able to determine all systems $\{H, K\}$ in a separable space $\mathfrak{H}$ such that $i(HK-KH) = L \geqslant 0$. Moreover, we shall show that L is necessarily in the trace class if H or K has finite spectral multiplicity. The problem is also related to *seminormal operators*. In this way we supplement and improve some of the results due to Putnam and others, for which we refer to a recent book [5].

We collect here some definitions and notations used in the paper (for details see [3] or [5]). Let T be a selfadjoint operator in $\mathfrak{H}$. We denote by $\{E_T(\lambda)\}$ the spectral family for T. We denote also by $E_T\colon S\to E_T(S)$ the corresponding spectral measure defined for Borel sets S on the real line R. $x\in\mathfrak{H}$ is *absolutely continuous* [*singular*] *with respect to T*, or *T-absolutely continuous* [*T-singular*] for short, if the measure $m_x\colon S\to m_x(S) = (E_T(S)x, x)$ is absolutely continuous [singular] with respect to the Lebesgue measure, which we denote by $|S|$. The set of all T-absolutely continuous [T-singular] vectors is the *subspace of T-absolute continuity* [*T-singularity*] and is denoted by $\mathfrak{H}_{ac}(T)$ [$\mathfrak{H}_s(T)$]. These two subspaces are orthogonal complements to each other. The associated projections are denoted by $P_{ac}(T)$ and $P_s(T) = 1-P_{ac}(T)$, and the parts of T in them by T_{ac} and T_s, respectively. T is (*spectrally*) *absolutely continuous* [*singular*] if $\mathfrak{H}_s(T)=\{0\}$ [$\mathfrak{H}_{ac}(T)=\{0\}$].

We denote by $\mathrm{sp}(T)$ the spectrum of T and by $\mathrm{int}(T)$ the smallest closed interval containing $\mathrm{sp}(T)$. When T is absolutely continuous, we define the *support* of T, denoted by $\mathrm{supp}(T)$, as a Borel set S_1 with $E_T(S_1) = 1$ and with the smallest possible $|S_1|$; S_1 is uniquely determined

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by T up to a null set. To construct S_1, consider the set of measures m_x introduced above for all $x \in \mathfrak{H}$. Since all m_x are absolutely continuous, $\{m_x\}$ has a supremum m in the sense of order relation for measures (see [2]). Since m is absolutely continuous too, its support S_1 can be defined as a Borel set such that m is equivalent to the Lebesgue measure restricted on S_1. S_1 is the desired support of T.

Clearly we have the inclusions $\operatorname{supp}(T) \subset \operatorname{sp}(T) \subset \operatorname{int}(T)$.

We denote by $\mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ the set of all bounded linear operators from $\mathfrak{H}$ to $\mathfrak{H}'$ with domain $\mathfrak{H}$, where $\mathfrak{H}'$ is another Hilbert space which may coincide with $\mathfrak{H}$. We write $\mathscr{B}(\mathfrak{H})$ for $\mathscr{B}(\mathfrak{H}, \mathfrak{H})$. If $A \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ and if $\mathfrak{H}_0$ is any subspace of $\mathfrak{H}$, the *part of A on $\mathfrak{H}_0$* is an operator $A_0 \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}')$ such that $A_0 x = Ax$ for all $x \in \mathfrak{H}_0$.

We also need some notions related to the measurability of complex-valued, $\mathfrak{H}$-valued or $\mathscr{B}(\mathfrak{H}, \mathfrak{H}')$-valued functions on R or on a Borel set S on R. Here $\mathfrak{H}$ and $\mathfrak{H}'$ are assumed to be separable and the measurability always refers to Borel sets. An $\mathfrak{H}$-valued function $u(\lambda)$ is *measurable* if $(u(\lambda), x)$ is measurable for each $x \in \mathfrak{H}$; we need not distinguish between strong and weak measurability. A $\mathscr{B}(\mathfrak{H}, \mathfrak{H}')$-valued function $B(\lambda)$ is *measurable* if $B(\lambda)x$ is measurable for each $x \in \mathfrak{H}$, i.e. if $(B(\lambda)x, x')$ is measurable for each $x \in \mathfrak{H}$ and $x' \in \mathfrak{H}'$. $B(\lambda)$ is measurable if and only if $B(\lambda)^*$ is measurable. If $u(\lambda)$ and $v(\lambda)$ are measurable, then $(u(\lambda), v(\lambda))$ is measurable. If $u(\lambda)$ and $B(\lambda)$ are measurable, then $B(\lambda)\, u(\lambda)$ is measurable.

2. Smooth operators and absolute continuity. Let $\mathfrak{H}$ and $\mathfrak{H}'$ be Hilbert spaces. In [4] we defined T-smooth operators when T is a densely defined, closed operator in $\mathfrak{H}$ with spectrum on the real axis. In the following theorem we reproduce the basic properties of a T-smooth operator A in the special case that T is bounded and selfadjoint. (For the proof see [4], Definition 1.2, Lemma 3.6, and Theorem 5.1. It should be remarked that in [4] $\mathfrak{H}$ and $\mathfrak{H}'$ are assumed to be separable, but this is not necessary. It is easy to see this at least in the special case considered here.)

(2.1) THEOREM. *Let $T \in \mathscr{B}(\mathfrak{H})$ be selfadjoint and $A \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$. Then we have, with $R(\zeta) = (T-\zeta)^{-1}$,*

$$
\begin{aligned}
(2.2)\qquad \|A\|_T^2 &= \sup_x \frac{1}{2\pi} \int_{-\infty}^{\infty} \|Ae^{-itT}x\|^2 dt / \|x\|^2 \\
&= \sup_{x,\varepsilon} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left(\|AR(\lambda+i\varepsilon)x\|^2 + \|AR(\lambda-i\varepsilon)x\|^2\right) d\lambda / \|x\|^2 \\
&= \sup_{x,\varepsilon} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \|AR(\lambda+i\varepsilon)x - AR(\lambda-i\varepsilon)x\|^2 d\lambda / \|x\|^2 \\
&= \sup_I \|AE_T(I)\|^2/|I|,
\end{aligned}
$$

where the suprema are taken over all $0 \neq x \in \mathfrak{H}$, $\varepsilon > 0$, and all finite intervals I on R. A is said to be T-smooth if $\|A\|_T < \infty$.

The following theorem is a simple consequence of the definition of $\|A\|_T$:

(2.3) THEOREM. *Let T, A be as in Theorem 2.1. Let T_0 be the part of T in a reducing subspace $\mathfrak{H}_0$ of $\mathfrak{H}$. If $A_0 \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}')$ is the part of A on $\mathfrak{H}_0$, then $\|A_0\|_{T_0} \leqslant \|A\|_T$. A_0 is T_0-smooth if A is T-smooth.*

The following theorem shows that T-smoothness is essentially related to T_{ac} only:

(2.4) THEOREM. *Let $T \in \mathscr{B}(\mathfrak{H})$ be selfadjoint and let $A \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ be T-smooth. Then A annihilates $\mathfrak{H}_s(T)$ (i.e. $Ax = 0$ for $x \in \mathfrak{H}_s(T)$). For any bounded measurable function Φ on R, we have*

$$(2.5)\qquad \|A\Phi(T)\| \leqslant \|A\|_T \|\Phi\|_{L^2(S_1)}, \qquad S_1 = \operatorname{supp}(T_{ac}).$$

In particular

$$(2.6)\qquad \|AE_T(S)\| \leqslant \|A\|_T |S \cap S_1|^{1/2}, \qquad \|A\| \leqslant \|A\|_T |S_1|^{1/2}.$$

Proof. The last equality in (2.2) implies that $\|AE_T(I)A^*\| = \|AE_T(I)\|^2 \leqslant \|A\|_{T^2} |I|$, so that $AE_T(\lambda)A^*$ is Lipschitz continuous in λ. It follows that

$$(2.7)\qquad \|AE_T(S)\|^2 = \|E_T(S)A^*\|^2 = \|AE_T(S)A^*\| \leqslant \|A\|_T^2 |S|$$

for any Borel set S. (2.7) implies that $Ax = 0$ if $x \in \mathfrak{H}_s(T)$, for then $x = E_T(S)x$ for some null set S (see [3], p. 517). Consequently, we may assume T to be absolutely continuous in the remainder of the proof. Let $S_1 = \operatorname{supp}(T_{ac}) = \operatorname{supp}(T)$.

Now $E_T(S) = E_T(S \cap S_1)$ so that (2.6) follows from (2.7). Similarly we obtain for each $x' \in \mathfrak{H}'$

$$
\begin{aligned}
\|\Phi(T)^* A^* x'\|^2 &= \int |\Phi(\lambda)|^2 d(E_T(\lambda)A^*x', A^*x') \\
&\leqslant \|A\|_T^2 \|x'\|^2 \int_{S_1} |\Phi(\lambda)|^2 d\lambda = \|A\|_T^2 \|x'\|^2 \|\Phi\|_{L^2(S_1)}^2.
\end{aligned}
$$

This proves (2.5).

3. The Γ -operation. Let T, $X \in \mathscr{B}(\mathfrak{H})$ with T selfadjoint. The following notation was introduced in [3], p. 552:

$$(3.1)\qquad \Gamma_T^{\pm} X = i \int_0^{\pm\infty} e^{itT} X e^{-itT} dt.$$

$\Gamma_T^{\pm} X$ is defined if and only if the integral in (3.1) exists as a strong limit. As was shown in [3], $\Gamma_T^{\pm}$ are inverse to the commutator operation in the sense that $Y^{\pm} = \Gamma_T^{\pm} X$ implies that $Y^{\pm} T - TY^{\pm} = X$.

(3.2) THEOREM. *Let T and X be as above. In order that both $\Gamma_T^{\pm} X$ exist, it is sufficient that $X = A_2^* A_1$, where A_1 and A_2 belong to $\mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ for some $\mathfrak{H}'$ and are T-smooth; then $\|\Gamma_T^{\pm} X\| \leqslant 2\pi \|A_1\|_T \|A_2\|_T$. If, in particular, X is selfadjoint and $X \geqslant 0$, then the stated condition is also necessary.*

Proof. We first consider the case $X \geqslant 0$. Then $e^{itT} X e^{-itT} \geqslant 0$ so that the two integrals in (3.1) exist in the strong sense if and only if the integral taken on any finite interval is uniformly bounded (see [6], p. 263). In view of the first expression for $\|A\|_T$ in (2.2), this is exactly the condition that $X^{1/2}$ be T-smooth. Setting $A_1 = A_2 = X^{1/2}$ and $\mathfrak{H}' = \mathfrak{H}$, we see that the condition of the theorem is necessary for the existence of both $\Gamma_T^{\pm} X$.

To prove the sufficiency of the condition, we note that for every $x, y \in \mathfrak{H}$

$$\left| \int_{s'}^{s''} (e^{itT} A_2^* A_1 e^{-itT} x, y)\, dt \right| \leqslant \Big[\int_{s'}^{s''} \|A_1 e^{-itT} x\|^2 dt\Big]^{1/2} \Big[\int_{s'}^{s''} \|A_2 e^{-itT} y\|^2 dt\Big]^{1/2}$$
$$\leqslant (2\pi)^{1/2} \|A_2\|_T \|y\| \Big[\int_{s'}^{s''} \|A_1 e^{-itT} x\|^2 dt\Big]^{1/2}.$$

Hence for $X = A_2^* A_1$

$$\left\| \int_{s'}^{s''} e^{itT} X e^{-itT} x\, dt \right\| \leqslant (2\pi)^{1/2} \|A_2\|_T \Big[\int_{s'}^{s''} \|A_1 e^{-itT} x\|^2 dt\Big]^{1/2}.$$

Since A_1 is T-smooth, the right member tends to zero as s', s'' both tend to ∞ or to $-\infty$. Hence $\Gamma_T^{\pm} X$ exist. It follows also from the last inequality that $\|\Gamma_T^{\pm} X\| \leqslant 2\pi \|A_1\|_T \|A_2\|_T$.

4. Representation as integral operators. We now determine the structure of all T-smooth operators when $T \in \mathscr{B}(\mathfrak{H})$ is selfadjoint and $\mathfrak{H}$ is separable. We shall show that A is T-smooth if and only if $B = A^* A$ is an integral operator with bounded kernel in a representation in which T is "diagonal". Thus we determine $B = A^* A$ rather than A itself. But this also determines A completely, for the polar decomposition of A shows that $A = W B^{1/2}$, where W is a partial isometry on $\mathfrak{H}$ to $\mathfrak{H}'$ with initial set equal to the closure of the range of B.

We first consider the diagonalization of T. Since we are interested in the T-smoothness of A, we may assume that T is absolutely continuous (see Theorem 2.4). Thus the system $\{\mathfrak{H}, T\}$ may be identified with the following canonical system (see [2] or [7]). $\mathfrak{H}$ is the direct sum of function spaces $\mathfrak{H}_j = L^2(M_j; \mathfrak{P}_j)$, $j = 1, 2, \ldots$, and ∞, where the M_j are disjoint Borel sets on a finite interval of R and the $\mathfrak{P}_j$ are separable Hilbert spaces with $\dim \mathfrak{P}_j = j$. $L^2(M; \mathfrak{P})$ denotes the L^2-space (with respect to the Lebesgue measure) of $\mathfrak{P}$-valued functions on M. T is the multiplication by the coordinate in each of the $\mathfrak{H}_j$: $Tu(\lambda) = \lambda u(\lambda)$ for $u \in \mathfrak{H}_j$ for all j.

We find it convenient to assume further that $\mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \ldots \subset \mathfrak{P}_\infty \equiv \mathfrak{P}$, so that all $\mathfrak{H}_j$ and $\mathfrak{H}$ itself may be regarded as subspaces of $\tilde{\mathfrak{H}} = L^2(S_1; \mathfrak{P})$, where $S_1 = \bigcup_j M_j = \operatorname{supp}(T)$ is a bounded Borel set. Thus $\mathfrak{H}$ is the subspace of $\tilde{\mathfrak{H}}$ consisting of all functions u such that $u(\lambda) \in \mathfrak{P}_j \subset \mathfrak{P}$ whenever $\lambda \in M_j$, $j = 1, 2, \ldots, \infty$. T is the part in $\mathfrak{H}$ of $\tilde{T}$ defined by $\tilde{T} u(\lambda) = \lambda u(\lambda)$ for all $u \in \tilde{\mathfrak{H}}$.

Now we can define exactly the integral operator in $\mathfrak{H}$ with bounded kernel. Let $B(\lambda, \mu)$ be a $\mathscr{B}(\mathfrak{P})$-valued measurable function on $S_1 \times S_1$ and assume that $\operatorname*{ess\,sup}_{\lambda,\mu} \|B(\lambda, \mu)\| = b < \infty$. Then

$$\tilde{B} u(\lambda) = \int_{S_1} B(\lambda, \mu) u(\mu)\, d\mu \tag{4.1}$$

obviously defines an operator $\tilde{B} \in \mathscr{B}(\tilde{\mathfrak{H}})$ such that

$$\|\tilde{B}\| \leqslant b\, |S_1|. \tag{4.2}$$

We shall call $\tilde{B}$ the *integral operator in* $\tilde{\mathfrak{H}}$ with kernel $B(\lambda, \mu)$.

Suppose now that the kernel $B(\lambda, \mu)$ is *adapted* to the subspace $\mathfrak{H}$ of $\tilde{\mathfrak{H}}$ in the sense that $\lambda \in M_j$ and $\mu \in M_k$ imply that $B(\lambda, \mu)$ annihilates $\mathfrak{P} \ominus \mathfrak{P}_k$ and has range in $\mathfrak{P}_j$ (so that $B(\lambda, \mu)$ is essentially in $\mathscr{B}(\mathfrak{P}_k, \mathfrak{P}_j)$). Then it is easy to see that B is reduced by $\mathfrak{H}$. The part B of $\tilde{B}$ in $\mathfrak{H}$ will be called the *integral operator in* $\mathfrak{H}$ with kernel $B(\lambda, \mu)$.

In what follows $\mathfrak{H}'$ is also assumed to be separable.

(4.3) THEOREM. *Let $\mathfrak{H}$ and T be as above. $A \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ is T-smooth if and only if $B = A^* A$ is an integral operator in $\mathfrak{H}$ with kernel $B(\lambda, \mu)$ which is adapted to $\mathfrak{H}$ and for which $\operatorname*{ess\,sup}_{\lambda,\mu} \|B(\lambda, \mu)\| = b < \infty$. In this case we have $\|A\|_T^2 = b$.*

(4.4) THEOREM. *Let $\mathfrak{H}$ and T be as in Theorem 4.3. If $A_1, A_2 \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ are T-smooth, then $B = A_2^* A_1$ is an integral operator in $\mathfrak{H}$ with kernel $B(\lambda, \mu)$ such that $\operatorname{ess\,sup} \|B(\lambda, \mu)\| \leqslant \|A_1\|_T \|A_2\|_T$.*

(4.5) THEOREM. *Let $\mathfrak{H}$ and T be as in Theorem 4.3 and let $A \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ be T-smooth. Then*

$$s(A)^2 \leqslant \|A\|_T^2 \sum_{j=1}^{\infty} j\, |M_j|, \tag{4.6}$$

where $s(A)$ denotes the Schmidt norm of A. A is in the Schmidt class, and $B = A^* A$ is in the trace class, if the right member of (4.6) is finite. In particular, this is true if T has finite spectral multiplicity.

(4.7) THEOREM. *In Theorem 4.5 assume that T has simple spectrum, so that $\mathfrak{H}$ may be identified with $L^2(S_1)$ of scalar functions. Then $\|A\| = \|A\|_T |S_1|^{1/2}$ is true if and only if A is of the form $A = (\ , f) g$ where $g \in \mathfrak{H}'$*

and $f \epsilon \mathfrak{H} = L^2(S_1)$ *is represented by a function* $f(\lambda)$ *with* $|f(\lambda)| =$ const. (*Note that in general* $\|A\| \leqslant \|A\|_T |S_1|^{1/2}$ *by* (2.6)).

(4.8) REMARKS. 1. Theorem 4.3 shows that there are many smooth operators. In fact for any bounded selfadjoint operator T in a separable space $\mathfrak{H}$ with $T_{ac} \neq 0$, there exist many non-trivial T-smooth operators. The boundedness of T and separability of $\mathfrak{H}$ are not essential, for one can always consider a bounded part in a separable subspace of an arbitrary selfadjoint operator.

2. An example of operators satisfying the condition of Theorem 4.7 is given in [5], p. 137.

3. For the B of Theorem 4.4, $\Gamma_T^{\pm} B$ exist by Theorem 3.2. It is easily seen that *formally* $\Gamma_T^{\pm} B$ are "integral operators" with kernels

$$\pm i\pi\delta(\lambda-\mu)\, B(\lambda, \lambda)+(\mu-\lambda)^{-1}B(\lambda, \mu).$$

The first term represents a multiplication operator in $\mathfrak{H}$. The second term is a singular kernel. Here we do not discuss the meaning of this kernel (cf. [1] and [5]).

Theorems 4.3 and 4.4 will be proved in the following section. Here we deduce Theorems 4.5 and 4.7 from Theorem 4.3.

To prove Theorem 4.5, we first prove (4.6) in the special case that T has simple spectrum, so that $|M_1| = |S_1|$ and $|M_j| = 0$ for $j \geqslant 2$. According to Theorem 4.3, $B = A^*A$ is then an integral operator with a scalar kernel bounded by $b = \|A\|_T^2$. Since $B \geqslant 0$, we have $s(A)^2 =$ trace $B \leqslant b|S_1| = \|A\|_T^2 |M_1|$.

In the general case, we note that $s(A)^2 = \sum \|Ae_n\|^2$ for any complete orthonormal family $\{e_n\}$ in $\mathfrak{H}$. Choosing $\{e_n\}$ adapted to the decomposition $\mathfrak{H} = \sum \oplus \mathfrak{H}_j$, we see that $s(A)^2 = \sum s(A_j)^2$, where A_j is the part of A on $\mathfrak{H}_j$. Since $\mathfrak{H}_j$ is in turn the direct sum of j copies H_{jk} of $L^2(M_j)$ each reducing T, we have

$$s(A_j)^2 = \sum_{k=1}^{j} s(A_{jk})^2,$$

where A_{jk} is the part of A on $\mathfrak{H}_{jk}$. Since the part T_{jk} of T in $\mathfrak{H}_{jk}$ has simple spectrum with $\mathrm{supp}(T_{jk}) = M_j$, we have

$$s(A_{jk})^2 \leqslant \|A_{jk}\|_{T_{jk}}^2 |M_j|$$

by what was proved above. Since $\|A_{jk}\|_{T_{jk}} \leqslant \|A\|_T$ by Theorem 2.3, we obtain (4.6) by adding up these results.

To prove Theorem 4.7, we note that $s(A)^2 \leqslant \|A\|_T^2\, |S_1|$ by (4.6). Since $\|A\| \leqslant s(A)$, we have $\|A\| = \|A\|_T |S_1|^{1/2}$ only if $\|A\| = s(A)$. This occurs only if A has rank one or zero so that it can be written $A = (\ \ , f)g$. Then

$A^*A = \|g\|^2(\ \ , f)f$ is an integral operator with kernel $\|g\|^2 f(\lambda)\overline{f(\mu)}$. It follows from Theorem 4.3 that

$$\|A\|_T^2 = \text{ess sup } \|g\|^2\, |f(\lambda)\overline{f(\mu)}| = \|g\|^2\, \|f\|_\infty^2,$$

where $\|\ \ \|_\infty$ denotes the L^∞-norm. On the other hand, $\|A\| = \|f\|\, \|g\|$. Thus $\|A\| = \|A\|_T |S_1|^{1/2}$ is true only if $\|f\|^2 = \|f\|_\infty^2 |S_1|$, which is the case only if $|f(\lambda)| =$ const almost everywhere on S_1. Conversely, it is easily seen that this condition is also sufficient for the equality.

5. Proof of the representation theorem. Define $\tilde{A} \epsilon \mathscr{B}(\tilde{\mathfrak{H}}, \mathfrak{H}')$ by $\tilde{A}x = Ax$ for $x \epsilon \mathfrak{H}$ and $\tilde{A}x = 0$ for $x \epsilon \tilde{\mathfrak{H}} \ominus \mathfrak{H}$. It is easy to see that it suffices to prove Theorem 4.3 for the system $\{\tilde{\mathfrak{H}}, \tilde{T}, \tilde{A}\}$ instead of $\{\mathfrak{H}, T, A\}$. Thus we may omit $\sim$ and assume that $\mathfrak{H} = L^2(S_1; \mathfrak{P})$. We use the notation $u = \{u(\lambda)\}$ to indicate that $u \epsilon \mathfrak{H}$ is represented by the function $u(\lambda)$.

(5.1) LEMMA. *Let* $\mathfrak{H} = L^2(S_1; \mathfrak{P})$, T *the multiplication operator by* λ *in* $\mathfrak{H}$, *and let* $A \epsilon \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ *be* T*-smooth. Then there is a* $\mathscr{B}(\mathfrak{H}', \mathfrak{P})$*-valued measurable function* $F(\lambda)$ *defined on* S_1 *such that* $\|F(\lambda)\| \leqslant \|A\|_T$ *and* $A^*x' = \{F(\lambda)x'\}$ *for every* $x' \epsilon \mathfrak{H}'$.

Proof. Set

$$A^*x' = y = \{y(\lambda)\} \epsilon \mathfrak{H}$$

and

$$z(I) = \int_{I \cap S_1} y(\lambda) d\lambda \,\epsilon\, \mathfrak{P}$$

for every interval I on R. Then

$$\text{(5.2)} \qquad \|z(I)\| \leqslant |I|^{1/2}\Big[\int_{I \cap S_1} \|y(\lambda)\|^2 d\lambda\Big]^{1/2} = |I|^{1/2}\|E_T(I \cap S_1)y\| = |I|^{1/2}\|E_T(I)A^*x'\| \leqslant \|A\|_T |I|\, \|x'\|$$

by (2.2) (note that $\|E_T(I)A^*\| = \|AE_T(I)\|$).

Since $z(I)$ is an indefinite Bochner integral of $y(\lambda)$ (extended by $y(\lambda) = 0$ to $\lambda \notin S_1$), it is strongly differentiable almost everywhere (see [8], p. 134), in the sense that

$$z'(\lambda) = \lim z(I)/|I| \,\epsilon\, \mathfrak{P}$$

exists and equals $y(\lambda)$ for almost all λ when $I = (\lambda-\varepsilon, \lambda+\varepsilon)$ and $\varepsilon \downarrow 0$ (which we shall express by saying that I *shrinks* to $\{\lambda\}$).

Let $\{x_n'\}$ be a countable set everywhere dense in $\mathfrak{H}'$, and define the corresponding y_n, $z_n(I)$, and $z_n'(\lambda)$ as above. There exists a null set N such that $z_n'(\lambda)$ exists for all n if $\lambda \notin N$.

Let $x' \epsilon \mathfrak{H}'$ and $\lambda \notin N$. For any $\varepsilon > 0$ choose x_n' such that $\|x' - x_n'\| < \varepsilon$. Since the dependence of $z(I)$ on x' is linear,

$$\|z(I)-z_n(I)\| \leqslant \|A\|_T |I| \varepsilon$$

by (5.2) and hence

$$\limsup \| \, |I|^{-1} z(I) - z_n'(\lambda) \| \leqslant \|A\|_T \varepsilon$$

as I shrinks to $\{\lambda\}$. Thus

$$\limsup \| \, |I|^{-1} z(I) - |J|^{-1} z(J) \| \leqslant 2 \|A\|_T \varepsilon$$

as I and J both shrink to $\{\lambda\}$. Since $\varepsilon > 0$ was arbitrary, the left-hand member of this inequality must vanish. This means that $z'(\lambda)$ exists for $\lambda \notin N$ for all $x' \in \mathfrak{H}'$.

Obviously $z'(\lambda)$ depends linearly on x', and $\|z'(\lambda)\| \leqslant \|A\|_T \, \|x'\|$ by (5.2). Thus setting $F(\lambda)x' = z'(\lambda)$ defines $F(\lambda) \in \mathscr{B}(\mathfrak{H}', \mathfrak{P})$ for $\lambda \notin N$, with $\|F(\lambda)\| \leqslant \|A\|_T$. We set $F(\lambda) = 0$ for $\lambda \in N$. Since $F(\lambda)x' = z'(\lambda) = y(\lambda)$ almost everywhere, $F(\lambda)x'$ is measurable in λ and $A^* x' = y = \{F(\lambda)x'\}$.

(5.3) Lemma. *Under the assumptions of Lemma* 5.1, *we have*

$$Ax = \int_{S_1} F(\lambda)^* x(\lambda) d\lambda \quad \textit{for } x = \{x(\lambda)\} \in \mathfrak{H}. \tag{5.4}$$

Proof. Note that $F(\lambda)^* \in \mathscr{B}(\mathfrak{P}, \mathfrak{H}')$ so that $F(\lambda)^* x(\lambda) \in \mathfrak{H}'$ for each λ. $F(\lambda)^* x(\lambda)$ is measurable in λ, for $F(\lambda)$ and $x(\lambda)$ are measurable (see the end of Section 1). Since

$$\|F(\lambda)^* x(\lambda)\| \leqslant \|A\|_T \|x(\lambda)\|$$

by Lemma 5.1, $F(\lambda)^* x(\lambda)$ is Bochner integrable on S_1. Then (5.4) follows from the equality

$$\int \big(F(\lambda)^* x(\lambda), x'\big) d\lambda = \int \big(x(\lambda), F(\lambda)x'\big) d\lambda = (x, A^* x') = (Ax, x'),$$

where use is made of $A^* x' = \{F(\lambda)x'\}$.

(5.5) Lemma. *Under the assumptions of Lemma* 5.1, $B = A^* A$ *is an integral operator in* $\mathfrak{H}$ *with kernel* $B(\lambda, \mu)$ *such that* $\|B(\lambda, \mu)\| \leqslant \|A\|_T^2$ *almost everywhere.*

Proof. It follows from the two preceding lemmas that if $y = A^* Ax$, then $y = \{y(\lambda)\}$ with

$$y(\lambda) = F(\lambda)Ax = F(\lambda) \int_{S_1} F(\mu)^* x(\mu) d\mu = \int_{S_1} B(\lambda, \mu) x(\mu) d\mu,$$

where $B(\lambda, \mu) = F(\lambda) F(\mu)^*$. Since

$$\|F(\lambda) F(\mu)^*\| \leqslant \|F(\lambda)\| \, \|F(\mu)\| \leqslant \|A\|_T^2,$$

the lemma is proved.

(5.6) Lemma. *Let* $B(\lambda, \mu)$ *be a* $\mathscr{B}(\mathfrak{P})$*-valued measurable function defined on* $S_1 \times S_1$ *and bounded:* $\|B(\lambda, \mu)\| \leqslant b < \infty$. *Then it defines an integral operator* B *in* $\mathfrak{H} = L^2(S_1; \mathfrak{P})$. *If* $B^* = B \geqslant 0$ *in addition, then* $B^{1/2}$ *is T-smooth with* $\|B^{1/2}\|_T^2 \leqslant b$.

Proof. We have already seen that B is well-defined (see (4.1) and (4.2)). If $x \in E_T(I)\mathfrak{H}$ so that $x(\lambda)$ has support on $I \cap S_1$, we have

$$|(Bx, x)| = \left| \int\int \big(B(\lambda, \mu) x(\lambda), x(\mu)\big) d\lambda d\mu \right| \leqslant b \Big(\int \|x(\lambda)\| d\lambda\Big)^2 \leqslant b|I| \, \|x\|^2.$$

If $B^* = B \geqslant 0$, it follows that $\|B^{1/2}x\|^2 \leqslant b|I| \, \|x\|^2$ for each $x \in E_T(I)\mathfrak{H}$. Hence $\|B^{1/2} E_T(I)\|^2 \leqslant b|I|$. By (2.2) this means that $\|B^{1/2}\|_T^2 \leqslant b$.

Combining the lemmas proved above, we obtain Theorem 4.3 in the case $\mathfrak{H} = L^2(S_1; \mathfrak{P})$. According to the remark given in the beginning of this section, this completes the proof of Theorem 4.3. To prove Theorem 4.4, it suffices to replace the $B(\lambda, \mu)$ of Lemma 5.5 by $B(\lambda, \mu) = F_2(\lambda) F_1(\mu)^*$ with an obvious notation.

6. Commutators. As an application of the results obtained above, we consider systems $\{H, K, L\}$ of bounded selfadjoint operators in $\mathfrak{H}$ such that

$$i(HK - KH) = L \geqslant 0. \tag{6.1}$$

(6.2) Theorem. *Let* H, $L \in \mathscr{B}(\mathfrak{H})$ *be selfadjoint with* $L \geqslant 0$. *In order that there exist a selfadjoint* $K \in \mathscr{B}(\mathfrak{H})$ *satisfying* (6.1), *it is necessary and sufficient that* $L^{1/2}$ *be H-smooth. If this condition is satisfied and if* K *is any such operator, then* $\mathfrak{H}_{ac}(H)$, $\mathfrak{H}_{ac}(K)$ *and their intersection* $\mathfrak{H}_0$ *all reduce the system* $\{H, K, L\}$, *and* L *is zero on* $\mathfrak{H} \ominus \mathfrak{H}_0$. *Furthermore,*

$$\|L^{1/2}\|_H^2 \leqslant |\mathrm{int}\,(K_{ac})|/2\pi$$

and

$$\|L^{1/2} E_H(S)\|^2 \leqslant \frac{1}{2\pi} |S \cap \mathrm{supp}(H_{ac})| \, |\mathrm{int}(K_{ac})| \tag{6.3}$$

for all Borel sets S *on* R. *In particular,*

$$\|L\| \leqslant \frac{1}{2\pi} |\mathrm{supp}(H_{ac})| |\mathrm{int}(K_{ac})|. \tag{6.4}$$

These inequalities are optimal in the following sense: for any H *and* L, *there is* K *for which* (6.3) *is optimal. For any* H *with simple spectrum, there are* L *and* K *for which* (6.4) *is optimal.*

(6.5) Remarks. 1. There is no restriction on the spectrum of H. For any H with $H_{ac} \neq 0$, there exists non-trivial $L \geqslant 0$ for which (6.1) has a solution. It follows that any H with $H_{ac} \neq 0$ is the real part of some semi-normal (but non-normal) operator. This answers affirmatively a question raised in [5], p. 43.

2. The reducibility of the system $\{H, K, L\}$ by the subspace $\mathfrak{H}_0$ somewhat strengthens the result of [5], p. 20. A similar remark applies to the optimality of the inequalities (6.3), (6.4).

Proof. Suppose that H, K, L satisfy (6.1). Then

$$(d/dt)e^{itH}Ke^{-itH} = ie^{itH}(HK-KH)e^{-itH} = e^{itH}Le^{-itH}$$

and so

$$\int_{s'}^{s''} e^{itH}Le^{-itH}dt = e^{is''H}Ke^{-is''H} - e^{is'H}Ke^{-is''H}.$$

The right-hand member is uniformly bounded with norm $\leqslant 2\|K\|$. Since $L \geqslant 0$, it follows that $L^{1/2}$ is H-smooth with

$$\|L^{1/2}\|_H^2 \leqslant \|K\|/\pi \tag{6.6}$$

(see the proof of Theorem 3.2), and letting $s' = 0$, $s'' \to \pm\infty$ we obtain

$$-i\Gamma_H^{\pm}L = K_{\pm} - K, \quad K_{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}Ke^{-itH}. \tag{6.7}$$

We note that $K_{\pm}$ commute with H, for $e^{isH}K_{\pm}e^{-isH} = K_{\pm}$ for all real s.

(6.1) is not changed when K is replaced by $K-a$ for any real a. Thus (6.6) implies

$$\|L^{1/2}\|_H^2 \leqslant \frac{1}{2\pi}|\mathrm{int}(K)|. \tag{6.8}$$

We shall now show that $\mathfrak{H}_{ac}(H)$ and $\mathfrak{H}_s(H)$ reduce not only H but K too. To this end it suffices to show that K sends $\mathfrak{H}_s(H)$ into itself. Let $x \in \mathfrak{H}_s(H)$. Then $e^{-itH}x \in \mathfrak{H}_s(H)$ and so $Le^{-itH}x = 0$, for L annihilates $\mathfrak{H}_s(H)$ (see Theorem 2.4). Thus $(\Gamma_H^{\pm}L)x = 0$ by (3.1), and $Kx = K_+x$ by (6.7). But since $x = E_H(S)x$ with some null set S (see the proof of Theorem 2.4) and since K_+ commutes with H as noted above, it follows that $Kx = K_+x = K_+E_H(S)x = E_H(S)K_+x \in \mathfrak{H}_s(H)$.

That $\mathfrak{H}_{ac}(H)$ reduces K implies that $P_{ac}(H)$ and $P_{ac}(K)$ commute (see [3], p. 517). Hence $P_0 = P_{ac}(H)P_{ac}(K)$ is a projection, with range $\mathfrak{H}_0 = \mathfrak{H}_{ac}(H) \cap \mathfrak{H}_{ac}(K)$.

Since $\mathfrak{H}_{ac}(H)$ reduces K, $P_{ac}(H)$ commutes with K. Hence P_0 commutes with K. By symmetry P_0 commutes also with H, hence with L too. Thus $\mathfrak{H}_0$ reduces the system $\{H, K, L\}$. On the other hand, L annihilates $\mathfrak{H}_s(H)$ as noted above, and by symmetry L annihilates $\mathfrak{H}_s(K)$ too. Since $\mathfrak{H}\ominus\mathfrak{H}_0 \subset \mathfrak{H}_s(H)+\mathfrak{H}_s(K)$, L annihilates $\mathfrak{H}\ominus\mathfrak{H}_0$.

Thus we may without loss of generality consider everything within the subspace $\mathfrak{H}_0$, in which both H and K are absolutely continuous. In particular, we obtain

$$\|L^{1/2}\|_H^2 \leqslant |\mathrm{int}(K_{ac})|/2\pi$$

from (6.8). Then (6.3) and (6.4) follow from (2.6).

Suppose now that H and L are given with $L \geqslant 0$ and with $L^{1/2}$ H-smooth. We shall show that there exists a selfadjoint K satisfying (6.1). One such K is given by $K_1 = i\Gamma_H^- L$, which exists by Theorem 3.2. All other possible K must have the form $K = K_1 + K_0$, where K_0 is an arbitrary selfadjoint operator commuting with H.

Theorem 3.2 shows that $K_1 \geqslant 0$ and $\|K_1\| \leqslant 2\pi\|L^{1/2}\|_H^2$. Hence

$$|\mathrm{int}(K_1)| \leqslant 2\pi\|L^{1/2}\|_H^2.$$

In view of the opposite inequality (6.8), we must have equality here. In particular, K_1 is optimal in this sense, and the optimality prevails also for (6.3) in virtue of (2.2).

It remains to prove the assertion about the optimality of (6.4). Again we may assume that H is absolutely continuous. If H has simple spectrum, there exists $L \geqslant 0$ such that $L^{1/2}$ is H-smooth and

$$\|L\| = \|L^{1/2}\|^2 = \|L^{1/2}\|_H^2\,|\operatorname{supp}(H)|$$

(see Theorem 4.7; L is necessarily of rank $\leqslant 1$). If we take the above-mentioned K_1 for this L, we have

$$\|L\| = |\mathrm{int}(K_1)|\,|\operatorname{supp}(H)|/2\pi.$$

This completes the proof of Theorem 6.2.

7. *U*-smooth operators. When U is a unitary operator in $\mathfrak{H}$, we can introduce the notion of *U-smooth operators* $A \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ by requiring that

$$\|A\|_U^2 \equiv \sup \sum_{n=-\infty}^{\infty} \|AU^n x\|^2/\|x\|^2 < \infty.$$

It can be shown, however, that A is U-smooth if and only if A is T-smooth, where T is a bounded selfadjoint operator related to U by the spectral formulas

$$U = \int_0^{2\pi} e^{i\theta}dE(\theta) \quad \text{and} \quad T = \int_0^{2\pi} \theta\, dE(\theta)$$

so that $U = e^{iT}$. Furthermore, we have $\|A\|_U = \|A\|_T$. Thus U-smoothness is a special case of T-smoothness.

The results can be applied to commutators of the form $D = U^{-1}JU - J$, where U is unitary and J, D are bounded selfadjoint. Given a pair U, D with $D \geqslant 0$, there exists J if and only if $D^{1/2}$ is U-smooth, and $\|D^{1/2}\|_U^2 \leqslant |\mathrm{int}(J)|/2\pi$. $\mathfrak{H}_{ac}(U)$ reduces both U and J. But there is no symmetry between U and J, and $|\mathrm{int}(J)|$ in the above-mentioned inequality cannot be replaced by $|\mathrm{int}(J_{ac})|$. (But it can be replaced by $|\mathrm{int}(J_{\mathrm{ess}})|$ where J_{ess} is the "essential part" of J, which is obtained from J after removing the part related to isolated eigenvalues with finite multiplicities; cf. [5], pp. 16, 22.)

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Metrics on an arc

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Consider the following two questions:

(A) Does there exist a "smallest" continuous metric on a given arc A, i.e. a metric ϱ such that, for any continuous metric σ on A, there exist a homeomorphism $\varphi: A \to A$ and a number $a > 0$ such that $a\delta\langle\varphi x, \varphi y\rangle \geqslant \varrho\langle x, y\rangle$ for every $x, y \in A$? If not, does there exist a "minimal" metric with respect to the order just described?

(B) If ϱ is a continuous metric on an arc A, does there exist a normed linear space E and a distance-preserving mapping $f: \langle A, \varrho\rangle \to E$ such that the arc $f[A]$ admits of a "coordinatization"? (We say that an arc $B \subset E$ admits of a "coordinatization" if there are a point $a \in E$ and a continuous mapping f of $[0, 1]$ into a closed hyperplane $L \subset E$, $a \notin L$, such that B consists of all $ta + ft$, $0 \leqslant t \leqslant 1$.)

Both questions seem to be rather elementary. However, I have not found any answer in the literature. So the present note appears though the results may be already known.

In § 1 some definitions and lemmas are given; § 2 contains some auxiliary concepts and propositions. In § 3 the main results are stated and proved.

1.

1.1. The terminology and notation of [1] is used. Since it does not differ substantially from current terms and symbols, only two points of difference should be mentioned: an ordered pair a, b is denoted by $\langle a, b\rangle$; the value of a mapping f at an element x is usually denoted simply by fx. As usual we often denote, e.g., a space and the set of its points by the same symbol. The letters $\mathcal{N}$ and $\mathcal{R}$ respectively stand for the set of all natural numbers $0, 1, 2, \ldots$ and the set of all reals.

1.2. Definition. If X is a set, we denote by $M(X)$ the set of all bounded pseudometrics on Y. If X is a topological (or uniform) space, we denote by $M_t(X)$ (or $M_u(X)$) the set of all continuous (or uniformly continuous) bounded pseudometrics on X.