On difference inequalities corresponding to partial differential inequalities of the first order

by A. PLIŚ (Kraków)

In this note we deal with a certain difference inequality that can be applied to a difference scheme for the partial differential equation of the first order.

This difference inequality plays the same role in estimating solutions for a difference equation as a differential inequality does for the partial differential equation of the first order. One can obtain the considered difference inequality (equation) from partial differential inequality (equation) replacing derivatives by suitable difference quotients.

Notations. We shall consider functions defined on the set of (nodal) points x^M of R^{n+1} , $0 \leq M \leq P$, where $M = (m_0, m)$, $m = (m_1, \dots, m_n)$, $P = (p_0, \dots, p_n)$ is a given system, $m_i, p_i > 0$ are integers $i = 0, 1, \dots, n$, $0 \leq M \leq P$ denotes $0 \leq m_i \leq p_i$, $i = 0, \dots, n$, k, h are positive numbers. (Without loss of generality Theorems 1, 2 can be formulated for $k = 1$ ($h = 1$). We have chosen our formulation because of applications to difference schemes (with steps h, k .) We denote by u^M, v^M values of considered functions at the point x^M . Write

$$+(M) = (m_0 + 1, m_1, \dots, m_n), \quad j(M) = (m_0, \dots, m_{j-1}, m_j - 1, m_{j+1}, \dots, m_n).$$

We shall use the following difference quotients

$$u^{M\sim} = k^{-1}(u^{+(M)} - u^M), \quad u^{M*} = (u^{M1}, \dots, u^{Mn}),$$

where $u^{Mj} = h^{-1}(u^M - u^{j(M)})$.

We start with a simpler case. We give now a theorem on the difference inequality corresponding to the ordinary differential inequality.

THEOREM 1. *Suppose that function $l(i, u)$ defined for $i = 0, \dots, q$, where q is a positive integer, and $u \in R$, satisfies the condition*

$$(1) \quad l(i, v) - l(i, u) \geq -k^{-1}(v - u) \quad \text{for} \quad v \geq u, \quad u, v \in R, \quad i = 0, \dots, q,$$

sequences $u^0, \dots, u^q, v^0, \dots, v^q$ satisfy the difference inequalities

$$(2) \quad u^{i\sim} \leq l(i, u^i), \quad v^{i\sim} \geq l(i, v^i), \quad i = 0, \dots, q-1,$$

where $u^{i\sim} = k^{-1}(u^{i+1} - u^i)$, and the initial inequality

$$(3) \quad u^0 \leq v^0.$$

Then the following inequalities

$$(4) \quad u^i \leq v^i \quad \text{for} \quad i = 0, 1, \dots, q,$$

hold true.

Remark 1. Theorem 1 is analogous to the theorem on ordinary differential inequalities. Assumption (1) has no counterpart in the latter but it can not be omitted. It follows from but is weaker than Lipschitz-continuity of $l(i, u)$ with coefficient k^{-1} (the coefficient tends to ∞ as $k \rightarrow 0$) or the assumption that $l(i, u)$ is (weakly) increasing in u .

We shall prove Theorem 1 by induction. Inequality (4) is satisfied for $i = 0$ in virtue of (3). Suppose it is satisfied for $i = j$. We have by (2)

$$u^{j+1} - u^j - (v^{j+1} - v^j) \leq k(l(j, u^j) - l(j, v^j)),$$

therefore in virtue of (1)

$$(u^{j+1} - v^{j+1}) - (u^j - v^j) \leq -(u^j - v^j)$$

and (4) results for $i = j+1$.

Now we formulate the main result.

THEOREM 2. Suppose that the function $f(M, u, w)$ is of class C^1 (in u and w) and satisfies the inequalities

$$(5) \quad f_{w_i} \leq 0, \quad i = 1, \dots, n,$$

$$(6) \quad h^{-1} \sum_{i=1}^n f_{w_i}(M, u, w) + f_u(M, u, w) \geq -k^{-1}$$

for $0 \leq M \leq P$, $u \in R$, $w = (w_1, \dots, w_n) \in R^n$.

Let u^M, v^M , $0 \leq M \leq P$, satisfy the difference inequalities

$$(7) \quad u^{M\sim} \leq f(M, u^M, u^{M*}), \quad v^{M\sim} \geq f(M, v^M, v^{M*})$$

for $0 \leq m_0 \leq p_0 - 1$, $1 \leq m_i \leq p_i$, $i = 1, \dots, n$, and the inequalities

$$(8) \quad u^M \leq v^M \quad \text{for} \quad m_0 = 0 \text{ or } m_1 = 0, \dots, \text{ or } m_n = 0.$$

Then the inequality

$$(9) \quad u^M \leq v^M \quad \text{for} \quad 0 \leq M \leq P,$$

holds true.

Remark 2. We can formulate Theorem 2 assuming (8) only for $m_0 = 0$ (and not for $m_1 = 0 \dots$ or $m_n = 0$) but in this case inequality (9) need not be satisfied on the whole set $0 \leq M \leq P$.

Proof of Theorem 2. Inequality (9) for $m_0 = 0$ results from (8). Suppose (9) holds for $m_0 = j$ ($0 \leq j < p_0$), $0 \leq M \leq P$. We shall show it for $m_0 = j+1$. Write $r^M = u^M - v^M$. Let

$$(10) \quad \max_{\substack{m_0=j+1 \\ 0 \leq M \leq P}} r^M = r^A,$$

where $A = (j+1, a)$, $a = (a_1, \dots, a_n)$. We assume $a_i \geq 1$, $i = 1, \dots, n$, if these inequalities do not hold inequality (9) for $m_0 = j+1$ results from (8). Inequality (9) being satisfied for $m_0 = j$, we have

$$(11) \quad r^M \leq 0 \quad \text{for} \quad m_0 = j.$$

It follows from definition of the difference quotient that $r^A = r^B + kr^{B\sim}$ for $B = (j, a)$. In virtue of (7) we have

$$r^{B\sim} \leq f(B, u_B, u^{B*}) - f(B, v^B, v^{B*}) = f_u(B, \bar{u}, \bar{w})r^B + \sum_{i=1}^n f_{w_i}(B, \bar{u}, \bar{w})r^{B_i}$$

and in virtue of (5), (11)

$$f_{w_i}(B, \bar{u}, \bar{w})r^{B_i} \leq f_{w_i}(B, \bar{u}, \bar{w})r^B h^{-1}.$$

We obtain

$$r^A \leq r^B + k(f_u(B, \bar{u}, \bar{w}) + \sum_{i=1}^n f_{w_i}(B, \bar{u}, \bar{w})h^{-1})r^B$$

and therefore in virtue of (6), (11) $r^A \leq 0$ and (9) for $m_0 = j+1$ results.

Remark 3. Consider the partial differential equation

$$z_{x_0} = g(x, z, z_{x_1}, \dots, z_{x_n})$$

with the initial condition given on the plane $x_0 = 0$. In [1] a difference scheme $z^{M\sim} = g(x^M, z^M, z^{M*})$ was dealt with and under suitable assumptions the convergence of the approximations was proved by means of particular case of Theorem 1. The convergence can be proved easier using Theorem 2.

Reference

[1] Z. Kowalski, *A difference method for the non-linear partial differential equation of the first order*, Ann. Polon. Math. 18 (1966), pp. 235-242.

Reçu par la Rédaction le 31. 12. 1966