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THE CLEVELAND STATE UNIVERSITY
Cleveland, Ohio

Reçu par la Rédaction le 27. 2. 1968

Fredholm σ-proper maps of Banach spaces

by

Kazimierz Gęba (Gdańsk)

Introduction. The theory of framed cobordism was introduced by L. Pontrjagin in order to study homotopy groups of spheres. Pontrjagin has shown in [9] that the problem of homotopy classification of continuous maps of S^m into S^n is equivalent to the problem of cobordism classification of $(m-n)$-dimensional framed submanifolds of S^m. Afterwards it has turned out to be easier to solve this homotopy classification problem by quite different methods. But it also turned out that Pontrjagin's methods allows to translate some problems in differential topology to homotopy theory.

Using the idea of Pontrjagin, S. Smale has suggested the following notion of degree for certain maps of differential Banach manifolds. Let X and Y be connected C^p Banach manifolds and $f\colon X \to Y$ a proper Fredholm C^p map of index n, with $p > n+1$. It follows from Smale's version of the Sard Theorem [10] that except for a set of the first category all points of Y are regular values of f. If y is a regular value of f, then $f^{-1}(y)$ is a C^p compact n-dimensional submanifold of X or is empty. Moreover, it is shown in [10] that if y_0 and y_1 are regular values of f, then $f^{-1}(y_0)$ and $f^{-1}(y_1)$ are cobordant as unoriented n-dimensional manifolds. Thus there is defined an element $\gamma(f)$ (generalized degree mod 2 of f) of the unoriented bordism group $\mathfrak{N}(X)$.

The purpose of this paper is to find a link between the invariant $\gamma(f)$ and the homotopy theory of so-called *compact fields* ([4], [5]). Instead of X and Y we consider two infinite dimensional Banach spaces E and F. We assume that there is given a subset Γ of the set $\Phi(E, F)$ of all Fredholm operators from E to F, satisfying certain conditions (see Section E). As an example of such a Γ we can take $V =$ a convex subset of $\Phi(E, F)$ and let $\Gamma = \{A \in \Phi(E, F);\ A = B + C,\ B \in V,\ C \text{ is compact}\}$.

Let U be an open subset of E. An n-dimensional C^p *Γ-framed submanifold* of U is a pair (M, φ) where M is an n-dimensional C^p submanifold of U and $\varphi\colon M \to \Gamma$ is a continuous map such that $\operatorname{Ker}\varphi(x) =$ the subspace tangent to M at x, for all $x \in M$. In the set of all Γ-framed compact submanifolds of U, there is a natural cobordism relation which we call *ω-cobordism*.

We denote by $\omega^p(U;\Gamma)$ the resulting collection of equivalence classes: $\omega^p(U;\Gamma)$ is an abelian group with respect to the addition induced by the disjoint union of submanifolds.

Let X be a bounded and closed subset of E. We consider a set $B_p(X;\Gamma)$ consisting of certain C^p Fredholm maps $f\colon E\to F$ such that f restricted to any closed ball is proper, $Df(x)\in\Gamma$ for all $x\in E$ and $0\notin f(X)$. We define also a suitable notion of homotopy of maps in $B_p(X;\Gamma)$ which we call *B_p-homotopy*. (For the precise definitions see Section F.)

Consider the simplest case $X=$ the unit sphere of E. Let U denote the interior of the unit ball in E. If $y\in F-f(X)$ is a regular value of $f\in B_p(X;\Gamma)$, then let $M=U\cap f^{-1}(y)$, $\varphi=Df_{|M}$. Then (M,φ) is a *Γ-framed submanifold* of U. It turns out that if we choose y sufficiently close to 0, then the class of (M,φ) in $\omega^p(U;\Gamma)$ depends only on the B_p-homotopy class of f. Thus there is defined a map Λ from the set $B_p[X;\Gamma]$ of B_p-homotopy classes into $\omega^p(U;\Gamma)$. Our main result is that the map Λ is bijective. This result generalizes to the case where X is an arbitrary closed and bounded subset of E but the corresponding theorem (Theorem H.4) is more complicated.

The main tool used in this paper is the following concept, first introduced by Neubauer [7]. Denote by $G(E)$ the set of all complementable closed linear subspaces of E. $G(E)$ is a metric space with a suitable defined metric. Then there exists a continuous function $\pi\colon G(E)\to G(E)$ such that for each $T\in G(E)$, T and $\pi(T)$ are complementary subspaces of E. $\pi(T)$ can be viewed as a generalization of the orthogonal complement in Hilbert space.

The contents of the various sections are as follows. In Section A we collect some known facts from functional analysis and introduce some notations and conventions we use in the paper. Section B is devoted to projections in Banach spaces. In Section C we give some technical lemmas on finite-dimensional submanifolds of Banach space. In Section D we introduce the notion of σ-proper Fredholm maps and give a modification of Smale's version of the Sard Theorem. In Section E we introduce the notion of Γ-framed submanifold and prove a few technical results. In this section we define also the group $\omega^p(U;\Gamma)$. In Section F we define B_p-maps and B_p-homotopies, the main theorem of this section is the Theorem F.10. Section G is devoted to a class of maps and homotopies which we call *admissible*, the main theorem of the section is the Theorem G.4. In Section H we define a map Λ and prove the main theorem of this paper. We end this section discussing the relation between B_p-maps and compact fields (which were considered in [4] and [5]). We show that there is a natural bijection from $B_p[X;\Gamma]$ to $\pi^{\infty-n}(X)$ - the generalized cohomotopy group defined in [5].

In the Appendix we consider the group $GL_c(E)$ consisting of all linear automorphisms of E which are of the form $I+A$, where I denotes the identity and A is compact. Using the function π we prove that the homotopy groups of $\mathrm{GL}_c(E)$ are given by the Bott periodicity theorem. This is a partial generalization of a theorem of Palais ([11], Theorem B) and a generalization of a theorem of Švarc ([12], Theorem 4).(¹)

The author is indebted to Professor A. Granas for many helpful suggestions.

A. Preliminaries. In this section we collect definitions and theorems of functional analysis we shall need in later sections.

If E and F are real Banach spaces, we denote by $L(E,F)$ the Banach space of continuous linear maps (= operators) of E into F with the norm $\|A\|=\sup\{\|Ax\|;\ \|x\|=1\}$. We put $L(E)=L(E,E)$; $L(E)$ is a Banach algebra. We denote by $I\colon E\to E$ the identity operator. We denote by $GL(E)$ the subset of $L(E)$ consisting of all invertible operators.

By $K(E,F)$ we denote the subset of $L(E,F)$ consisting of all compact (= completely continuous) operators; $A\in K(E,F)$ if and only if A maps the unit ball of E onto a relatively compact subset of F. $K(E,F)$ is a closed linear subspace of $L(E,F)$. Put $K(E)=K(E,E)$, $K(E)$ is a closed ideal in the algebra $L(E)$.

By $L_c(E)$ we denote the subset of $L(E)$ consisting of all operators of the form $I+A$, where $A\in K(E,F)$. We put $GL_c(E)=L_c(E)\cap GL(E)$.

An operator $A\in L(E,F)$ is called a *Fredholm operator* if $\operatorname{Ker}A=A^{-1}(0)$ and $\operatorname{Coker}A=F/A(E)$ are both finite dimensional. Denote by $\Phi(E,F)$ the set of all Fredholm operators from E to F. If $A\in\Phi(E,F)$ then the *index* of A is defined by

$$\operatorname{ind}A=\dim\operatorname{Ker}A-\dim\operatorname{Coker}A\,.$$

We recall the basic facts about Fredholm operators (see e.g. [8], Ch. VII).

1. *If $A\in\Phi(E,F)$, then $A(E)$ is closed in F.*
2. *If $A\in\Phi(E,F)$ and $B\in K(E,F)$, then $A+B\in\Phi(E,F)$.*
3. *$\Phi(E,F)$ is an open subset of $L(E,F)$.*
4. *If $A\in\Phi(E,F)$ and $B\in\Phi(F,G)$, then $B\circ A\in\Phi(E,G)$.*
5. *The index function is constant on each component of $\Phi(E,F)$.*
6. *If $A\in\Phi(E,F)$ and $B\in K(E,F)$, then $\operatorname{ind}(A+B)=\operatorname{ind}A$.*

An operator $P\in L(E)$ is called a *projection* if $P^2=P$. If P is a projection, then every element $x\in E$ can be written uniquely as a sum $x=x_1+x_2$, where $x_1=Px_1$ and $Px_2=0$.

Two closed subspaces R,T are *complementary* if every $x\in E$ can be written uniquely as $x=x_1+x_2$, where $x_1\in R$ and $x_2\in T$. In this case we write $R\oplus T=E$.

(¹) Added in proof. The same theorem was proved in [13].

If R and T are complementary, then $P = P(R, T)$ defined by $Px = x_1$ is a projection. We call $P(R, T)$ the *projection on* R *along* T. There is a ono-to-one correspondence between projections and direct sum decompositions of E into two closed subspaces.

We recall the following well-known facts ([3], p. 480).

1. *Every linear finite dimensional subspace of* E *is closed.*

2. *If* R *is a closed linear subspace of* E *and if either* $\dim R$ *or* $\operatorname{codim} R$ *is finite, then there is a closed linear subspace* T *such that* R *and* T *are complementary.*

3. *If* R *and* T *are closed linear subspaces of* E *such that* $\operatorname{codim} R = \dim T$ *is finite and* $R \cap T = \{0\}$, *then* $R \oplus T = E$.

4. *If* $R \oplus T = E$ *and* $L \oplus T = E$, *then* $P(R, T)$ *maps* L *isomorphically onto* R.

5. *Let* $T_0 \oplus R_0 = E$ *and* $T \oplus R = E$. *Put* $P_0 = P(T_0, R_0)$ *and* $P = P(T, R)$. Then

(a) $P \circ P_0 = P_0$ *if and only if* $T_0 \subset T$,

(b) $P_0 \circ P = P_0$ *if and only if* $R \subset R_0$.

6. *If* $R_i \oplus T = E$ *for* $i = 1, 2, \ldots, k$ *and* $t_1 + t_2 + \ldots + t_k = 1$, *then* $P = \sum t_i P(T, R_i)$ *is a projection on* T.

Thus $Q = I - P = \sum t_i P(R_i, T)$ *is a projection on* $\operatorname{Ker} P$ *along* T.

Let X be an arbitrary subset of a metric space Z and let $x \in Z$, we set

$$\delta(x, X) = \inf\{\varrho(x, y);\ y \in X\},$$
$$B(X, \varepsilon) = \{y \in E;\ \delta(y, X) < \varepsilon\}$$

If Y is another subset of Z we set

$$\delta(X, Y) = \sup\{\delta(x, Y);\ x \in X\}.$$

If R is a closed linear subspace of E, we let $D_r(R) = \{x \in R;\ \|x\| < r\}$ and $\mathrm{C}D_r(R) = \{x \in R;\ \|x\| \leqslant r\} =$ the closure of $D_r(R)$. By $S_r(R)$ we denote the sphere of R of radius r, i.e. $S_r(R) = \{x \in R;\ \|x\| = r\}$.

Denote by $G(E)$ the set of all direct summands of E; $R \in G(E)$ if and only if there is T such that $R \oplus T = E$. $G(E)$ is called the *Grassmanian* of E.

For $R, T \in G(E)$ and $R \neq \{0\}$, $T \neq \{0\}$ let

$$d(R, T) = \max\{\delta(S_1(R), S_1(T)),\ \delta(S_1(T), S_1(R))\}.$$

Moreover,

$$d(\{0\}, T) = d(T, \{0\}) = 1 \quad \text{for} \quad T \neq \{0\} \quad \text{and} \quad d(\{0\}, \{0\}) = 0.$$

It is known that d is a metric on $G(E)$ (see [1], where d is denoted by $\tilde{\vartheta}$).

We denote by $G_*(E)$ (resp., $G^*(E)$) the subset of $G(E)$ consisting of all finite dimensional (resp., finite codimensional) subspaces of E.

For $R, T \in G(E)$, $T \neq \{0\}$, we define

$$a(T, R) = \inf\{\delta(x, R);\ x \in S_1(T)\}.$$

Note that $\delta(ax, R) = |a| \cdot \delta(x, R)$, thus

$$a(T, R) = \inf\left\{\frac{1}{\|x\|}\delta(x, R);\ x \in T,\ x \neq 0\right\}.$$

We end this section by giving some notations and conventions we follow.

We denote by $\mathbf{R}^n$ the n-dimensional Euclidean space with the usual norm. The unit interval $[0, 1]$ will be denoted by J. If h: $X \times J \to Y$ is a map, then h_t: $X \to Y$ denotes the map defined by $h_t(x) = h(x, t)$. By μ: $J \to J$ we denote a C^∞ monotonic function such that $\mu(t) = 0$ for $0 \leqslant t \leqslant 1/3$ and $\mu(t) = 1$ for $2/3 \leqslant t \leqslant 1$.

Let $\{U_i\}$ be an open covering of a topological space X. A system of continuous functions $\{\vartheta_i\}$, ϑ_i: $X \to J$, is called a *partition of unity subordinated to* $\{U_i\}$ if

(a) $\operatorname{Supp}\vartheta_i =$ the closure of $\{x \in X;\ \vartheta_i(x) \neq 0\} \subset U_i$,

(b) *each point* $x \in X$ *has an open neighbourhood which meets* $\operatorname{Supp}\vartheta_i$ *for only finitely many* i,

(c) $\sum \vartheta_i(x) = 1$ *for all* $x \in X$ (the sum can be formed because of (b)).

It is well known that if X is a paracompact Hausdorff space, then every open covering of X admits a subordinated partition of unity.

Given two Banach spaces R and T A: $T \approx R$ stands for "A is a linear isomorphism of T onto R".

Throughout the rest of paper E and F denote two infinite dimensional Banach spaces.

By Θ we denote the empty set.

B. Projections.

Proposition B.1. *Let* $A \in L(E, F)$ *and suppose* $A(E) = F$. *Suppose further that there exist* $R \in G(E)$, $R_0 \in G(F)$, $T \in G(E)$, $T_0 \in G(F)$ *such that*

(a) $T \oplus R = E$,

(b) $T_0 \oplus R_0 = F$,

(c) $A_{|R}$: $R \approx R_0$.

Let $P = P(T, R)$, $Q = I - P$, $P_0 = P(T_0, R_0)$ *and* $A_t = (1-t)A + t[A \circ Q + P_0 \circ A \circ P]$.

Then $A_0 = A$, $A_1(R) = R_0$, $A_1(T) = T_0$ *and* $A_t(E) = F$ *for all* $t \in J$.

Proof. Let $Q_0 = I - P_0$. Since $A(R) = R_0$, $P_0 \circ A \circ Q = 0$. Thus $A \circ Q = (Q_0 + P_0) A \circ Q = Q_0 \circ A \circ Q$. Hence we have

$$\begin{aligned} A_t &= (1-t)A + t[Q_0 \circ A \circ Q + P_0 \circ A \circ P] \\ &= Q_0 \circ A \circ Q + P_0 \circ A \circ P + (1-t) Q_0 \circ A \circ P. \end{aligned}$$

Take an arbitrary element $y \in F$. Then $y = y_1+y_2$, $y_1 = Q_0 y$, $y_2 = P_0 y$. Since A is surjective, there is $x_0 \in E$ such that $A(x_0) = y_2$. Let $x_2 = Px_0$. Then $Ax_0 = (A\circ Q)x_0+(A\circ P)x_0 = y_2$. Thus $(P_0\circ A)x_0 = (P_0\circ A\circ Q)x_0 + (P_0\circ A)x_2 = y_2$. But $(P_0\circ A\circ Q)x_0 = 0$ hence $(P_0\circ A)x_2 = y_2$. Let $y_0 = y_1-(1-t)(Q_0\circ A)x_2$. Since $y_0 \in R_0$, there is $x_1 \in R$ such that $A(x_1) = y_0$. Let $x = x_1+x_2$. Then

$$A_t(x) = A_t(x_1+x_2) = (Q_0AQ)x_1+(P_0AP)x_2+(1-t)(Q_0AP)x_2$$
$$= y_0+y_2(1-t)(Q_0A)x_2 = y_1+y_2 = y.$$

This proves our proposition.

Proposition B.2. *Let $T, T_0, R \in G(E)$. We have the following inequality*

$$\alpha(T, R) \leqslant \alpha(T_0, R)+d(T_0, T).$$

Proof. Take an arbitrary $\varepsilon > 0$. We can find $x_0 \in S_1(T_0)$, $x \in S_1(T)$ and $y \in R$ such that $\|x_0-y\| \leqslant \alpha(T_0, R)+\varepsilon$ and $\|x-x_0\| \leqslant d(T_0, T)+\varepsilon$. From this we have

$$\alpha(T, R) \leqslant \|x-y\| \leqslant \|x-x_0\|+\|x_0-y\| \leqslant \alpha(T_0, R)+d(T_0, T)+2\cdot\varepsilon$$

and the inequality follows.

Corollary B.3. *The assignment $(T, R)\mapsto\alpha(T, R)$ defines a continuous function α: $G(E)\times G(E)\to\mathbf{R}$.*

Proposition B.4. *Let $T_0, T_1, R \in G(E)$ and suppose $T_0\oplus R = E$, $T_1\oplus R = E$. Let $P_0 = P(T_0, R)$, $P_1 = P(T_1, R)$ and $T_t = [(1-t)\cdot P_0 + tP_1](E)$. Then*

$$\alpha(T_t, R) \geqslant \min\{\alpha(T_0, R), \alpha(T_1, R)\}.$$

Proof. Take $x \in T_t$, $x \neq 0$. Let $x_0 = P_0x$, $x_1 = P_1x$; then $x = (1-t)x_0 + tx_1$; $x-x_0$, $x-x_1 \in R$. Thus $\delta(x, R) = \delta(x_0, R) = \delta(x_1, R)$. Since

$$\|x\| \leqslant \max\{\|x_0\|, \|x_1\|\},$$
$$\|x\|^{-1}\cdot\delta(x, R) \geqslant \min\{\|x_0\|^{-1}\cdot\delta(x_0, R), \|x_1\|^{-1}\cdot\delta(x_1, R)\}$$
$$\geqslant \min\{\alpha(T_0, R), \alpha(T_1, R)\}$$

and the conclusion follows.

Theorem B.5. *Let $R\oplus T_0 = E$ and let $P_0 = P(T_0, R)$. Then there is an $\varepsilon > 0$ such that $T \in G(E)$ and $d(T, T_0) < \varepsilon$ imply $T\oplus R = E$. Moreover, $T\mapsto P(T, R)$ is a continuous map from $U = \{T \in G(E);\ d(T, T_0) < \varepsilon\}$ into $L(E)$.*

Proof. This follows from the Theorem 5.2. of [1].

$G(E)$ is metric hence paracompact. We can find a locally finite covering $\{U_i\}$ of $G(E)$ such that for every i there is $R_i \in G(E)$ with $R_i\oplus T = E$ for all $T \in U_i$. Take a partition of unity $\{\vartheta_i\}$ subordinated to $\{U_i\}$. Define

$$\Pi(T) = \sum\vartheta_i(T)\cdot P(T, R_i), \qquad \pi(T) = \operatorname{Ker}\Pi(T) = [I-\Pi(T)](E).$$

Theorem B.6 (Neubauer, [7]). *The operator $\Pi(T)$ is a projection on T along $\pi(T)$ and the assignment $T\mapsto\Pi(T)$ is a continuous map Π: $G(E)\to L(E)$.*

Proof. It is clear from the definition of Π.

Lemma B.7. *Let $x, y \in E$, $\|x\| = 1$. If $y \neq 0$, then*

$$\left\|\|y\|^{-1}y-x\right\| \leqslant 2\cdot\|x-y\|.$$

Proof. We have $\|x-y\| \geqslant |\|x\|-\|y\|| = |1-\|y\||$. Thus

$$\left\|\|y\|^{-1}y-x\right\| \leqslant \left\|y-\|y\|^{-1}\cdot y\right\|+\|y-x\| = |1-\|y\||+\|x-y\| \leqslant 2\cdot\|x-y\|.$$

From the lemma above we have

Corollary B.8. *Let $R, T \in G(E)$. If $x \in S_1(T)$, then*

$$\delta\big(x, S_1(R)\big) \leqslant 2\cdot\delta(x, R),$$

Proposition B.9. *Denote by $PL(E)$ the subset of $L(E)$ consiting of all projections. Then the assignment $P\mapsto P(E)$ is a continuous map from $PL(E)$ into $G(E)$.*

Proof. Take $P, Q \in PL(E)$. For an arbitrary $x \in S_1(P(E))$

$$\|x-Qx\| = \|Px-Qx\| \leqslant \|P-Q\|.$$

$Qx \in Q(E)$, thus by B.8 $\delta\big(x, S_1(Q(E))\big) \leqslant 2\cdot\|P-Q\|$ and the conclusion follows.

Theorem B.10. *The assignment $T\mapsto\pi(T)$ defines a continuous map π: $G(E)\to G(E)$.*

Proof. By B.6 it suffices to prove that the assignment $\Pi(T)\mapsto\pi(T)$ is continuous. Since $\pi(T) = [I-\Pi(T)](E)$ and $I-\Pi(T)$ is a projection, this map is continuous by B.9.

Through the rest of paper we assume that we are given fixed Π and π in both E and F.

Lemma B.11. *Let $A, B \in L(E, F)$. Suppose that there is $R \in G(E)$ such that $R\cap\operatorname{Ker}A = R\cap\operatorname{Ker}B = \{0\}$ and $A(R)$, $B(R) \in G(F)$. Thus A and B map R isomorphically onto $A(R)$ and $B(R)$, respectively. Let $A_0 = (A|_R)^{-1}$, $B_0 = (B|_R)^{-1}$. Then*

(a) $d\big(A(R), B(R)\big) \leqslant 2\cdot\|A-B\|\max\{\|A_0\|, \|B_0\|\}$,

(b) $\|A_0\|\cdot\|A-B\| < 1$ *implies* $\|B_0\| \leqslant \|A_0\|\cdot[1-\|A_0\|\cdot\|A-B\|]^{-1}$.

Proof. To prove (a), take $y \in S_1\big(A(R)\big)$ and let $x = A_0y$, $z = Bx$. We have

$$\|y-z\| = \|Ax-Bx\| \leqslant \|A-B\|\,\|x\| \leqslant \|A_0\|\cdot\|A-B\|$$

and (a) follows from B.7.

To prove (b) take $z \in B(R)$ and let $x = B_0 z$, $y = Ax$. Now

$$\|z\| = \|Bx\| = \|Ax-(A-B)x\| \geqslant \|Ax\| - \|(A-B)x\| \geqslant \|Ax\| - \|A-B\|\cdot\|x\|.$$

Clearly $\|Ax\| \geqslant \|A_0\|^{-1}\cdot\|x\|$ and then $\|z\| \geqslant [\|A_0\|^{-1} - \|A-B\|]\|x\|$. From this

$$\|B_0 z\| = \|x\| \leqslant \|A_0\|\cdot[1-\|A_0\|\cdot\|A-B\|]^{-1}\|z\|$$

and thus (b) is proved.

As an immediate consequence of the lemma above we obtain the following two corollaries.

Corollary B.12. *Let X be a subset of $L(E,F)$. Suppose that there is a subspace $R \in G(E)$ such that for all $A \in X$, $A(R) \in G(F)$ and $R \cap \operatorname{Ker} A = \{0\}$. Then the assignment $A \mapsto A(R)$ defines a continuous map from X into $G(F)$.*

Corollary B.13. *Let X be a compact subset of $GL(E)$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|A-B\| < \delta$ implies $d(A(R), B(R)) < \varepsilon$ for $A, B \in X$ and all $R \in G(E)$.*

Proposition B.14. *Let $A \in L(E,F)$. Suppose that $R \in G^*(E)$, $A(R) \in G^*(F)$ and $R \oplus \operatorname{Ker} A = E$. Then there exists an $\varepsilon > 0$ such that $B \in L(E,F)$ and $\|A-B\| < \varepsilon$ imply $R \cap \operatorname{Ker} B = \{0\}$.*

Proof. $A_0 = A|_R$: $R \approx A(R)$ is an isomorphism; hence A_0^{-1} is continuous. Set $\varepsilon = \|A_0^{-1}\|^{-1}$. If $x \in R$, $x \neq 0$ and $\|A-B\| < \varepsilon$, then

$$\|Bx\| \geqslant \|Ax\| - \|(A-B)x\| \geqslant \|A_0^{-1}\|^{-1}\|x\| - \|A-B\|\cdot\|x\|$$
$$= (\|A_0^{-1}\|^{-1} - \|A-B\|)\cdot\|x\| > 0$$

and the conclusion follows.

Proposition B.15. *If $X \subset \Phi(E,F)$ is a compact subset, then there is $R \in G^*(E)$ such that $R \cap \operatorname{Ker} A = \{0\}$ for all $A \in X$. Moreover, the assignment $A \mapsto A(R)$ is a continuous map from X into $G^*(F)$.*

Proof. If $A \in \Phi(E,F)$ and $R \in G^*(E)$, then $A_{|R}$: $E \to F$ is a Fredholm operator. Thus $A(R) \in G^*(F)$. It follows from B.14 that we can find an open covering $U_1, U_2, \ldots, U_k$ of X and $R_1, R_2, \ldots, R_k \in G^*(E)$ such that $\operatorname{Ker}(A \cap R_i) = \{0\}$ for all $A \in U_i$. Then $R = \bigcap R_i$ is the required subspace. The map $A \mapsto A(R)$ is continuous by B.12.

Proposition B.16. *Let X be a compact metric space. Let h: $X \times J \to G^*(E)$ (resp., h: $X \times J \to G_*(E)$) be a continuous map and suppose that there is $R \in G^*(E)$ (resp., $R \in G_*(E)$) such that $h_1(x) = R$ for all $x \in X$. Then there is a continuous map H: $X \times J \to GL_c(E)$ such that $H(x,0) = I$ and $H(x,t)(h_0 x) = h_t(x)$ for every $x \in X$ and $t \in J$.*

Proof. Define k: $X \times J \to G_*(E)$ (resp., k: $X \times J \to G^*(E)$) by $k(x,t) = \pi(h(x,t))$. k is continuous by B.10. Consider the function β: $X \times J \times J \to \boldsymbol{R}$ defined by $\beta(x,t,s) = a(k(x,t), h(x,s))$; β is continuous by B.3. Since $\beta(x,t,t) > 0$ and X is compact, there is a number $\delta > 0$ such that $\beta(x,t,s) > 0$ for $|t-s| < \delta$.

Let

$$P(x,t) = \Pi(h(x,t)), \quad Q(x,t) = I - P(x,t).$$

Then $P(x,t)$ (resp., $Q(x,t)$) maps $h(x,s)$ (resp., $k(x,s)$) isomorphically onto $h(x,t)$ (resp., $k(x,t)$) if $|t-s| < \delta$. Take $t_0 = 0 < t_1 < \ldots < t_{k-1} < t_k = 1$ such that $|t_i - t_{i-1}| < \delta$ for $i = 1, 2, \ldots, k$.

Define A, B: $X \times J \to L(E)$ by

$$A(x,t) = P(x,t) \circ P(x,t_{i-1}) \circ \ldots \circ P(x,t_1) \circ P(x,t_0),$$
$$B(x,t) = Q(x,t) \circ Q(x,t_{i-1}) \circ \ldots \circ Q(x,t_1) \circ Q(x,t_0),$$

for $t_{i-1} \leqslant t \leqslant t_i$.

Then A and B are continuous by B.6. Note, that $A(x,t)$ maps $h_0 x$ isomorphically onto $h_t x$ and $B(x,t)$ maps $k_0 x$ isomorphically onto $k_t x$. Moreover, if h: $X \times J \to G^*(E)$, then $A(x,t) \in L_c(E)$ and $B(x,t) \in K(E)$ for every $x \in X$ and $t \in J$. If h: $X \times J \to G_*(E)$, then $A(x,t) \in K(E)$ and $B(x,t) \in L_c(E)$ for every $x \in X$ and $t \in J$. Define H by $H(x,t) = A(x,t) + B(x,t)$. Then H is the required map.

Proposition B.17. *Let X be a metric space and let f: $X \to L(E,F)$ be a continuous map. Suppose that there is $T \in G_*(E)$ such that $\operatorname{Ker} f(x) \subset T$ for all $x \in X$. If $\dim \operatorname{Ker} f(x)$ is constant, then the map φ: $X \to G_*(F)$ defined by $\varphi(x) = (fx)(T)$ is continuous.*

Proof. Take $x_0 \in X$. There are an open neighbourhood U of x_0 in X and a linear subspace $T_1 \subset T$ such that $\operatorname{Ker} f(x) \cap T_1 = \{0\}$, $(fx)(T) = (fx)(T_1)$ for $x \in U$. By B.12, φ is continuous on U and the proposition follows.

C. Submanifolds.

Proposition C.1. *Let X be a metric space and let $Y \subset X$ be a closed subset of X which is C^p finite dimensional manifold, possible with the boundary. If $\{U_i\}$ is an open covering of X, then there exists a subordinated partition of unity $\{\vartheta_i\}$ such that $\vartheta_{i|Y}$ is a C^p function for every i.*

Proof. Let $\{V_i\}$ be a locally finite refinement of $\{U_i\}$. Find an open covering $\{W_i\}$ of X such that $\overline{W}_i \subset V_i$. There exists a real valued non-negative C^p function λ_i on Y which is identically 1 on $\overline{W}_i \cap Y$ and identically 0 on $(X - V_i) \cap Y$.

Define ϑ'_i on $\overline{W}_i \cup (X - V_i) \cup Y$ by

$$\vartheta'_i(x) = \begin{cases} \lambda_i(x) & \text{for } x \in Y, \\ 1 & \text{for } x \in \overline{W}_i, \\ 0 & \text{for } x \in X - V_i. \end{cases}$$

Clearly, ϑ_i' is continuous. By the Titze extension theorem there exists a continuous function ϑ_i'': $X \to \mathbf{R}$ which is an extension of ϑ_i'. Since $\{W_i\}$ is a locally finite covering of X, the sum $\sum \vartheta_i''(x)$ is never zero continuous function. Define ϑ_i by $\vartheta_i(x) = \vartheta_i''(x)/\sum \vartheta_i''(x)$. Then $\{\vartheta_i\}$ is the required partition of unity.

PROPOSITION C.2. *Let Y and Z be two closed subsets of a compact metric space X. Suppose that Y is a C^p manifold. Let V be a convex subset of a Banach space E. Let U be an open subset of V. Suppose further that there is given a continuous map f: $(X, Z) \to (U, u_0)$, $u_0 \in U$. Then for a given number $\varepsilon > 0$ there exists a continuous map g: $(X, Z) \to (U, u_0)$ such that $g_{|Y}$ is C^p and $\|f(x) - g(x)\| < \varepsilon$ for all $x \in X$.*

Proof. $f(X)$ is a compact subset of U; hence we can find an $\varepsilon_1 > 0$ such that $B(f(X), \varepsilon_1) \cap V \subset U$. Set $\varepsilon_2 = \min(\varepsilon, \varepsilon_1)$. Choose a finite covering $U_1, U_2, \ldots, U_k$ of X and $x_1, x_2, \ldots, x_k \in X$ such that

(a) $x_i \in U_i$, $i = 1, 2, \ldots, k$,

(b) $Z \cap U_i \neq \Theta$ *implies* $x_i \in Z$,

(c) $\|f(x) - f(x_i)\| < \varepsilon_2$ *if* $x \in U_i$.

Let $\{\vartheta_i\}$ be a partition of unity subordinated to $\{U_i\}$. By C.1 we may assume that $\vartheta_{i|Y}$ are C^p. Define g by $g(x) = \sum \vartheta_i(x) \cdot f(x_i)$. Then g is the required map.

PROPOSITION C.3. *Let Z be a compact metric space and let Y, W be two closed subsets of Z. Suppose that Y is a C^p manifold. Suppose further that f: $Z \to \Phi(E, F)$ is a continuous map such that*

(a) *f is homotopic to a constant map,*

(b) *there is $L \in G_*(E)$ such that* $\operatorname{Ker} f(x) \subset L$ *for all $x \in Z$,*

(c) *$f(x)$ is surjective for all $x \in W$.*

Then there exist $T \in G_*(E)$, $R \in G^*(E)$, $T_0 \in G_*(F)$, $R_0 \in G^*(F)$ and a continuous map χ: $Z \times J \to GL_c(F)$ such that

(1) $L \subset T$, $T \oplus R = E$, $T_0 \oplus R_0 = F$,

(2) $\chi(x, 0) = I$ *for all* $x \in Z$,

(3) $\chi_{|Y \times J}$ *is a C^p map,*

(4) *let* $P = P(T, R)$, $Q = I - P$, $P_0 = P(T_0, R_0)$, $Q_0 = I - P_0$, *then* $(1-t)\chi_1(x) \cdot f(x) + t[Q_0 \cdot \chi_1(x) \cdot f(x) \cdot Q + P_0 \cdot \chi_1(x) \cdot f(x) P]$ *is a surjective map of E onto F for all $x \in W$ and $t \in J$,*

(5) $Q_0 \chi_1(x) f(x) \cdot Q$ *maps R isomorphically onto R_0 for all $x \in Z$.*

Proof. Let h: $Z \times J \to \Phi(E, F)$ be a homotopy such that $h_0 = f$ and $h(x, 1) = A_0$ for all $x \in Z$. By B.15 we can find $R \in G^*(E)$ such that $R \cap \operatorname{Ker} h(x, t) = \{0\}$ for every $x \in Z$ and $t \in J$. We may assume that $R \cap L = \{0\}$ and then find $T \in G_*(E)$ with $L \subset T$, $T \oplus R = E$.

Let $R_0 = A_0(R)$ and choose $T_0 \in G_*(F)$ such that $T_0 \oplus R_0 = F$. Consider the assignment $(x, t) \mapsto h(x, t)(R)$. By B.15 this is a continuous map of $Z \times J$ into $G^*(F)$ and since $h(x, 1)(R) = A_0(R) = R_0$, there exists, by B.16, a continuous map H: $Z \times J \to GL_c(F)$ such that $H(x, 0) = I$ and $[(H_1 x) \cdot (fx)](R) = R_0$ for all $x \in Z$.

Define λ: $W \to G_*(F)$ by $\lambda(x) = [(H_1 x)(fx)](T)$. By B.17 it follows from (b) that λ is continuous. Moreover, $R_0 \oplus \lambda(x) = F$ for all $x \in W$. Thus $\alpha(\lambda(x), R_0) > 0$ for all $x \in W$. Since W is compact and λ is continuous, it follows from B.3 that we can find an $\varepsilon > 0$ such that $\alpha(\lambda(x), R_0) > 2 \cdot \varepsilon$ for all $x \in W$ and $\alpha(R_0, T_0) > 2 \cdot \varepsilon$.

It follows from B.13 and C.2 that we can find a continuous map χ: $Z \times J \to GL_c(F)$ such that

(i) $\chi(x, 0) = I$ *for all* $x \in Z$,

(ii) χ *is a C^p map on* $Y \times J$,

(iii) $d(\eta(x), R_0) < \varepsilon$ *for all* $x \in Z$, *where* $\eta(x) = [\chi_1(x) \cdot (fx)](R)$,

(iv) $d(\xi(x), \lambda(x)) < \varepsilon$ *for all* $x \in W$, *where* $\xi(x) = [\chi_1(x) \cdot (fx)](T)$.

Note that the assignment $x \mapsto \eta(x)$ defines a map η: $Z \to G^*(F)$ which is continuous by B.15. Similarly, the assignment $x \mapsto \xi(x)$ defines a map ξ: $W \to G_*(F)$ which is continuous by B.17.

We are going to prove that χ is the required map. It is clear that χ satisfies (1), (2) and (3). To prove (4) and (5) note first that by B.2

$$\alpha(R_0, T_0) \leqslant d(R_0, \eta(x)) + \alpha(\eta(x), T_0) \quad \text{for all } x \in Z.$$

Thus

$$\alpha(\eta(x), T_0) \geqslant \alpha(R_0, T_0) - d(R_0, \eta(x)) > 2\varepsilon - \varepsilon > 0.$$

Similarly, for all $x \in W$,

$$\alpha(\lambda(x), R_0) \leqslant d(\lambda(x), \xi(x)) + \alpha(\xi(x), R_0),$$

and hence

$$\alpha(\xi(x), R_0) \geqslant \alpha(\lambda(x), R_0) - d(\lambda(x), \xi(x)) > \varepsilon > 0.$$

Thus it follows from B.4 that for all $x \in Z$ and $t \in J$

$$[(1-t) \cdot P(\eta(x), T_0) + tQ_0] \circ \chi_1(x) f(x) Q$$

maps R isomorphically onto

$$[(1-t) \cdot P(\eta(x), T_0) + tQ_0](R).$$

Similarly, for all $x \in W$ and $t \in J$,

$$[(1-t) \cdot P(\xi(x), R_0) + t \cdot P_0] \chi_1(x) \circ f(x) P$$

maps T isomorphically onto

$$[(1-t) \cdot P(\xi(x), T_0) + t \cdot P_0](T).$$

But
$$(1-t)\chi_1(x)f(x)+t[Q_0\cdot\chi_1(x)f(x)\cdot Q+P_0\chi_1(x)\circ f(x)\circ P]$$
$$=\big[(1-t)\cdot P(\eta(x),T_0)+tQ_0\big]\chi_1(x)\circ f(x)\circ Q+$$
$$+\big[(1-t)\cdot P(\xi(x),R_0)+tP_0\big]\chi_1(x)\circ f(x)\circ P\,.$$
This completes the proof.

PROPOSITION C.4. *Let M be an n-dimensional C^p submanifold of a Banach space E. Let $K\subset M$ be a compact subset. Then there exist $T\in G_*(E)$ and $R\in G^*(E)$ such that*

(a) *T and R are complementary,*

(b) *$P=P(T,R)$ restricted to K is a homeomorphism,*

(c) *if $x,y\in K$, $x\neq y$, then $x+R$ and $y+R$ are disjoint.*

Proof. For $x\in K$ let $N_x=\pi(T_xM)$. Then it follows from B.5 that there is an open neighbourhood U_x of x in M such that $T_yM\oplus N_x$ for all $y\in U_x$. Choose a finite subcovering $U_1,U_2,\ldots,U_k$, $U_i=U_{x_i}$.

Let $R_0=N_{x_1}\cap N_{x_2}\cap\ldots\cap N_{x_k}$ and choose $T_0\in G_*(E)$ such that $T_0\oplus R_0=E$. Let $P_0=P(T_0,R_0)$. Since $T_xM\cap R_0=\{0\}$ for all $x\in K$, P_0 is a local embedding on K. Then, for a given $x\in K$, there is only finite number of points $y_1,y_2,\ldots,y_p\in K$ such that $P_0x=P_0y_i$. Denote by L_x the finite dimensional subspace spanned by T_0 and $y_1-x,y_2-x,\ldots,y_p-x$. Choose $R_x\in G^*(E)$ such that $R_x\subset R_0$ and $R_x\oplus L_x=E$. Let $P_x=P(L_x,R_x)$. Since $R_x\subset R_0$, $P_0P_x=P_0$ and thus $P_xy=P_xx$ implies $y=x$ for $y\in K$. It is easy to check that there is an open neighbourhood V_x of x in K such that $y\in V_x$, $z\in K$ and $P_xz=P_xy$ imply $z=y$.

Now once again choose a finite covering $V_{x_1},V_{x_2},\ldots,V_{x_q}$. Define R by $R=\bigcap_{i=1}^{q}R_{x_i}$ and take T such that $T\oplus R=E$. Let $P=P(T,R)$. Then $P_xP=P_x$. Thus (a) and (b) are clearly satisfied. Finally, observe that $Px=Py$ if and only if $x+R=y+R$. Thus (c) follows from (b) and the proof is completed.

PROPOSITION C.5. *Let M be an n-dimensional C^p submanifold of a Banach space E. Let $K\subset M$ be a compact subset of M. Suppose further that there is a closed subset $K_0\subset K$ and $L\in G_*(E)$ such that $K_0\subset L$. Then, given a number $\varepsilon>0$, there exist $T\in G_*(E)$ and a C^p map $H\colon E\times J\to E$ such that*

(a) *each H_t is a diffeomorphism,*

(b) *$DH_t(x)\in GL_c(E)$ for every $x\in E$ and $t\in J$,*

(c) *$\|H_t(x)-x\|<\varepsilon$ for every $x\in E$ and $t\in J$,*

(d) *$H_0=I$,*

(e) *$H_t(x)=x$ for every $x\in K_0$ and $t\in J$,*

(f) *$H_1(K)\subset T$.*

Proof. Choose an open neighbourhood U of K in M such that $K_1=$ the closure of U is a compact subset of M. Applying C.4 we can find complementary subspaces T_0 and R_0 such that T_0 is finite dimensional and $P_0=P(T_0,R_0)$ restricted to K_1 is a homeomorphism. For each $x\in K_1$ define $\varphi_x\colon K_1\to E$ by
$$\varphi_x(y)=P_0(y-x)+x\,.$$
Since
$$\varphi_x(y)=y+P_0(y-x)+x-y=y+(I-P_0)(x-y)\in y+R_0\,,$$
φ_x is an imbedding; moreover, $\varphi_x(x)=x$. We may assume without a loss of generality that $L\subset T_0$. Hence $\varphi_x(y)=y$ for $x,y\in K_0$.

It is clear from the definition of φ_x that we can find an open covering $U_1,U_2,\ldots,U_k$ of K_1 such that $\|\varphi_i(y)-y\|<\varepsilon$ for every $y\in U_i$, where $\varphi_i=\varphi_{x_i}$, $x_i\in U_i$. Moreover, we may assume that if U_i meets K_0, then $x_i\in K_0$. Let $\{\vartheta_i\}$ be a subordinated C^p partition of unity. Define $\varphi\colon K_1\to E$ by $\varphi(x)=\sum\vartheta_i(x)\cdot\varphi_i(x)$. It is clear from the definition that $\|\varphi(x)-x\|<\varepsilon$ for all $x\in K_1$ and $\varphi(x)=x$ for $x\in K_0$. Set $T=$ the subspace spanned by T_0 and $x_1,x_2,\ldots,x_k$. Then $\varphi(K_1)\subset T$, because $\varphi_i(y)=P_0(y-x_i)+$ $+x_i\in T_0+x_i\subset T$.

Define $\eta_0\colon P_0(K_1)\to R_0$ by $\eta(P_0x)=x-\varphi(x)$. Since $P_0(U)$ is a C^p submanifold of T_0, there is a C^p map $\eta\colon T_0\to R_0$ such that $\eta(x)=\eta_0(x)$ for $x\in P_0(K)$. Define H by $H(x,t)=x-t\eta(P_0x)$. For each $t\in J$, H_t is a homeomorphism, because $H_t(z+R)\subset z+R$ for every $z\in T_0$ and on each $z+R$ H_t is clearly a homeomorphism. The remaining conclusions are evident.

LEMMA C.6. *Let M be a C^p n-dimensional submanifold of $\mathbf{R}^m$. Suppose that there is a continuous map $\varphi\colon M\to L(\mathbf{R}^m,\mathbf{R}^{m-n})$ such that $\operatorname{Ker}\varphi(x)=T_xM$ for every $x\in M$. Then for a given compact subset $K\subset M$ there exist a C^p map $f\colon\mathbf{R}^m\to\mathbf{R}^{m-n}$ such that*

(a) *$f(x)=0$ for every $x\in K$,*

(b) *$\operatorname{Ker}[(1-t)\cdot\varphi(x)+t\cdot Df(x)]=T_xM$ for every $x\in K$ and $t\in J$.*

Proof. Since the subset of all surjective linear maps is open in $L(\mathbf{R}^m,\mathbf{R}^{m-n})$ and $\varphi(x)$ is surjective for all $x\in M$, there is an $\varepsilon>0$ such that $x\in K$ and $\|\varphi(x)-A\|<\varepsilon$ imply that A is surjective. Choose $x_1,x_2,\ldots$ $\ldots,x_k\in K$, an open covering $U_1,U_2,\ldots,U_k$ of K in M and $N_1,N_2,\ldots$ $\ldots,N_k\in G_n(\mathbf{R}^m)$ such that

(1) $x_i\in U_i$,

(2) $\|\varphi(x)=\varphi(x_i)\|<\frac{1}{2}\varepsilon$ for $x\in U_i$,

(3) $N_i\oplus T_xM=\mathbf{R}^m$ for $x\in U_i$,

(4) $\|Q_i(x)\|\leqslant 2$ for $x\in U_i$, where $Q_i(x)=P(N_i,T_xM)$,

(5) for each i the map $\gamma_i\colon U_i\times N_i\to\mathbf{R}^m$ defined by $\gamma_i(x,v)=x+v$ maps $U_i\times N_i$ homeomorphically onto an open subset of $\mathbf{R}^m$.

Let $\{\vartheta_i\}$ be a C^p partition of unity on K subordinated to $U_1, U_2, \ldots, U_k$ (i.e. each ϑ_i is a C^p function defined on M with the support contained in U_i and $\sum \vartheta_i(x) = 1$ for $x \in K$). Define λ_i: $\boldsymbol{R}^m \to \boldsymbol{R}$ by

$$\lambda_i(y) = \begin{cases} \vartheta_i(x) & \text{if} \quad y = \gamma_i(x, v), \\ 0 & \text{if} \quad y \notin \gamma_i(U_i \times N_i). \end{cases}$$

Then $\lambda_1, \lambda_2, \ldots, \lambda_k$ are C^p functions and $\sum \lambda_i(x) = 1$ for $x \in K$.

Define f_i: $\gamma_i(U_i \times N_i) \to \boldsymbol{R}^{m-n}$ by $f_i(\gamma_i(x, v)) = \varphi(x_i)(v)$. Then $Df_i(x) = \varphi(x_i)Q_i(x)$ and $f_i(x) = 0$ for $x \in U_i$. Define f by $f(x) = \sum \lambda_i(x) f_i(x)$. Since $f_i(x) = 0$ for $x \in K$.

$$Df(x) = \sum \lambda_i(x) Df_i(x) \quad \text{for} \quad x \in K.$$

It is clear from the definition of f that $T_x M \subset \operatorname{Ker}[(1-t)\varphi(x) + t \cdot Df(x)]$. Thus it suffices to prove that $(1-t)\varphi(x) + t \cdot Df(x)$ is surjective for all $x \in K$ and $t \in J$.

Note, that $\varphi(x) = \varphi(x)Q_i(x)$ for $x \in U_i$. Thus

$$\|\varphi(x) - Df_i(x)\| < \|\varphi(x) - \varphi(x_i)\| \cdot \|Q_i(x)\|.$$

From this it follows at once that $\|\varphi(x) - Df(x)\| < \varepsilon$ for $x \in K$. Thus

$$(1-t)\varphi(x) + t \cdot Df(x) = \varphi(x) + t[Df(x) - \varphi(x)]$$

is surjective. This completes the proof.

D. Proper Fredholm maps. Recall that a continuous map f: $X \to Y$ of metric spaces is *proper* if for every compact subset $K \subset Y$, $f^{-1}(K)$ is a compact subset of X.

LEMMA D.1. *If f: $X \to Y$ is proper, then for every $y \in Y$ and every $\varepsilon > 0$ there is $\eta > 0$ such that*

$$\varrho(f(x), y) < \eta \quad \textit{implies} \quad \delta(x, f^{-1}(y)) < \varepsilon.$$

Proof. Suppose that this is not true. Then there exist $\varepsilon > 0$, $y \in Y$ and a sequence $\{x_n\}$ in X such that $\varrho(f(x_n), y) < n^{-1}$ and $\delta(x_n, f^{-1}(y)) \geqslant \varepsilon$ for all n. Since f is proper and $f(x_n) \to y$, then by passing to a subsequence if necessary we can suppose that $x_n \to x$. Then $f(x) = \lim f(x_n) = y$. Thus $x \in f^{-1}(y)$ which contradicts our assumption. Thus the lemma is proved.

As an immediate consequence we have the following

COROLLARY D.2. *Let f: $X \to Y$ be a proper map. Let $y \in Y$. Then, given a number $\varepsilon > 0$, there is $\delta > 0$ such that if g: $X \to Y$ is another map with $\varrho(f(x), g(x)) < \delta$ for all $x \in X$, then $g(x) = y$ implies $\delta(x, f^{-1}(y)) < \varepsilon$.*

PROPOSITION D.3. *If f: $X \to Y$ is proper, then f is closed, i.e. maps closed subsets of X onto closed subsets of Y.*

Proof. The proof consists of a standard argument and is omitted.

DEFINITION D.4. Let E and F be two Banach spaces. We say that a map f: $E \to F$ is *σ-proper* if $f_{|CD_r(E)}$ is proper for every $r > 0$.

LEMMA D.5. *Let f: $E \to F$ be a σ-proper map and let φ: $E \to GL(F)$ be a continuous map such that for every $r > 0$ $\varphi(CD_r(E))$ is a compact subset of $GL(F)$. Then g: $E \to F$ defined by $g(x) = (\varphi x)(fx)$ is σ-proper.*

Proof. Let $\{x_n\}$ be an arbitrary sequence in $CD_r(E)$ and suppose that $y_n = g(x_n) \to y$ in F. Thus $f(x_n) = (\varphi x_n)^{-1}(y_n)$. By passing to a subsequence if necessary we can assume that $\varphi(x_n) \to A_0 \in GL(F)$. Hence $f(x_n) \to A_0^{-1}(y)$. Since f is proper on $CD_r(E)$, $\{x_n\}$ is a relatively compact subset of $CD_r(E)$. This proves our lemma.

LEMMA D.6. *If f: $E \to F$ is σ-proper and g: $E \to F$ is a continuous map such that $g(CD_r(E))$ is relatively compact for every $r > 0$, then $f + g$ is σ-proper.*

Proof. Suppose that $\{x_n\}$ is a sequence in $CD_r(E)$ such that $y_n = f(x_n) + g(x_n)$ converges in F. We may assume without loss of generality that $g(x_n) \to z \in F$. Let $y = \lim y_n$. Then $f(x_n) \to y - z$. Since f is proper on $CD_r(E)$, $\{x_n\}$ is relatively compact. This proves our lemma.

PROPOSITION D.7. *If f: $E \to F$ and φ: $E \to L(F)$ are C^p maps, then the map g: $E \to F$ defined by $g(x) = (\varphi x)(fx)$ is a C^p map and*

$$[Dg(x)](h) = [\varphi x \circ Df(x)](h) + [D\varphi(x, h)](fx).$$

Proof. The proof consists of a standard argument and is omitted. We remark that $D\varphi$: $E \to L(E, L(F))$ and $D\varphi(x, h) = [D\varphi(x)](h)$ denotes an element of $L(F)$. $D\varphi(x, h)$ is linear with respect to h.

DEFINITION D.8. (Smale [10]). A map f: $E \to F$ is called a *Fredholm map* if it is C^1 and $Df(x) \in \Phi(E, F)$ for every $x \in E$.

If f: $E \to F$ is a Fredholm map the *index of* f is defined to be the index of $Df(x)$ for some $x \in E$. Since E is connected, this definition does not depend on x. A point $x \in E$ is called a *regular point* of f if $Df(x)$ is surjective and *singular* (or *critical*) if not regular. The images of the singular points under f are called *singular* (or *critical*) *values*. An element $y \in F$ is called *regular value* of f if it is not singular value.

We use "almost all" instead of "except for a set of first category".

THEOREM D.9. *Let f: $E \to F$ be a C^p σ-proper Fredholm map with $p > \max\{\operatorname{index} f, 0\}$. Then the regular values are almost all in F.*

Proof. It is proved in [10] that for each point $x \in E$ there exists an open neighbourhood U of x in E such that the regular values of $f_{|U}$ are almost all in F (see the proof of 1.3 in [10] and note that the proof of this local lemma does not use separability of E). Let y be an arbitrary element of F and take $r > 0$. Let $K = f^{-1}(y) \cap CD_r(E)$. Since K is compact, there is $\varepsilon > 0$ such that the regular values of $f_{|B(K,\varepsilon)}$ are almost all in F.

By D.1 there is an open neighbourhood V of y such that

$$f(CD_r(E)) \cap V \subset f(B(K,\varepsilon)).$$

Thus the regular values of $f_{|CD_r(E)}$ are almost all in V. This shows that the set of critical values of $f_{|CD_r(E)}$ is locally of first category in F. Hence it is globally of first category in F. Since the first category is closed under countable union, the set of critical values of f is of first category in F. Thus the theorem is proved.

It is a standard fact that if y is a regular value of f, then $f^{-1}(y)$ is a C^p submanifold of E (see [6], Ch. II, § 2). Thus we receive

COROLLARY D.10. *Let $f\colon E\to F$ be as in D.9. Then for almost all $y \in F$, $f^{-1}(y)$ is a C^p submanifold of E whose dimension is equal to the index of f or is empty.*

THEOREM D.11. *Let $f\colon E\to F$ be a C^p σ-proper Fredholm map with $p>\max\{1+\operatorname{index} f, 0\}$. Let $r>0$ be a given number. Suppose that y_0 and y_1 are regular values of f. Then for a given $\varepsilon>0$ there exists a C^∞ map $\eta\colon J\to F$ such that 0 is a regular value of the map $h_{|CD_r(E)\times J}$, where $h(x,t) = f(x)-\eta(t)$, and $\delta(\eta(J),[y_0,y_1])<\varepsilon$.*

Proof. We may assume without loss of generality that $y_0=0$. Since the set of singular points of $f_{|CD_r(E)}$ is a closed subset of $CD_r(E)$ and $f_{|CD_r(E)}$ is a closed map, the set of critical values of $f_{|CD_r(E)}$ is a closed subset of F. Thus the set of regular values of $f_{|CD_r(E)}$ is an open subset of F. Hence we can find $\delta>0$ with $\varepsilon>\delta$ such that all points of $B(0,\delta)$ and $B(y_1,\delta)$ are regular points of $f_{|CD_r(E)}$. Let $L=$ the one dimensional subspace spanned by y_1. Let $F_0=F/L$ and denote by $A\colon F\to F_0$ the projection map. Define $g\colon E\to F_0$ by $g=Af$. Then g is a C^p σ-proper Fredholm map of index $n+1$. It follows from D.9 that there exists a regular value $z\in F_0$ of g such that $\|z\|<\delta$. Find $z_0\in F$ such that $Az_0=z$ and $\|z_0\|<\delta$. Then $z_0=0+z_0$ and $z_1=y_1+z_0$ are regular values of f. Now let $\eta\colon J\to F$ be a C^∞ map such that

(a) $\|\eta(t)\|<\delta$ for $0\leqslant t\leqslant 1/3$,

(b) $\|\eta(t)-y_1\|<\delta$ for $2/3\leqslant t\leqslant 1$,

(c) $\eta(3t-1)=(1-t)z_0+tz_1$ for $1/3\leqslant t\leqslant 2/3$.

Then η is the required map.

E. Framed submanifolds. Through the rest of paper we will assume that there is given a subset $\Gamma\subset\Phi(E,F)$ satisfying the following conditions

(Γ_1) $\operatorname{ind} A = n\geqslant 0$ *is constant on* Γ,

(Γ_2) *if* $A\in\Gamma$ *and* $B\in K(E,F)$, *then* $A+B\in\Gamma$,

(Γ_3) Γ *is contractible.*

We will assume that A_0 is a fixed element of Γ such that $A_0(E)=F$. Such an element exists because of (Γ_2). Let $\Gamma_0=\{A\in\Gamma;\ A(E)=F\}$, Γ_0 is an open subset of Γ.

By E_1 we will denote the Banach space $E\oplus\mathbf{R}$ with the norm defined by $\|(x,t)\|=\max\{\|x\|,|t|\}$. Define $I_1\colon E\to E_1$ by $I_1(x)=(x,0)$ and $P_1\colon E_1\to E$ by $P_1(x,t)=x$. In what follows we identify points under I_1; so we regard E as a one-codimensional subspace of E_1. Let

$$\Gamma_1=\{A\in\Phi(E_1,F);\ AI_1\in\Gamma\};$$

the assignment $A\mapsto AP_1$ defines an embedding. We regard Γ as a subset of Γ_1. It is clear that Γ is a strong deformation retract of Γ_1. Thus Γ_1 is also contractible.

For $R\in G^*(E)$, $R_0\in G^*(F)$ denote by $\Gamma(R,R_0)$ the subset of $\Phi(R,R_0)$ defined by

$$\Gamma(R,R_0)=\{A\in\Phi(R,R_0);\ \text{there is } B\in\Gamma \text{ such that } B_{|R}=A\}.$$

Let Q be a projection of E onto R; the assignment $A\mapsto A\cdot Q$ defines an embedding $\Gamma(R,R_0)\subset\Gamma$. Thus $\Gamma(R,R_0)$ is also contractible. Finally, let

$$\Gamma_0(R,R_0)=\{A\in\Gamma(R,R_0);\ A(R)=R_0\}.$$

Let U be an open subset of E. Denote by $M^p(U)$ the set consisting of all n-dimensional C^p submanifolds (without boundaries) of U. By $M^p(U\times J)$ we denote the set of all $(n+1)$-dimensional submanifolds of $U\times J$ whose boundaries lie in $U\times\{0\}\cup U\times\{1\}$. An n-dimensional C^p *Γ-framed submanifold* of U is a pair (M,φ) consisting of $M\in M^p(U)$ together with a continuous map $\varphi\colon M\to\Gamma_0$ such that $\operatorname{Ker}\varphi(x)=T_xM$ for all $x\in M$. We denote the collection of all such pairs by $FM^p(U;\Gamma)$ or shortly $FM^p(U)$. Similarly, we denote by $FM^p(U\times J)$ the collection of pairs (W,η) such that $W\in M^p(U\times J)$ and $\eta\colon W\to\Gamma_1$ is a continuous map such that $\operatorname{Ker}\eta(x,t)=T_{(x,t)}W$.

DEFINITION E.1. Two Γ-framed submanifolds $(M_0,\varphi_0),(M_1,\varphi_1)\in FM^p(U)$ are called *Γ-cobordant* if there exists $(W,\eta)\in FM^p(U\times J)$ such that

(a) *there is a number* $\varepsilon>0$ *such that*

$$W\cap(U\times[0,\varepsilon))=M_0\times[0,\varepsilon)$$

$$W\cap(U\times(1-\varepsilon,1])=M_1\times(1-\varepsilon,1]$$

(b)

$$\varphi_0(x)=\eta(x,0)\circ I_1 \quad \textit{for} \quad x\in M_0,$$

$$\varphi_1(x)=\eta(x,1)\circ I_1 \quad \textit{for} \quad x\in M_1.$$

(W,η) is called a *Γ-cobordism from* (M_0,φ_0) *to* (M_1,φ_1).

PROPOSITION E.2. *The relation of Γ-cobordism is an equivalence relation.*

Proof. Symmetry and reflexivity are clear. Transitivity follows at once from condition (a) of E.1.

PROPOSITION E.3. *Let $(M, \varphi_0), (M, \varphi_1) \in FM^p(U)$. Suppose that there is a continuous map $\chi\colon M\times J \to \Gamma_0$ such that*

(a) $\chi_0 = \varphi_0$, $\chi_1 = \varphi_1$,

(b) $\operatorname{Ker}\chi(x, t) = T_xM$ *for every* $x \in M$ *and* $t \in J$.

Then (M, φ_0) and (M, φ_1) are Γ-cobordant.

Proof. Let $W = M\times J$. Define $\eta\colon W \to \Gamma_1$ by $\eta(x, t) = \chi(x, t)\circ P_1$. Then (W, η) is the required Γ-cobordism.

DEFINITION E.4. Let $T \in G_*(E)$, $R \in G^*(E)$, $R \oplus T = E$. We say that a framed submanifold $(M, \varphi) \in FM^p(U)$ is *(T, R)-admissible* if

(a) $\operatorname{Ker}A_0 \subset T$,

(b) $M \subset T$,

(c) $\varphi(x)(T) = A_0(T)$ *for every* $x \in M$,

(d) $\varphi(x)_{|R} = A_{0|R}$ *for every* $x \in M$.

Similarly, $(W, \eta) \in FM^p(U\times J)$ is called *(T, R)-admissible* if

(a′) $\operatorname{Ker}A_0 \subset T$,

(b′) $W \subset T\times J$,

(c′) $\eta(x, t)(T\oplus \mathbf{R}) = A_0(T)$ *for every* $(x, t) \in W$,

(d′) $\eta(x, t)_{|\mathbf{R}} = A_{0|\mathbf{R}}$ *for every* $(x, t) \in W$.

We say that $(M, \varphi) \in FM^p(U)$ (resp., $(W, \eta) \in FM^p(U\times J)$) is *admissible* if there exist $T \in G_*(E)$ and $R \in G^*(E)$ such that (M, φ) (resp., (W, η)) is (T, R)-admissible.

PROPOSITION E.5. *Let $(M, \varphi) \in FM^p(U)$. Suppose there is $L \in G_*(E)$ such that $M \cap CD_s(E)$ is a compact subset of L. Let $M_0 = M\cap D_s(E)$. Then there exist $T \in G_*(E)$, $R \in G^*(E)$ and a continuous map $\chi\colon M_0\times J \to \Gamma_0$ such that*

(a) $\chi_0 = \varphi_{|M_0}$,

(b) $\operatorname{Ker}\chi(x, t) = T_xM$ *for every* $x \in M_0$ *and* $t \in J$,

(c) (M_0, χ_1) *is* (T, R)*-admissible.*

Note, that the condition (b) implies $(M_0, \chi_t) \in FM^p(U)$ for every $t \in J$.

Proof. Let $K = M \cap CD_s(E)$. Since Γ is contractible, there exists $\varkappa\colon K\times J \to \Gamma$ such that $\varkappa_0 = \varphi_{|K}$ and $\varkappa(x, 1) = A_0$ for all $x \in K$. Since K is compact, it follows from B.15 that we can find $R \in G^*(E)$ such that $\operatorname{Ker}\varkappa(x, t)\cap R = \{0\}$ for every $x \in K$ and $t \in J$. We may assume that $L\cap R = \{0\}$. Choose $T \in G_*(E)$ such that $L \subset T$ and $T\oplus R = E$. Define $h\colon K\times J \to G_*(F)$ by $h(x, t) = \pi\big(\varkappa(x, t)(R)\big)$. This map is continuous by B.6; moreover, $h_1(x) = \pi\big(A_0(R)\big)$. Thus by B.16 there exists a continuous map $H\colon K\times J \to GL_c(F)$ such that $H(x, 0) = I$ and $H(x, t)(h_0x) = h_t(x)$ for every $x \in K$ and $t \in J$.

Let $Q(x) = \Pi\big(\varkappa(x, t)(R)\big)$, $Q(x)$ is a projection of F onto $\varkappa(x, t)(R)$; $Q(x)$ is continuous by B.15 and B.6. Let $P(x) = I - Q(x)$; $P(x)$ is the projection on $\pi\big(\varkappa(x, t)(R)\big)$ along $\varkappa(x, t)(R)$. Define $\gamma(x, t)$ by

$$\gamma(x, t) = (1-t)\cdot\varphi(x) + t\cdot[Q(x)\circ\varphi(x)\circ Q + P(x)\varphi(x)\circ Q]\,.$$

It follows from B.1 that $\gamma(x, t)$ is surjective for all $x \in K$ and $t \in J$. Thus

$$\gamma\colon K\times J \to \Gamma_0\,.$$

For $(x, t) \in K\times J$ define $\lambda(x, t)$ by

$$\lambda(x, t) = \varkappa(x, t)\circ Q + H(x, t)\circ\gamma_1(x)\circ P\,.$$

It is clear from the definition that $\lambda(x, t) \in \Gamma$. Moreover, since $\lambda(x, t)$ maps R onto $\varkappa(x, t)(R)$ and T onto $\pi\big(\varkappa(x, t)(R)\big)$, $\lambda(x, t)$ is surjective for all $(x, t) \in K\times J$. Note that $\lambda_1(x) = A_0\circ Q + H_1(x)\circ\gamma_1(x)\circ P$ and thus $\lambda_1(x)$ maps R isomorphically onto $R_0 = A_0(R)$ and T onto $\pi(R_0)$. Let $P_0 = P(T_0, R_0)$, $Q_0 = I - P_0$. Define $\xi\colon K\times J \to \Gamma_0$ by

$$\xi(x, t) = A\circ Q + [(1-t)\cdot I + tP_0]H_1(x)\circ\gamma_1(x)\circ P$$

hen $\xi_0 = \lambda_1$ and ξ_1 is (T, R)-regular. Finally, define the required χ by

$$\chi(x, t) = \begin{cases} \gamma(x, 3t) & \text{for} \quad 0 \leqslant t \leqslant 1/3\,,\\ \lambda(x, 3t-1) & \text{for} \quad 1/3 \leqslant t \leqslant 2/3\,,\\ \xi(x, 3t-2) & \text{for} \quad 2/3 \leqslant t \leqslant 1\,. \end{cases}$$

PROPOSITION E.6. *Let $(M, \varphi), (N, \psi) \in FM^p(U)$ be (T, R)-admissible ramed submanifolds. Let $(W, \eta) \in FM^p(U\times J)$ be a Γ-cobordism from (M, φ) to (N, ψ). Suppose further that there is $L \in G_*(E)$ such that $W\cap (CD_s(E)\times J)$ is a compact subset of $L\times J$. Then there exist $T_1 \in G_*(E)$, $T \subset T_1$, $R_1 \in G^*(E)$, $R_1 \subset R$ and a (T_1, R_1)-admissible Γ-cobordism (W_0, ξ) from $(M_0, \varphi_{|M})$ to $(N_0, \psi_{|N_0})$, where $W_0 = W\cap\big(D_s(E)\times J\big)$, $M_0 = M\cap D_s(E)$ and $N_0 = N\cap D_s(E)$.*

Proof. The proof is analogous to the proof of E.5.

Denote by $CFM^p(U)$ the subset of $FM^p(U)$ consisting of all framed submanifolds (M, φ) such that M is compact. Similarly, $CFM^p(U\times J)$ denotes the subset of $FM^p(U\times J)$ consisting of all (W, η) such that W is compact. Say that $(M_0, \varphi_0), (M_1, \varphi_1) \in CFM^p(U)$ are ω-cobordant if there is a Γ-cobordism from (M_0, φ_0) to (M_1, φ_1) which belongs to $CFM^p(U\times J)$. It is clear that ω-cobordism defines an equivalence relation in $CFM^p(U)$. We denote by $\omega^p(U; \Gamma)$, or $\omega^p(U)$, the set of all ω-cobordism classes.

PROPOSITION E.7. *Let $(M, \varphi) \in CFM^p(U)$. Then there exists an admissible framed submanifold $(N, \psi) \in CFM^p(U)$ such that (M, φ) and (N, ψ) are ω-cobordant.*

Proof. Since M is compact, there is $\varepsilon > 0$ such that $B(M, \varepsilon) \subset U$. Applying C.5 with $K = M$, we can find a C^p map $H: E \times J \to E$ and $T \in G_*(E)$ satisfying conditions (a)-(f) of C.5. Set

$$W = \{(x, t) \in E_1;\ x = H_t(y),\ y \in M\}.$$

Let $M_t = H_t(M)$, M_t is a C^p submanifold. Note that $W = \{(x, t) \in E_1; x \in M_t\}$. From this it follows at once that W is a C^p submanifold. From the definition of H it follows that $T_{(x,t)}W = T_xM + L(x, t)$, where $L(x, t)$ is the one-dimensional subspace of E_1 spanned by $(H_t(x) - x, t)$. For $(x, t) \in W$ let $Q(x, t) = P(E, L(x, t))$. Define $\eta: W \to \Gamma_1$ by

$$\eta(x, t) = \varphi(H_t^{-1}x) \circ [DH_t(x)]^{-1} Q(x, t).$$

Set $N = W \cap (U \times \{1\})$ and define $\varphi: N \to \Gamma_1$ by $\psi(x) = \eta(x, 1) I_1$. Then $N \subset T$. Now the proposition follows from E.5.

PROPOSITION E.8. *Let $(M, \varphi), (N, \psi) \in CFM^p(U)$ and let (W, η) be an ω-cobordism from (M, φ) to (N, ψ). Suppose that there is $L \in G_*(E)$ such that $M, N \subset L$. Then there exist $T \in G_*(E)$ and an ω-cobordism (W_0, ξ) from (M, φ) to (N, ψ) such that $W_0 \subset T \times J$.*

Proof. Since W is a compact subset of $U \times J$, there is an $\varepsilon > 0$ such that $B(W, \varepsilon) \subset U \times J$. Applying C.5 with $K = W$ and $K_0 = M \times \times\{0\} \cup N \times \{1\}$ we can find a diffeomorphism $H: E_1 \to E_1$ such that $\|H(x, t) - (x, t)\| < \varepsilon$ for all $(x, t) \in E_1$, $H(x, t) = (x, t)$ for $(x, t) \in K_0$ and $H(W) \subset T \times J$. Set $W_0 = H(W)$ and define $\xi: W_0 \to \Gamma_1$ by $\xi(x, t) = \eta(H^{-1}(x, t)) \circ [DH^{-1}(x, t)]$. Then (W_0, ξ) is the required cobordism.

From the above proposition and from E.6 we obtain

COROLLARY E.9. *If $(M, \varphi), (N, \psi) \in CFM^p(U)$ are admissible ω-cobordant framed submanifolds then there exists an admissible ω-cobordism from (M, φ) to (N, ψ).*

For $\alpha, \beta \in \omega^p(U)$ we can assume, in view of E.7 that there is $T \in G_*(E)$ and $(M, \varphi) \in \alpha$, $(N, \psi) \in \beta$ with $M, N \subset T$. Since N is a compact subset of U there is $x_0 \in T$ such that $[0, x_0] + N \subset U$. Let $N_0 = x_0 + N$ and define $\psi_0: N_0 \to \Gamma_0$ by $\psi_0(x_0 + x) = \psi(x)$. Then $(N_0, \psi_0) \in CFM^p(U)$, (N, ψ) and (N_0, ψ_0) are ω-cobordant and $N_0 \cap M = \Theta$. Define $\alpha + \beta =$ the ω-cobordism class of $(M \cup N_0, \xi)$ where $\xi: M \cup N_0 \to \Gamma_0$ is defined by

$$\xi(x) = \begin{cases} \varphi(x) & \text{if } x \in M, \\ \psi_0(x) & \text{if } x \in N_0. \end{cases}$$

In view of E.8 $\alpha + \beta$ is independent of the choice of (M, φ) and (N, ψ).

THEOREM E.10. *$\omega^p(U)$ is an abelian group with respect to the above defined addition.*

Proof. It is evident that this addition is commutative and associative. The zero class consists of those Γ-framed submanifolds which ω-bord, i.e. are ω-cobordant to the empty submanifold. It remains to prove that given an element $\alpha \in \omega^p(U)$ there is $\beta \in \omega^p(U)$ such that $\alpha + \beta = 0$.

Choose (M, φ) such that $M \subset T$ for some $T \in G_*(E)$. Choose $x_0 \in T$ such that $x + tx_0 \in U$ for all $x \in M$, $t \in J$. Let $N = M + x_0$. Denote by L the one-dimensional subspace of E spanned by x_0. Choose $R \in G^*(E)$ such that $T \subset R$ and $L \oplus R = E$. Let $P = P(L, R)$, $Q = I - P$. Define $\psi: N \to \Gamma_0$ by $\psi(x) = \varphi(x) \cdot Q - \varphi(x) \circ P$. It is evident that $(N, \psi) \in CFM^p(U)$. Set $\beta =$ the ω-cobordism class of (N, ψ).

Consider the subspace $L_1 = L \oplus \mathbf{R}$ of E_1. We can find an one-dimensional submanifold W_0 of L_1 and a continuous map $\varphi_0: W_0 \to L(L_1, L)$ such that

(a) $\partial W_0 = W_0 \cap L = \{0, 0\} \cup \{x_0, 0\}$,

(b) there exists a number $\varepsilon > 0$ such that

$$W_0 \cap (L \times [0, \varepsilon)) = \{0\} \times [0, \varepsilon) \cup \{x_0\} \times [0, \varepsilon),$$

(c) if $(x, t) \in W_0$, then $0 \leqslant t < 1$, $0 \leqslant \|x\| \leqslant \|x_0\|$,

(d) $\operatorname{Ker} \varphi_0(x, t) = T_{(x,t)}W_0$ for all $(x, t) \in W_0$,

(e) $\varphi_0(0, 0)(v, t) = v$ for $(v, t) \in L_1$,
$\varphi_0(x_0, 0)(v, t) = -v$ for $(v, t) \in L_1$.

Define $W \subset U \times J$ by $W = M \times W_0 = \{x + y \in T + L;\ x \in M, y \in W_0\}$ and define $\eta: W \to \Gamma_1$ by

$$\eta(x, t)(v, t) = \varphi(x) \cdot (Qv) + [\varphi(x) \cdot \varphi_0(Px, t)](Pv, t)$$

for $v \in E$, $t \in \mathbf{R}$. Then (W, η) is an ω-cobordism from $(M \cup N, \xi)$ to the empty submanifold. Thus $\alpha + \beta = 0$. This completes the proof.

F. The set $B_p[X; \Gamma]$. In this section X will denote a bounded and closed subset of E. We set $U = E - X$, $U_r = U \cap D_r(E) = \{x \in U; \|x\| < r\}$. We will assume also that $p > \max(1 + n, 0)$.

We denote by $A_p(E, F; \Gamma)$ or shortly $A_p(E, F)$ the set consisting of all C^p: σ-proper maps $f: E \to F$ such that $Df(x) \in \Gamma$ for all $x \in E$. By $A_p(E \times J, F; \Gamma)$ or $A_p(E \times J, F)$ we denote the set consisting of all C^p σ-proper maps $h: E \times J \to F$ such that $Dh(x, t) \in \Gamma$, for all $(x, t) \in E \times \times J \subset E_1$.

Denote by $B_p(X; \Gamma)$ or, shortly, $B_p(X)$ the subset of $A_p(E, F)$ consisting of all maps $f: E \to F$ such that

(a) $0 \notin f(X)$,

(b) there is $h \in A_p(E \times J, F)$ such that $h_0 = f$ and $h_1 = A_0$.

Two maps $f, g \in B_p(X)$ are called *B_p-homotopic* if there is $h \in A_p(E\times J, F)$ such that $h_0 = f$, $h_1 = g$ and $h_t \in B_p(X)$ for all $t \in J$. We call h a *B_p-homotopy connecting f and g*. We denote by $B_p(X\times J)$ the subset of $A_p(E\times J, F)$ consisting of all B_p-homotopies.

PROPOSITION F.1. *Let $h, k \in A_p(E\times J, F)$ and suppose $h_1 = k_0$. Then there exists $l \in A_p(E\times J, F)$ such that $l_0 = h_0$ and $l_1 = k_1$. Moreover, if $h, k \in B_p(X\times J)$, then $l \in B_p(X\times J)$.*

Proof. The required l is defined by

$$l(x,t) = \begin{cases} h\big(x, \mu(2t)\big) & \text{for} \quad 0 \leqslant t \leqslant 1/2\,, \\ k\big(x, \mu(2t-1)\big) & \text{for} \quad 1/2 \leqslant t \leqslant 1\,. \end{cases}$$

The relation of B_p-homotopy is clearly symmetric and reflexive. By the above proposition it is also transitive, hence

COROLLARY F.2. *The relation of B_p-homotopy is an equivalence relation.*

For $f \in B_p(X)$ we denote by $[f]$ the equivalence class of f. We denote by $B_p[X]$ the collection of all B_p-homotopy classes.

Let $f \in B_p(X)$. By D.3 $f(X)$ is a closed subset of $F-\{0\}$. Let $\delta(f) = \delta\big(0, f(X)\big)$. Then $\delta(f) > 0$. Similarly, if $h \in B_p(X\times J)$, then $\delta(h) = \delta\big(0, h(X\times J)\big) > 0$.

PROPOSITION F.3. *If $f \in B_p(X)$, then there is $g \in B_p(X)$ such that f and g are B_p-homotopic and 0 is a regular value of g.*

Proof. Find a regular value y of f with $\|y\| < \delta(f)$. Define h by $h(x,t) = f(x) - t\cdot y$. Let $g = h_1$. Then h is a B_p-homotopy and g is the required map.

PROPOSITION F.4. *Let $f \in B_p(X)$. Then, given an arbitrary number $r > 0$, there exist $T \in G_*(E)$ and $g \in B_p(X)$ such that 0 is a regular value of g, $g^{-1}(0) \subset D_r(E) \cap T$ and g is B_p-homotopic to f.*

Proof. By F.3 we may assume that 0 is a regular value of f. Choose $\varepsilon > 0$ such that $B\big(f^{-1}(0), \varepsilon\big) \subset U$. Applying C.5 with $K =$ the closure of $f^{-1}(0) \cap U_{r+\varepsilon}$ we find $T \in G_*(E)$ and a C^p map $H\colon E\times J \to E$ satisfying conditions (a)-(f) of C.5. Define $h\colon E\times J \to F$ by $h(x,t) = f\big(H_t^{-1}(x)\big)$. It follows from C.5 that h is a B_p-homotopy and $h_0 = f$. Set $g = h_1$; g is the required map.

PROPOSITION F.5. *Let $h \in A_p(E\times J, F)$ and suppose $h_0, h_1 \in B_p(X)$. Then, given two numbers $r > 0$ and $\varepsilon > 0$, there is $T \in G_*(E)$ and $k \in A_p(E\times J, F)$ such that*

(a) *0 is a regular value of k,*

(b) *$k^{-1}(0) \cap \big(D_r(E)\times J\big) \subset T\times J$,*

(c) *$\delta\Big(k^{-1}(0) \cap \big(CD_r(E)\times J\big),\ h^{-1}(0) \cap \big(CD_r(E)\times J\big)\Big) < \varepsilon$,*

(d) *$k_i \in B_p(X)$ and k_i is B_p-homotopic to h_i for $i = 0, 1$.*

Proof. The proof is analogous to the proof of the preceding proposition.

LEMMA F.6. *Let $X \subset D_r(E)$. Let $h \in A_p(E\times J; F)$ be such that $h_0, h_1 \in B_p(X)$. Suppose that there is $(W, \eta) \in FM^p(U\times J)$, where $s > r$, such that*

(a) *$W \subset h^{-1}(0)$,*

(b) *$W \cap \big(CD_{2s}(E)\times J\big)$ is a compact subset of E,*

(c) *$Dh_{|W} = \eta$,*

(d) *$\big(h_0^{-1}(0) \cap D_{2s}(E)\big) \times \{0\} = W \cap (U_{2s}\times\{0\})$,*
$\big(h_1^{-1}(0) \cap D_{2s}(E)\big) \times \{1\} = W \cap (U_{2s}\times\{1\})$.

Then h_0 and h_1 are B_p-homotopic.

Proof. Let $Z_0 = W \cap \big(CD_{2s}(E)\times J\big)$, $Z_1 = h^{-1}(0) \cap \big(CD_{2s}(E)\times J\big) - Z_0$. Since W consits of regular points of h, Z_0 and Z_1 are disjoint. Moreover, $Z_0 \subset U\times J$ and $Z_1 \subset E\times(0,1)$. Since Z_0 and Z_1 are compact, we can choose $\varepsilon > 0$ so small that

(1) $\varepsilon < s - r$,

(2) $B(Z_0, \varepsilon) \cap B(Z_1, \varepsilon) = \Theta$,

(3) $B(Z_0, \varepsilon) \subset U\times J$,

(4) $B(Z_1, \varepsilon) \subset E\times(0,1)$.

Applying F.5 with r replaced by $2s$ we can find k and T satisfying conditions (a)-(d) of F.5. From conditions (c) and (1), (2) it follows that $k^{-1}(0) \cap \big(CD_{s+r}(E)\times J\big) = Y_0 \cup Y_1$ where Y_0 and Y_1 are compact and disjoint subsets with $\delta(Y_0, Z_0) < \varepsilon$, $\delta(Y_1, Z_1) < \varepsilon$. Thus by (3) and (4) $Y_0 \subset U\times J$ and $Y_1 \subset E\times(0,1)$. Find $x_0 \in E$ such that $\|x_0\| \leqslant r+\varepsilon$ and $\|x_0 + y\| \geqslant r$ for all $y \in T$. Choose $R \in G^*(E)$ such that $x_0 \in R$ and $T\oplus R = E$. Let $P = P(T, R)$. Let $\lambda\colon T\times J \to J$ be a C^∞ function such that

$$\lambda(x,t) = \begin{cases} 0 & \text{for} \quad (x,t) \in Y_0\,, \\ 0 & \text{for} \quad \|x\| \geqslant 2s\,, \\ 0 & \text{for} \quad t = 0, 1, \\ 1 & \text{for} \quad (x,t) \in Y_1\,. \end{cases}$$

Define $l \in A_p(E\times J, F)$ by $l(x,t) = k\big(x - \lambda(Px, t)\cdot x_0, t\big)$.

Since $\lambda(x, 0) = \lambda(x, 1) = 0$, $l_0 = k_0$ and $l_1 = k_1$. To complete the proof we will show that $l \in B_p(X\times J)$. To prove this assume that $\|x\| \leqslant r$ and $l(x,t) = k\big(x - \lambda(Px,t)x_0, t\big) = 0$. Thus $(y,t) = \big(x - \lambda(Px,t)x_0, t\big) \in Y_0 \cup Y_1$. If $(y,t) \in Y_0$ then, since $Px = Py$, $\lambda(Px, t) = 0$ and $x = y$. Hence $(x,t) \in Y_0 \subset U\times J$. If $(y,t) \in Y_1$, then $y = x - x_0$. Thus $x = y + x_0$ and hence $\|x\| \geqslant r$. Since $X \subset D_r(E)$, $(x,t) \in U\times J$. This completes the proof.

DEFINITION F.7. Let $M \in M^p(U_s)$. We say that a map $f \in A_p(E, F)$ is *M-regular* if

(a) $M \subset f^{-1}(0)$,

(b) *all points of M are regular points of f.*

Let $f, g \in A_p(E, F)$ be two M-regular maps. We say that f and g are *M-equivalent* if there is $h \in A_p(E \times J, F)$ such that $h_0 = f$, $h_1 = g$ and h is $M \times J$-regular.

Note that the relation of M-equivalence is an equivalence relation in the set of M-regular maps. In fact, let $h, k \in A_p(E \times J, F)$ and suppose that h, k are $M \times J$-regular and $h_1 = k_0$. Let l be the same as defined in the proof of $F \cdot 1$. Then l is $M \times J$-regular and $l_0 = h_0$, $l_1 = k_1$. This shows that M-equivalence is transitive. Clearly it is symmetric and reflexive.

LEMMA F.8. *Let $f, g \in B_p(X)$. Suppose that there is $s > r$ and $M \in M^p(U_{2s})$ such that*

(a) $f^{-1}(0) \cap U_{2s} = g^{-1}(0) \cap U_{2s} = M$,

(b) *f and g are M-regular and M-equivalent.*

Then f and g are B_p-homotopic.

Proof. If follows from (a) that $M \cap CD_q(E)$ is compact for $q < 2s$. Since f and g are M-equivalent, there exists $h \in A_p(E \times J, F)$ such that $h_0 = f$, $h_1 = g$ and $Dh(x, t)$ is surjective for all $(x, t) \in M \times J$. Let $W = M \times J$ and define η: $W \to \Gamma_1$ by $\eta(x, t) = Dh(x, t)$. Then f and g are B_p-homotopic by F.6.

LEMMA F.9. *Let $h \in A_p(E \times J, F)$, $f = h_0$, $g = h_1$. Suppose that there exist $T \in G_*(E)$, $R \in G^*(E)$, $T_0 \in G_*(F)$, $R_0 \in G^*(F)$, $M \in M^p(U_s)$ and $(W, \eta) \in FM^p(U_s \times J)$ such that*

(a) $T \oplus R = E$, $T_0 \oplus R_0 = F$,

(b) $W = M \times J \subset h^{-1}(0)$,

(c) $M \subset T$,

(d) $\left.\begin{aligned} &\eta(x, t)\colon \quad R \approx R_0, \\ &\eta(x, t)(T) \ = T_0, \\ &Dh(x, t)\colon \ R \approx R_0, \end{aligned}\right\}$ *for all* $(x, t) \in W$,

(e) *f and g are M-regular,*

(f) *if $q < s$, then $M \cap CD_q(E)$ is compact.*

Then for a given $q < s$ there exists $N \in M^p(U_q)$ such that f and g are N-regular and N-equivalent.

Proof. Set $K = (M \times J) \cap (CD_q(E) \times J)$, $N = D_q(E) \cap M$. By (f), K is a compact subset of $M \times J$. Define ξ: $W \to L(T \oplus \mathbf{R}, T_0)$ by $\xi(x, t)(v, \tau) = \eta(x, t)(v, \tau)$ for $(x, t) \in W$, $(v, \tau) \in T \oplus \mathbf{R} \subset E_1$. By C.6 there exists a C^p map λ: $T \times J \to T_0$ such that $(1-\tau)\lambda(x, t) + \tau \cdot D\lambda(x, t)$ is surjective for all $(x, t) \in K$. Let $P = P(T, R)$, $Q = I - P$. Define k: $E \times J \to F$ by

$$k(x, t) = \begin{cases} f(x) + \mu(3t)[\lambda(Px, 0) - f(Px)] & \text{for} \quad 0 \leqslant t \leqslant 1/3 \\ h(x, 3t-1) \circ Q + \lambda(Px, 3t-1) & \text{for} \quad 1/3 \leqslant t \leqslant 2/3\,, \\ g(x) + \mu(3-3t)[\lambda(Px, 1) - g(Px)] & \text{for} \quad 2/3 \leqslant t \leqslant 1\,.\,, \end{cases}$$

Then $k \in A_p(E \times J, F)$, $k_0 = f$ and $k_1 = g$. Moreover, $Dk(x, t)$ is surjective for all $(x, t) \in N \times J$. Thus f and g are N-equivalent.

THEOREM F.10. *Let $X \subset D_r(E)$. Let $f, g \in B_p(X)$ and suppose that 0 is a regular value of f and g. Suppose further that there exist $L \in G_*(E)$, a number $s > r$ and $(W, \eta) \in FM^p(U \times J)$ such that*

(a) $f^{-1}(0) \cap U_{2s} = g^{-1}(0) \cap U_{2s} = M \subset L$,

(b) $W = M \times J$,

(c) *(W, η) is a Γ-cobordism from $(M, Df_{|M})$ to $(M, Dg_{|M})$.*

Then f and g are B_p-homotopic.

Proof. From the definition of $B_p(X)$ it follows that there is $h \in A_p(E \times J, F)$ such that $h_0 = f$, $h_1 = g$. Set $W_0 = W \cap (CD_{s+r}(E) \times J)$, $Y = (L \cap CD_{s+r}(E)) \times J$. Then W_0 and Y are compact. Take the disjoint union of Y and W_0 and identify the points of the form $(x, 0)$ and $(x, 1)$ of W_0 with the corresponding points of Y. Denote the obtained space by Z. Define l: $Z \to \Gamma_1$, by

$$l(x, t) = \begin{cases} Dh(x, t) & \text{for} \quad (x, t) \in Y\,, \\ \eta(x, t) & \text{for} \quad (x, t) \in W_0\,. \end{cases}$$

Since $Dh(x, 0) = \eta(x, 0)$ and $Dh(x, 1) = (x, 1)$ for $x \in M$, l is a continuous function.

Applying C.3 we find $T \in G_*(E)$ with $L \subset T$, $R \in G^*(E)$, $R_0 \in G^*(F)$, $T_0 \in G_*(F)$ and χ: $Z \times J \to GL_c(F)$ which satisfy the conditions (a)-(e) of C.3. Set $q = 2r + 1/3(s-r)$. Let ϑ: $L \to J$ be a C^∞ function such that $\vartheta(x) = 1$ if $\|x\| \leqslant q$ and $\vartheta(x) = 0$ if $\|x\| \geqslant 2r + 2/3(s-r)$. Define γ: $L \times J \to GL_c(F)$ by

$$\gamma(x, t) = \begin{cases} \chi(x, t \cdot \vartheta(x)) & \text{if} \quad \|x\| \leqslant s + r\,, \\ I & \text{if} \quad \|x\| \geqslant s + r\,. \end{cases}$$

Let P' be a projection of E onto L. Set $k(x, t) = [\gamma(P'x, t)](h(x, t))$. $k \in A_p(E \times J, F)$. It is clear from the definition of k that k_i is B_p-homotopic for $i = 0, 1$.

Set $N = M \cap D_q(E)$, $W_1 = N \times J$ and define ξ: $W_1 \to \Gamma_1$ by $\xi(x, t) = \chi(x, t) \cdot \eta(x, t)$ (we regard W_1 as a subset of Z). Set

$$h^{(0)}(x, t) = (1-t)k_0(x) + t \cdot [(Q_0 \circ k_0 \circ Q)(x) + (P_0 \circ k_0 \circ P)(x)]\,,$$
$$h^{(1)}(x, t) = Q_0(k(Qx, t)) + P_0(k(Px, t))\,,$$
$$h^{(2)}(x, t) = tk_1(x) + (1-t)[(Q_0 \circ k_1 \circ Q)(x) + (P_0 \circ k_1 \circ P \cdot)(x)]\,.$$

Define $\eta^{(1)}\colon W_1 \to \Gamma_1$ by

$$\eta^{(1)}(x,t) = Q_0 \circ \xi(x,t) \circ Q + P_0 \circ \xi(x,t) \circ P .$$

Then $k_0 = h_0^{(0)}$, $h_1^{(0)} = h_0^{(1)}$, $h_1^{(1)} = h_0^{(2)}$, $h_1^{(2)} = k_1$. Moreover, it follows from C.3 that $h^{(0)}, h^{(1)}, h^{(2)}$ are $N\times J$-regular. Thus k_0 is N-equivalent to $h_0^{(1)}$ and $h_1^{(1)}$ is N-equivalent to k_1. Moreover, $h^{(1)}$ and $(W_1, \eta^{(1)})$ satisfy the assumptions of F.9. Thus $h_0^{(1)}$ and $h_1^{(1)}$ are N-equivalent. Therefore k_0 and k_1 are N-equivalent. Hence, by F.8, k_0 and k_1 are B_p-homotopic. Thus f and g are B_p-homotopic. This completes the proof.

G. Admissible maps. In this section we keep the notation of the preceding section. We assume $X \subset D_r(E)$.

DEFINITION G.1. Let $T \in G_*(E)$, $R \in G^*(E)$ and $T \oplus R = E$. Let $P = P(T,R)$, $Q = I - P$. We say that a map $f \in B_p(X)$ is (T,R)-*admissible* if

(a) $\operatorname{Ker} A_0 \subset T$,

(b) $f(T) \subset A_0(T)$,

(c) $f = A_0 \circ Q + f \circ P$.

Similarly, a homotopy $h \in B_p(X\times J)$ is called (T,R)-*admissible* if

(a′) $\operatorname{Ker} A_0 \subset T$,

(b′) $h(T\times J) \subset A_0(T)$,

(c′) $h(x,t) = (A_0 \circ Q)(x) + h(Px, t)$ for all $(x,t) \in E\times J$.

We call $f \in B_p(X)$ (resp., $h \in B_p(X\times J)$) *admissible* if f (resp., h) is (T,R)-admissible for some T and R.

Note that if $f \in B_p(X)$ (resp., $h \in B_p(X\times J)$) is (T,R)-admissible and $T_1 \oplus R_1 = E$, $T \subset T_1$ and $R_1 \subset R$, then f (resp., h) is (T_1, R_1)-admissible.

Denote by $R_p(X)$ (resp., $R_p(X\times J)$) the subset of $B_p(X)$ (resp., $B(X\times J)$) consisting of all admissible B_p-maps (resp., B_p-homotopies).

LEMMA G.2. *Let $h \in B_p(X\times J)$ and suppose $h_0, h_1 \in R_p(X)$. Let $s > 0$. Then there exists $k \in B_p(X\times J)$ such that*

(a) *0 is a regular value of k,*

(b) *$k_0, k_1 \in R_p(X)$,*

(c) *0 is a regular value of k_i, $i = 0, 1$,*

(d) *for $i = 0,1$ let $M_i = k_i^{-1}(0) \cap D_s(E)$, $\varphi_i = Dk_{i|M_i}$ then (M_i, φ_i) is admissible,*

(e) *let $W = k^{-1}(0) \cap \big(D_s(E)\times J\big)$, $\eta = Dk_{|W}$; then (W,η) is a Γ-cobordism from (M_0, φ_0) to (M_1, φ_1).*

Proof. Define $l(x,t)$ by $l(x,t) = h\big(x, \mu(t)\big)$. Then l is a B_p-homotopy and $l_t = h_0$ for $0 \leqslant t \leqslant 1/3$, $l_t = h_1$ for $2/3 \leqslant t \leqslant 1$. Find a regular value of l with $\|y\| < \delta(l)$. Define k by $k(x,t) = l(x,t) - y$. Assume that h_0 and h_1 are (T,R)-admissible. Let $T_0 =$ the subspace of F spanned by $A_0(T)$ and y. Set $T_1 = A_0^{-1}(T)$ and choose $R_1 \in G^*(E)$ such that $R_1 \subset R$ and $T_1 \oplus R_1 = E$. Then k_0, k_1 are (T_1, R_1)-admissible. Clearly k is the required homotopy.

From E.6 and G.2 we have

COROLLARY G.3. *Let $f_0, f_1 \in R_p(X)$ and let h be a B_p-homotopy connecting f_0 and f_1. Let $s > 0$. Then there exist $g_0, g_1 \in R_p(X)$ and $(W,\eta) \in FM^p(U\times J)$ such that*

(a) *g_i is R_p-homotopic to f_i, $i = 0,1$,*

(b) *0 is a regular value of g_i,*

(c) *let $M_i = g_i^{-1}(0) \cap D_s(E)$, $\varphi_i = Dg_{i|M_i}$, then (M_i, φ_i) is admissible,*

(d) *(W,η) is an admissible cobordism from (M_0, φ_0) to (M_1, φ_1),*

(e) *$W \cap \big(CD_q(E)\times J\big)$ is compact for $q < s$.*

Note that the R_p-homotopy defines an equivalence relation in $R_p(X)$. Symmetry and reflexivity are clear. To prove transitivity assume that we are given $h, k \in R_p(X\times J)$ such that h is (T_1, R_1)-admissible and k is (T_2, R_2)-admissible. Let $T_0 = T_1 + T_2$. Since R_1, R_2 are finite-codimensional, $R_1 \cap R_2 \in G^*(E)$. Thus there is $R \in G^*(E)$ such that $R \subset R_1 \cap R_2$ and $R \cap T_0 = \{0\}$. Hence there is $T \in G_*(E)$ such that $T_0 \subset T$ and $T \oplus R = E$. Thus k and h are both (T,R)-admissible. This implies that our relation is transitive.

Denote by $R_p[X]$ the set of all R_p-homotopy classes of admissible maps. Then there is a natural map

$$\varkappa\colon R_p[X] \to B_p[X]$$

induced by the inclusion $R_p(X) \subset B_p(X)$.

THEOREM G.4. *The map $\varkappa\colon R_p[X] \to B_p[X]$ is bijective.*

Before proving the theorem we prove the following two lemmas.

LEMMA G.5. *Let $s > r$. Let $(M,\varphi) \in FM^p(U)$ be a (T,R)-admissible framed submanifold such that $M \cap CD_s(E)$ is compact. Then there exist $f \in R_p(X)$ and $(W,\eta) \in FM^p(U\times J)$ such that*

(a) *0 is a regular value of f,*

(b) *let $M_0 = M \cap D_s(E)$; then $f^{-1}(0) \cap D_s(E) = M_0$,*

(c) *(W,η) is a Γ-cobordism from $(M_0, \varphi_{|M_0})$ to $(M_0, Df_{|M_0})$, $W = M_0 \times J$,*

(d) *if $r < q < s$, then $W \cap \big(CD_q(E)\times J\big)$ is compact.*

Proof. Let $T_0 = A_0(T)$ and define $\xi\colon M \to L(T, T_0)$ by $\xi(x)(v) = \varphi(x)(v)$ for $x \in M$, $v \in T$. Applying C.6 with $K = M \cap CD_s(E)$ we find a C^p map $g_0\colon T \to T_0$ such that

(1) $K \cap g_0^{-1}(0)$ and $\operatorname{Ker}[(1-t)\cdot\xi(x) + t\cdot Dg_0(x)] = T_x M$ for every $x \in K$ and $t \in J$.

Choose $x_0 \in R$, $x_0 \neq 0$ and let $T^{(1)} =$ the subspace spanned by T and x_0. Let $T_0^{(1)} = A_0(T^{(1)})$. Define $g_1\colon T^{(1)} \to T_0^{(1)}$ by $g_1(x) = g_0(Px) + A_0(Qx)$, where $P = P(T, R)$, $Q = I - P$. Evidently, $g_1(x) = g_0(x)$ for $x \in T$ and $g_1^{-1}(0) = g_0^{-1}(0) \subset T$. Choose $x_1 \in T^{(1)}$ such that $\|x_1 + y\| \geqslant s$ for all $y \in \dot{T}$ and $\|t \cdot x_1 + y\| \geqslant s$ for all $y \in T$ such that $\|y\| \geqslant s$ and all $t \in \mathbf{R}$. Since all points of K are regular points of g_1, then $g_1^{-1}(0) \cap CD_s(T) = K \cup Y$, where K and Y are closed disjoint. Let $\lambda\colon T^{(1)} \to J$ be a C^∞ function such that $\lambda(x) = 0$ for $x \in K$ and $\lambda(x) = 1$ for $x \in Y$. Define $f_0\colon T^{(1)} \to T_0^{(1)}$ by $f_0(x) = g_1(x - \lambda(Px) \cdot x_1)$. Let $P^{(1)} = P(T^{(1)}, R^{(1)})$, $Q^{(1)} = I - P^{(1)}$, where $R^{(1)} \subset R$. Define $f\colon E \to F$ by $f = A_0 \circ Q^{(1)} + f_0 \cdot P^{(1)}$. Let $W = M_0 \times J$ and define η by $\eta(x, t) = (1-t) \cdot \xi(x) + t \cdot Df(x)$. Then f is the required map an it follows easily from (1) that (W, η) is the required cobordism.

LEMMA G.6. *Let $f_0, f_1 \in R_p(X)$. Let $s > r$ and suppose that 0 is a regular value of f_0 and f_1. Suppose further that there is an admissible Γ-cobordism (W, η) from $(M_0, Df_{0|M_0})$ to $(M_1, Df_{1|M_1})$, where $M_i = f_i^{-1}(0) \cap D_s(E)$, such that $W \cap \big(CD_q(E) \times J\big)$ is compact for some $s > q > r$. Then f_0 and f_1 are R_p-homotopic.*

Proof. The proof is analogous to the proof of the preceding lemma.

Proof of Theorem G.4. To prove that $\varkappa$ is surjective, take $\alpha \in B_p[X]$. Take $s > r$. Choose f which represents α. By F.4 we may assume that 0 is a regular value of f and there is $L \in G_*(E)$ such that $f^{-1}(0) \cap D_{2s} \subset L$. Let $M = f^{-1}(0) \cap D_{2s}(E)$. It follows from E.5 that there is a Γ-cobordism (W, η) from $(M, Df_{|M})$ to (M, φ), where (M, φ) is admissible and $W = M \times J$. Take a number q with $s > q > r$. Then it follows from G.5 that there is $g \in R_p(X)$ such that 0 is a regular value of g and there is a Γ-cobordism (W_0, ξ) from $(N, \varphi_{|N})$ to $(N, Dg_{|N})$, where $N = M \cap D_{2q}(E)$. Thus, by F.8, f and g are B_p-homotopic. Hence $\varkappa$ is surjective.

To prove that $\varkappa$ is injective take $f_0, f_1 \in R_p(X)$ and let $h \in B_p(X \times J)$ be a B_p-homotopy connecting f_0 and f_1. Then it follows from G.2, G.3 and G.5 that f_0 and f_1 are R_p-homotopic. This completes the proof of our theorem.

H. The main theorem. In this section we keep the notations from the sections E, F and G. For a while we restrict our attention to X which satisfy the following condition.

CONDITION (r). *$X \subset CD_r(E)$ and $S_r(E) \subset X$.*

Let $f \in B_p(X)$. By D.9 we can find a regular value y of f with $\|y\| < \delta(f)$. Since X satisfies Condition (r), $f^{-1}(y) \cap D_r(E)$ is a compact n-dimensional C^p submanifold of U_r. Define $(M, \varphi) \in CFM^p(U_r)$ by $M = f^{-1}(y) \cap U_r$, $\varphi = Df_{|M}$. Then it follows from D.11 that the class of (M, φ) in $\omega^p(U_r)$ is independent of y. Let $f, g \in B_p(X)$ and let h be B_p-homotopy connecting f and g. Define k by $k(x, t) = h\big(x, \mu(t)\big)$. Then k is a B_p-homotopy connecting f and g. Choose a regular value y of k such that $\|y\| < \delta(k)$. Note that from the definition of k it follows at once that y is a regular value of f and g. Set

$$W = k^{-1}(0) \cap \big(D_r(E) \times J\big), \quad \eta = Dk_{|W}, \quad M = f^{-1}(0) \cap D_r(E), \quad \varphi = Df_{|M},$$

$$N = g^{-1}(0) \cap D_r(E), \quad \psi = Dg_{|N}.$$

Then (M, φ), $(N, \psi) \in CFM^p(U_r)$ and (W, η) is an ω-cobordism from (M, φ) to (N, ψ). Thus the assignment $f \to (M, \varphi)$ induces a map

$$\Lambda\colon B_p[X] \to \omega^p(U_r).$$

THEOREM H.1. *Let X satisfy Condition (r). Then the map $\Lambda\colon B_p[X] \to \omega^p(U_r)$ is bijective.*

Proof. To prove that Λ is surjective take $\alpha \in \omega^p(U_r)$. By E.7 we can find an admissible Γ-framed submanifold (M, φ) which represents α. Then by G.5 there is $f \in B_p(X)$ such that $\Lambda[f] = \alpha$.

To prove that Λ is injective take $f_0, f_1 \in B(X)$ with $\Lambda[f_0] = \Lambda[f_1]$. We may assume without loss of generality that 0 is a regular value of f_0 and f_1. Let $M_i = f_i^{-1}(0) \cap D_r(E)$, $i = 0, 1$. By F.5 we can assume that there is $L \in G_*(E)$ such that $M_0, M_1 \subset L$. Let (W, η) be an ω-cobordism from $(M_0, Df_{|M_0})$ to $(M_1, Df_{|M_1})$. By E.8 we can assume that $W \subset L \times J$. Thus f_0 and f_1 are B_p-homotopic by F.8. This completes the proof of our theorem.

Now, passing to the general case we relax Condition (r), instead we assume that $X \subset D_r(E)$.

Let V_1, V_2 be two open subsets of E. Assume $V_1 \subset V_2$ and denote by $i\colon V_1 \to V_2$ the inclusion map. Thus i induces in a natural way the map

$$i_\#\colon \omega^p(V_1) \to \omega^p(V_2).$$

Clearly, $i_\#$ is a homomorphism.

Take a number $s > r$ and let $V = D_s(E) - CD_r(E)$. Consider the following commutative diagram in which i, j, k denote corresponding inclusions

$$\begin{tikzcd} & U_s \arrow[dr, "i"] & \\ V \arrow[ur, "j"] \arrow[rr, "k"'] & & D_s(E) \end{tikzcd}$$

Thus we obtain the following commutative diagram

$$\begin{tikzcd} & \omega^p(U_s) \arrow[dr, "i_\#"] & \\ \omega^p(V) \arrow[ur, "j_\#"] \arrow[rr, "k_\#"'] & & \omega^p\big(D_s(E)\big) \end{tikzcd}$$

It is easily seen that $k_\#$ is an isomorphism. Thus $\operatorname{Ker} i_\#$ is a direct summand of $\omega^p(U_s)$. Set $\widetilde{\omega}^p(U_s) = \operatorname{Ker} i_\#$.

PROPOSITION H.2. *Let $s > r$. If $f, g \in R_p(X)$ are (T, R)-admissible and $f_{|D_s(T)} = g_{|D_s(T)}$, then f and g are R_p-homotopic.*

Proof. Let $T_0 = A_0(T)$. Since f and g are (T, R)-admissible, we can find a regular value $y \in T_0$ with $\|y\| \leqslant \max(\delta(f), \delta(g))$. Define f_1, g_1 by $f_1(x) = f(x) - y$, $g_1(x) = g(x) - y$. Then f_1, g_1 are (T, R)-admissible, 0 is a regular value of f_1 and g_1 and $f_{1|D_s(T)} = g_{1|D_s(T)}$. Thus, by G.6, f_1 and g_1 are R_p-homotopic. Since f is R_p-homotopic to f_1 and g_1 is R_p-homotopic to g, the proposition is proved.

Now let $q > r$ be a fixed number. Take $f \in B_p(X)$ and suppose that f is (T, R)-admissible. Let $P = P(T, R)$, $Q = I - P$. Denote by f_0: $T \to T_0$ the map defined by $f_0(x) = f(x)$ for $x \in T$. Take s with $r < s < q$ and let g_0: $T \to T_0$ be a C^p map such that $g_{0|D_s(T)} = f_{0|D_s(T)}$ and $g_0(x) = y_0 \neq 0$ for $x \in S_q(T)$. Define g by $g = A_0 \circ Q + g_0 \circ P$. Then g is (T, R)-admissible map which, by H.2, is R_p-homotopic to f. Moreover, there exists a C^p map χ: $T \times J \to T_0$ such that

(a) $\chi_0 = g_0$,

(b) $\chi(x, 1) = y_0$ for all $x \in T$,

(c) $\chi(x, t) = y_0$ for all $x \in S_q(T)$ and $t \in J$.

Define h by $h(x, t) = A_0 \circ Q + \chi(Px, t)$. Then $h \in R_p(S_q(E) \times J)$, $h_0 = g$ and $0 \notin h_1(S_q(E))$. Hence if we consider g as a map in $B_p(X \cup S_q(E))$, then $\Lambda[g] \in \widetilde{\omega}(U_q)$. It is evident that $\Lambda[g]$ depends only on the B_p-class of f. Thus the assignment $f \mapsto g$ define a map

$$\lambda_q\colon R_p[X] \to \widetilde{\omega}^p(U_q).$$

PROPOSITION H.3. *The map λ_q is bijective.*

Proof. It follows at once from the inclusions $R_p(S_q(E) \cup X) \subset R_p(X)$, $R_p((S_q(E) \cup X) \times J) \subset R_p(X \times J)$.

Define Λ_q: $B_p[X] \to \widetilde{\omega}^p(U_q)$ by $\Lambda_q = \lambda_q \varkappa^{-1}$. From the preceding proposition we have

COROLLARY H.4. *The map Λ_q: $B_p[X] \to \widetilde{\omega}^p(U_q)$ is bijective.*

This fact can be interpreted as follows. Suppose that $f, g \in B_p(X)$, 0 is a regular value of f and g, $0 \notin f(S_q(E)) \cup g(S_q(E))$. Then $M = f^{-1}(0) \cap D_q(E)$, $N = g^{-1}(0) \cap D_q(E)$ are compact submanifolds of U_q. Let $\alpha =$ the class of $(M, Df_{|M})$ in $\omega^p(U_q)$, $\beta =$ the class of $(N, Dg_{|N})$ in $\omega^p(U_q)$. Then f and g are B_p-homotopic if and only if $\alpha - \beta \in \operatorname{Ker} i_\#$, where i denotes the inclusion i: $U_q \to D_q(E)$.

Finally we discuss briefly the connection between B_p-maps and compact fields considered in [4] and [5]. We assume that $n \geqslant 0$ and identify F with an n-codimensional subspace of E. In this case we may assume that A_0 is a projection of E onto F along an n-dimensional subspace of E. In what follows we use terminology of [5].

For $p \geqslant 0$ we denote by $C_p(X)$ (resp., $C_p(X \times J)$) the subset of $B_p(X)$ (resp., $B_p(X \times J)$) consisting of all C^p compact fields (resp., of all C^p compact homotopies). Denote by $C_p[X]$ the corresponding set of homotopy classses. From the inclusions

$$R_p(X) \subset C_p(X) \subset B_p(X), \qquad R_p(X \times J) \subset C_p(X \times J) \subset B_p(X \times J)$$

and G.4 we obtain the following theorem

THEOREM H.5. *If $p > n+1$, then the inclusion $C_p(X) \subset B_p(X)$ induces a bijective map β: $C_p[X] \to B_p[X]$.*

On the other hand, it follows from (4.3) in ([5], II) that

THEOREM H.6. *For every $p \geqslant 0$ the inclusion $C_p(X) \to C_0(X)$ induces a bijective map γ: $C_p[X] \to C_0[X]$.*

Thus $\beta\gamma^{-1}$: $C_0[X] \to B_p[X]$ is a bijective map. The set $C_0[X]$ has been denoted by $\pi^{\infty - n}(X)$ in [4]; it has an abelian group structure with addition patterned after Borsuk's cohomotopy addition. On the other hand $B_p[X]$ can be regarded as an abelian group with the addition induced by the bijection Λ: $B_p[X] \to \widetilde{\omega}^p(U_s)$. It turns that $\beta\gamma^{-1}$ is an isomorphism of abelian groups, the proof of this will be published elsewhere.

I. Appendix. For a notational convenience we establish a one-to-one correspondence between the symbols $\alpha, \beta, \ldots$ and the elements $R_\alpha, R_\beta, \ldots$ of $G^*(E)$. Set $\alpha \leqslant \beta$ if $R_\beta \subset R_\alpha$ and denote by $\mathfrak{A}$ the set $G^*(E)$ ordered by the relation $\leqslant$. Clearly, $\mathfrak{A}$ is a directed set. For $\alpha \in \mathfrak{A}$ let

$$L_\alpha(E) = \{A \in L_c(E);\ A(x) = x \text{ for } x \in R_\alpha\} \quad \text{and} \quad GL_\alpha(E) = GL(E) \cap L_\alpha(E).$$

Clearly, $\alpha \leqslant \beta$ implies $L_\alpha(E) \subset L_\beta(E)$ and $GL_\alpha(E) \subset GL_\beta(E)$.

PROPOSITION I.1. *Let X and Y be two compact subsets of $GL_c(E)$ with $Y \subset X \cap GL_\alpha(E)$. Then there exist $\beta \geqslant \alpha$ and a continuous map h: $(X \times J, Y \times J) \to (GL_c(E), GL_\beta(E))$ (a deformation of the pair (X, Y)) such that for all $A \in X$, $h(A, 0) = A$ and $h(A, 1) \in GL_\beta(E)$.*

Proof. For $A \in X$ set $k(A, t) = I - (1-t)(I - A)$. Then $k(A, 0) = A$, $k(A, 1) = I$. Moreover, if $A \in Y$, then for all $t \in J$, $k(A, t) \in L_\alpha(E)$. Thus we have a continuous map k: $(X \times J, Y \times J) \to (L_c(E), L_\alpha(E))$. By B.15 we can find $R_\beta \in G^*(E)$ such that $R_\beta \cap \operatorname{Ker} k(A, t) = \{0\}$ for all $(A, t) \in X \times J$. Without loss of generality we may assume that $R_\beta \subset R_\alpha$; hence $\alpha \leqslant \beta$.

Set $T = \pi(R_\beta)$, $Q = \Pi(R_\beta) = P(R_\beta, T)$, $P = I - Q$. Consider the map λ: $X \times J \to G_*(E)$ defined by $\lambda(A, t) = \pi(k(A, t)(R))$. By B.10 and B.15, λ is continuous. By B.16 there exists a continuous map γ: $X \times J \to K(E)$

such that $\gamma(A, 0)$ is a projection onto $\lambda(A, 0)$ and $\gamma(A, t)$ maps $\lambda(A, 0)$ isomorphically onto $\lambda(A, t)$. Define h by

$$h(A, t) = \begin{cases} A\circ Q+(1-2t)A\circ P+2t\cdot\gamma(A, 0)\cdot A\circ P & \text{for} \quad 0 \leqslant t \leqslant 1/2\,, \\ k(A, 2t-1)\circ Q+\gamma(A, 2t-1)\cdot A\circ P & \text{for} \quad 1/2 \leqslant t \leqslant 1\,. \end{cases}$$

It is easy to check that h is the required map.

For $\alpha \leqslant \beta$ denote by $i_{\alpha\beta}$: $GL_\alpha(E)\to GL_\beta(E)$ and i_α: $GL_\alpha(E)\to GL_c(E)$ the corresponding inclusions. For every $p \geqslant 0$ $\{\pi_p(GL_\alpha(E));\,(i_{\alpha\beta})_*\}$ is a direct system of abelian groups indexed by $\mathfrak{A}$. From I.1 we obtain

PROPOSITION I.2. *For every* $p \geqslant 0$

$$\varinjlim_{\alpha}\{(i_\alpha)_*\}\colon\ \varinjlim_{\alpha}\{\pi_p(GL_\alpha(E));\,(i_{\alpha\beta})_*\} \approx \pi_p(GL_c(E))\,.$$

For $\alpha \in \mathfrak{A}$ let

$$T_\alpha = \pi(R_\alpha)\,, \qquad Q_\alpha = \Pi(R_\alpha) = P(R_\alpha, T_\alpha)\,, \qquad P_\alpha = I - Q_\alpha = P(T_\alpha, R_\alpha)\,.$$

For $A \in GL(T_\alpha)$ define $j_\alpha(A) = Q_\alpha + A\circ P_\alpha$. The assignment $A \mapsto j_\alpha(A)$ defines a continuous map j_α: $GL(T_\alpha)\to GL_\alpha(E)$.

PROPOSITION I.3. *The map* j_α: $GL(T_\alpha)\to GL_\alpha(E)$ *is a homotopy equivalence.*

Proof. For $A \in GL_\alpha(E)$, $t \in J$, let

$$\varkappa(A, t) = Q_\alpha + (1-t)A\circ P_\alpha + t\cdot P_\alpha\circ A\circ P_\alpha\,.$$

Clearly $\varkappa(A, 0) = A$ for all $A \in GL_\alpha(E)$. By B.1 $\varkappa(A, t) \in GL_\alpha(E)$ for all $(A, t) \in GL_\alpha(E)\times J$. Moreover, $\varkappa(A, 1) \in j_\alpha\,(GL(T_\alpha))$ for all $A \in GL_\alpha(E)$ and $\varkappa(A, t) = A$ for all $A \in j_\alpha(GL(T_\alpha))$. Thus $\varkappa$ is a deformation retraction of $GL_\alpha(E)$ onto $j_\alpha(GL(T_\alpha))$. Since j_α is an embedding, this proves our proposition.

THEOREM I.4. *For every* $p \geqslant 0$,

$$\pi_p(GL_c(E)) \approx \mathrm{Lim}\,\{\pi_p(GL(\boldsymbol{R}^n))\}$$

Proof. For $\alpha \leqslant \beta$ define $k_{\alpha\beta}$: $GL(T_\alpha)\to GL(T_\beta)$ by $k_{\alpha\beta}(A) = (j_\alpha A)_{|T_\beta}$. Then we have the following commutative diagram

$$\begin{array}{ccc} \pi_p((GL(T_\alpha)) & \xrightarrow{(j_\alpha)_*} & \pi_p(GL_\alpha(E)) \\ \big\downarrow{\scriptstyle (k_{\alpha\beta})_*} & & \big\downarrow{\scriptstyle (i_{\alpha\beta})_*} \\ \pi_p(GL(T_\beta)) & \xrightarrow{(j_\beta)_*} & \pi_p(GL_\beta(E)) \end{array}$$

Thus, by I.3, if $\dim T_\alpha$ is large enough, $(i_{\alpha\beta})_*$ is an isomorphism. Now the theorem follows from I.2.

Finally note that the above proof carries over to the case where E is a complex Banach space. Thus

THEOREM I.5. *If E is a complex Banach space, then for every* $p \geqslant 0$

$$\pi_p(GL_c(E)) \approx \{\mathrm{Lim}\,\pi_p(GL(\boldsymbol{C}^n))\}\,.$$

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Reçu par la Rédaction le 26. 4. 1968