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Reçu par la Rédaction le 8. 8. 1967

Fundamental retracts and extensions of fundamental sequences

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In order to extend some standard notions of the homotopy theory onto arbitrary compacta X, Y lying in the Hilbert space H, I introduced in [2] the notion of the *fundamental sequence from X to Y*, defined as an ordered triple $\underline{f} = \{f_k, X, Y\}$ consisting of X, Y and of a sequence $\{f_k\}$ of (continuous) maps of H into itself satisfying the following condition:

For every neighborhood V of Y (neighborhoods are understood here always in the space H) there exists a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \text{ in } V \quad \text{for almost all } k.$$

The set X will be said to be the *domain*, and the set Y—the *range* of the fundamental sequence $\underline{f}$.

Setting $i_k(x) = x$ for every point $x \in H$, we immediately see that for every compactum $X \subset H$ the triple $\{i_k, X, X\}$ is a fundamental sequence $\underline{i}_X$, called the *fundamental identity sequence for X*.

If c is a point of a compactum $X \subset H$, then setting $c(x) = c$ for every point $x \in H$, we get a fundamental sequence $\underline{c}_X = \{c, X, X\}$ called a *constant fundamental sequence for X*.

Let us observe that if $\hat{X}$ is a closed subset of a compactum $X \subset H$, and Y is a closed subset of a compactum $\hat{Y} \subset H$, and if $\underline{f} = \{f_k, X, Y\}$ is a fundamental sequence, then $\underline{\hat{f}} = \{f, \hat{X}, \hat{Y}\}$ is also a fundamental sequence.

Two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, X, Y\}$ are said to be *homotopic* (in symbols: $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there exists a neighborhood U of X such that

$$f_k/U \simeq g_k/U \text{ in } V \quad \text{for almost all } k.$$

The fundamental sequences from X to Y may be considered as a generalization of the maps of X into Y, and the classes of all homotopic fundamental sequences from X to Y (called *fundamental classes from X to Y*) may be considered as a generalization of the homotopy classes of maps of X into Y.

It is known ([2], p. 242) that every fundamental sequence $\underline{f} = \{f_k, X, Y\}$ induces a homomorphism

$$\underline{f}_*\colon H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A}),$$

where $H_n(X, \mathfrak{A})$ denotes the nth homology group (in the sense of Vietoris or of Čech) of X over the group of coefficients $\mathfrak{A}$, and the homotopic fundamental sequences induce the same homomorphism.

It x_0 is a point of a compactum $X \subset H$ and y_0 is a point of a compactum $Y \subset H$, then a sequence of maps $f_k\colon (H, x_0) \to (H, y_0)$ is said to be a *pointed sequence from* (X, x_0) *to* (Y, y_0) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/(U, x_0) \simeq f_{k+1}/(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k.$$

We denote this pointed sequence by $\{f_k, (X, x_0), (Y, y_0)\}$, or shortly by $\underline{f}$. Manifestly every pointed sequence from (X, x_0) to (Y, y_0) is also a fundamental sequence from X to Y.

Two pointed sequences $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ and $\underline{g} = \{g_k, (X, x_0), (Y, y_0)\}$ are said to be *homotopic* (in symbols: $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there exists a neighborhood U of X such that

$$f_k/(U, x_0) \simeq g_k/(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k.$$

One proves ([2], p. 253) that each pointed sequence $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$ induces a homomorphism

$$\underline{f}_*\colon \underline{\pi}_n(X, x_0) \to \underline{\pi}_n(Y, y_0) \quad \text{for} \quad n = 1, 2, \ldots,$$

where $\underline{\pi}_n(X, x_0)$ denotes the *n-th fundamental group* of (X, x_0), which is an appropriate generalization of the nth homotopy group (see [2], p. 251).

In the present note I consider the concept of the extension of a fundamental sequence, which permits the introduction of some generalizations of many notions and results of the theory of retracts.

The author acknowledges his gratitude to Dr. A. Lelek and to Dr. H. Patkowska, who read the manuscript and made several important suggestions.

1. Restriction and extension of fundamental and of pointed sequences. Let X, X', Y be compacta in the real Hilbert space H and let $X \subset X'$. Let us consider two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{f}' = \{f_k', X', Y\}$ (which in particular can be pointed sequences $\{f_k, (X, x_0), (Y, y_0)\}$ and $\{f_k', (X', x_0), (Y, y_0)\}$). We say that $\underline{f}$ is a *restriction* of $\underline{f}'$ to X, or that $\underline{f}'$ is an *extension* of $\underline{f}$ onto X' if

$$f_k(x) = f_k'(x) \quad \text{for every point } x \in X \text{ and } k = 1, 2, \ldots$$

It is clear that for every fundamental sequence $\underline{f}' = \{f_k', X', Y\}$ and for every compactum $X \subset X'$ there exist restrictions of $\underline{f}'$ to X. In fact, one of them is the fundamental sequence $\{f_k', X, Y\}$. The question of the existence of an extension is more delicate. First let us prove that

(1.1) *If two fundamental sequences $\underline{f} = \{f_k, X, Y\}$, $\underline{g} = \{g_k, X, Y\}$ satisfy the condition $f_k(x) = g_k(x)$ for every point $x \in X$ and for $k = 1, 2, \ldots$, then $\underline{f} \simeq \underline{g}$.*

Proof. Let V be an open neighborhood of Y. Then there exist a neighborhood U of X and an index k_0 such that

(1.2) $$f_k/U \simeq f_{k_0}/U \text{ in } V \quad \text{and} \quad g_k/U \simeq g_{k_0}/U \text{ in } V \quad \text{for every } k \geq k_0.$$

It follows, in particular, that V is a neighborhood of the compact set $f_{k_0}(X) = g_{k_0}(X)$. Since $f_{k_0}(x) = g_{k_0}(x)$ for every point $x \in X$, there exists a neighborhood $U_0 \subset U$ of X such that

(1.3) $$f_{k_0}/U_0 \simeq g_{k_0}/U_0 \text{ in } V.$$

The inclusion $U_0 \subset U$ and the homotopies (1.2) and (1.3) imply that

$$f_k/U_0 \simeq f_{k_0}/U_0 \simeq g_{k_0}/U_0 \simeq g_k/U_0 \text{ in } V \quad \text{for every } k \geq k_0,$$

and consequently $\underline{f} \simeq \underline{g}$.

Now let us consider a compactum X contained in a compactum $X' \subset H$ and let $j\colon X \to X'$ be the inclusion map. Moreover, let $\underline{f}' = \{f_k', X', Y\}$ and $\underline{f} = \{f_k, X, Y\}$ be two fundamental sequences. Consider the homomorphism

$$j_*\colon H_n(X, \mathfrak{A}) \to H_n(X', \mathfrak{A})$$

induced by the map j and the homomorphisms

$$\underline{f}_*\colon H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A}) \quad \text{and} \quad \underline{f}'_*\colon H_n(X', \mathfrak{A}) \to H_n(Y, \mathfrak{A})$$

induced by the fundamental sequences $\underline{f}$ and $\underline{f}'$. Let us prove that

(1.4) *If $\underline{f}'$ is an extension of $\underline{f}$, then $\underline{f}_* = \underline{f}'_* j_*$.*

Proof. Setting $j_k(x) = x$ for every point $x \in H$, we get a fundamental sequence $\underline{j} = \{j_k, X, X'\}$ such that the composition $\underline{f}'\underline{j} = \{f_k' j_k, X, Y\}$ satisfies the condition $f_k' j_k(x) = f_k(x)$ for every point $x \in X$. It follows by (1.1) that $\underline{f} \simeq \underline{f}'\underline{j}$ and consequently ([2], p. 242) $\underline{f}_* = \underline{f}'_* \underline{j}_*$. It remains to observe ([2], p. 242) that the homomorphism $\underline{j}_*$ coincides with the homomorphism j_* induced by the map j.

Hence by (1.4) we obtain the following

(1.5) THEOREM. *Let X, X', Y be compacta in H such that $X\subset X'$. If a fundamental sequence $\underline{f}$ from X to Y has an extension onto X', then the kernel of the homomorphism $\underline{f}_*\colon H_n(X,\mathfrak{A})\to H_n(Y,\mathfrak{A})$ induced by $\underline{f}$ contains the kernel of the homomorphism $j_*\colon H_n(X,\mathfrak{A})\to H_n(X',\mathfrak{A})$ induced by the inclusion map $j\colon X\to X'$.*

By an analogous argument we infer that if $x_0\in X\subset X'$ and $y_0\in Y$ and if $\underline{f}=\{f_k,(X,x_0),(Y,y_0)\}$ and $\underline{f}'=\{f'_k,(X',x_0),(Y,y_0)\}$ are two pointed sequences, then

(1.6) *If $\underline{f}'$ is an extension of $\underline{f}$, then the induced homomorphisms $\underline{f}_*\colon \underline{\pi}_n(X,x_0)\to\underline{\pi}_n(Y,y_0)$ and $\underline{f}'_*\colon \underline{\pi}_n(X',x_0)\to\underline{\pi}_n(Y,y_0)$ satisfy the condition $\underline{f}_*=\underline{f}'_* j_*$, where j_* denotes the homomorphism of $\underline{\pi}_n(X,x_0)$ into $\underline{\pi}_n(X',x_0)$ induced by the inclusion map $j\colon X\to X'$.*

As an immediate consequence of (1.4) and (1.6), one gets the following

(1.7) THEOREM. *Let X, X', Y be compacta in H, $X\subset X'$, $x_0\in X$, $y_0\in Y$ and let $\underline{f}=\{f_k,(X,x_0),(Y,y_0)\}$ be a pointed sequence. If $\underline{f}$ has an extension $\underline{f}'=\{f'_k,(X',x_0),(Y,y_0)\}$, then the kernels of the homomorphisms of the groups $H_n(X,\mathfrak{A})$ and $\underline{\pi}_n(X,x_0)$ into $H_n(X',\mathfrak{A})$ and $\underline{\pi}_n(X',x_0)$ respectively induced by the inclusion map $j\colon X\to X'$ are contained in the kernels of the homomorphisms of these groups induced by $\underline{f}$.*

2. Weak and fundamental retractions and retracts. Let X be a closed subset of a compactum $X'\subset H$. A fundamental sequence $\underline{r}=\{r_k,X',X\}$ is said to be a *fundamental retraction of X' to X* if $r_k(x)=x$ for every point $x\in X$. Thus the fundamental retractions of X' to X are the same as the fundamental sequences from X' to X, being extensions of the fundamental identity sequence for X.

If $x_0\in X$, then a pointed sequence $\underline{r}=\{r_k,(X',x),(X,x_0)\}$ is said to be a *fundamental retraction of the pointed compactum* (X',x_0) to the pointed compactum (X,x_0) if $r_k(x)=$ for every point $x\in X$, that is if $\underline{r}$ is an extension of the pointed identity sequence for (X,x_0).

Let us observe that

(2.1) *If $\underline{r}=\{r_k,X',X\}$ is a fundamental retraction and $\{n_k\}$ is a sequence of indices with $\lim\limits_{k=\infty} n_k=\infty$, then setting $r'_k=r_{n_k}$ for $k=1,2,\ldots$ we get a fundamental retraction of X' to X.*

(2.2) *If $\underline{r}=\{r_k,X',X\}$ and $r'=\{r'_k,X'',X'\}$ are fundamental retractions then $\underline{r}\,\underline{r}'=\{r_kr'_k,X'',X\}$ is a fundamental retraction.*

(2.3) *If $\underline{r}=\{r_k,(X',x_0),(X,x_0)\}$ is a fundamental retraction and $\{n_k\}$ is a sequence of indices with $\lim\limits_{k=\infty} n_k=\infty$ then setting $r_k=r_{n_k}$ for $k=1,2,\ldots$, one gets a fundamental retraction of (X',x_0) to (X,x_0).*

(2.4) *If $\underline{r}\colon X'\to X$, and $\underline{r}'\colon X''\to X'$ (or $\underline{r}\colon (X',x_0)\to(X,x_0)$ and $\underline{r}'\colon(X'',x_0)\to(X',x_0)$) are fundamental retractions, then $\underline{r}\,\underline{r}'$ is a fundamental retraction.*

A fundamental sequence $f\colon X\to X$, is said to be a *h-fundamental sequence* if there exists a fundamental sequence $\underline{g}\colon Y\to X$ such that the composition $\underline{f}\,\underline{g}\colon Y\to Y$ is homotopic to the fundamental identity sequence $\underline{i}_Y$. Replacing in this definition X and Y by pointed compacta (X,x_0) and (Y,y_0) one gets the notion of a *pointed h-fundamental sequence*. Let us observe that

(2.5) *Every fundamental retraction is a h-fundamental sequence.*

In fact, $\underline{r}=\{r_k,X',X\}$ is a fundamental retraction then setting $g_k=i\colon H\to H$ for every $k=1,2,\ldots$, one gets a fundamental sequence $g=\{g_k,X,X\}$ such that $\underline{r}\underline{g}=\{r_kg_k,X,X\}$ is homotopic to the fundamental identity sequence $\underline{i}_X=\{i,X,X\}$. Hence $\underline{r}$ is an h-fundamental sequence. The same argument holds also in the case when compacta X and X' are pointed.

If there exists a fundamental retraction of X' to X (or of (X',x_0) to (X,x_0)) then X is said to be a *fundamental retract* of X'((X,x_0) is said to be a *fundamental retract of* (X',x_0), respectively).

A closed subset X_0 of a compactum $X\subset H$ is said to be a *fundamental neighborhood retract of X*, if there exists a closed neighborhood W of X_0 such that X_0 is a fundamental retract of the set $W\cap X$.

If $r\colon X'\to X$ is a retraction of a compactum $X'\subset H$, then there exits a map $f\colon H\to H$ such that $f(x)=r(x)$ for every point $x\in X'$. Setting $r_k\colon H\to H$ $r_k(x)=f(x)$ for every point $x\in H$, we get a sequence of maps $r_k\colon H\to H$ such that $\underline{r}=\{r_k,X',X\}$ is a fundamental retraction of X' to X. Hence

(2.6) *Every retract of a compactum $X'\subset H$ is a fundamental retract of this compactum.*

By an analogous argument one shows that

(2.7) *Every neighborhood retract of a compactum $X'\subset H$ is a fundamental neighborhood retract of this compactum.*

(2.8) *Every fundamental retract of a fundamental retract of a compactum is a fundamental retract of this compactum.*

Moreover,

(2.9) *If $\underline{r}'=\{r_k,X',X\}$ is a fundamental retraction of X' to X and X'' is a compactum such that $X\subset X''\subset X'$, then $\underline{r}=\{r_k,X'',X\}$ is a fundamental retraction of X'' to X.*

Let us show that the notion of the fundamental retract belong to topological invariants. More exactly, let us prove the following

(2.10) THEOREM. *Let X', Y' be two compacta in the Hilbert space H and let h be a homeomorphism mapping X' onto Y'. Then the set $Y = h(X)$ is a fundamental retract of Y' if X is a fundamental retract of X'.*

Proof. Consider a map $\alpha\colon H \to H$ such that $\alpha(x) = h(x)$ for every point $x \in X'$, and a map $\beta\colon H \to H$ such that $\beta(y) = h^{-1}(y)$ for every point $y \in Y'$. Now let us assume that there exists a fundamental retraction $\underline{r} = \{r_k, X', X\}$. Setting

$$s_k(y) = \alpha r_k \beta(y) \quad \text{for every point } y \in H \text{ and } k = 1, 2, \ldots,$$

we get a sequence of maps $s_k\colon H \to H$. Let us show that $\{s_k, Y', Y\}$ is a fundamental sequence.

Consider a neighborhood V of Y. Since $\alpha(X) = Y$, the set $U = \alpha^{-1}(V)$ is a neighborhood of X. Since $\underline{r}$ is a fundamental sequence from X' to X, there exists a neighborhood U' of X' (in H) such that

(2.11) $$r_k/U' \simeq r_{k+1}/U' \text{ in } U \quad \text{for almost all } k.$$

Since $\beta(Y') = X'$, the set $V' = \beta^{-1}(U')$ is a neighborhood of Y' (in H). It follows by (2.11) that

$$r_k\beta/V' \simeq r_{k+1}\beta/V' \text{ in } U \quad \text{for almost all } k,$$

whence also

$$\alpha r_k\beta/V' \simeq \alpha r_{k+1}\beta/V' \text{ in } V \quad \text{for almost all } k.$$

Thus we have shown that $\underline{s} = \{s_k, Y', Y\}$ is a fundamental sequence. Moreover, for every point $y \in Y$, we have

$$\beta(y) = h^{-1}(y) \in X,$$

whence

$$r_k\beta(y) = \beta(y) = h^{-1}(y) \quad \text{and} \quad s_k(y) = \alpha h^{-1}(y) = hh^{-1}(y) = y.$$

It follows that $\underline{s}$ is a fundamental retraction of Y' to Y.

By an analogous argument, we get the following

(2.12) THEOREM. *Let (X', x_0), (Y', y_0) be two pointed compacta in H and let h be a homeomorphism mapping (X', x_0) onto (Y', y_0). If (X, x_0) is a fundamental retract of (X', x_0), then (Y, y_0), where $Y = h(X)$, is a fundamental retract of (Y', y_0).*

(2.13) THEOREM. *If $\underline{r}$ is a fundamental retraction of X' to X, then the homomorphism $\underline{r}_*$ of the group $H_n(X', \mathfrak{A})$ into the group $H_n(X, \mathfrak{A})$ induced by $\underline{r}$ is an r-homomorphism.*

Proof. Since the fundamental identity sequence $\underline{i}_X = \{i_k, X, X\}$ induces the identity homomorphism for every group $H_n(X, \mathfrak{A})$, and since $\underline{r}$ is an extension of $\underline{i}_X$, we infer by (1.4) that the composition $\underline{r}_* j_*$ of the homomorphism $\underline{r}_*$ and of the homomorphism j_* induced by the inclusion map $j\colon X \to X'$ is the identity. Hence the homomorphism j_* is right-inverse to the homomorphism $\underline{r}_*$, and we infer that $\underline{r}_*$ is an r-homomorphism.

(2.14) COROLLARY. *If X is a fundamental retract of a compactum $X' \subset H$, then every homology group $H_n(X, \mathfrak{A})$ is isomorphic to a factor of the group $H_n(X', \mathfrak{A})$.*

By an analogous argument we infer by (1.6)

(2.15) THEOREM. *If $\underline{r}$ is a fundamental retraction of (X', x_0) to (X, x_0) then the homomorphism $\underline{r}_*$ of the group $\underline{\pi}_n(X', x_0)$ into the group $\underline{\pi}_n(X, x_0)$ induced by $\underline{r}$ is an r-homomorphism.*

(2.16) COROLLARY. *If X is a fundamental retract of a compactum $X' \subset H$ and $x_0 \in X$, then for $n > 1$ the group $\underline{\pi}_n(X, x_0)$ is isomorphic to a factor of the group $\underline{\pi}_n(X', x_0)$.*

3. Fundamental retractions for plane continua. Let E^2 denote the Euclidean plane which we consider as identical with the subset of the Hilbert space H consisting of all points of the form $(x_1, x_2, 0, 0, \ldots)$, and let p denote the projection of H onto E^2 given by the formula

$$p(x_1, x_2, x_3, \ldots) = (x_1, x_2, 0, 0, \ldots).$$

Let us prove the following

(3.1) THEOREM. *Let X, X' be two continua lying in E^2 such that $X \subset X'$. X is a fundamental retract of X' if and only if no component of the set $E^2 - X$ is contained in X'.*

Proof. First let us assume that there exists a fundamental retraction $\underline{r} = \{r_k, X', X\}$ of X' to X and let G be a bounded component of the set $E^2 - X$. Then there exists in the boundary $F = \overline{G} - G$ of G a 1-dimensional true cycle $\underline{\gamma}$ (over the group $\mathfrak{N}$ of integers) homologous to zero in $\overline{G}$, but not homologous to zero in X. If we denote by $(\underline{\gamma})$ the element of the group $H_1(X, \mathfrak{N})$ with the representative $\underline{\gamma}$, and by $(\underline{\gamma})'$ the element of the group $H_1(X', \mathfrak{N})$ with the representative $\underline{\gamma}$, then the homomorphism $j_*\colon H_1(X, \mathfrak{N}) \to H_1(X, \mathfrak{N})$, induced by the inclusion map $j\colon X \to X'$, assings $(\underline{\gamma})'$ to $(\underline{\gamma})$. Since the homomorphism $\underline{i}_*\colon H_1(X, \mathfrak{N}) \to H_1(X, \mathfrak{N})$ induced by the fundamental identity sequence $\underline{i}$ for X is the identity and $\underline{r}$ is an extension of $\underline{i}$, we infer by (1.4) that $(\underline{\gamma}) = \underline{r}((\underline{\gamma})')$. It follows that $\underline{\gamma} \not\sim 0$ in X'. But $\underline{\gamma} \sim 0$ in $\overline{G}$, whence $\overline{G}$ is not contained in X'. It suffices to observe that also the unbounded component of $E^2 - X$ is not contained in X', in order to obtain the first part of Theorem (3.1).

Now let us assume that no component of the set E^2-X is contained in X'. The collection of all components of the set E^2-X is finite or countable. We shall consider only the second case, because the proof in the first one is analogous, but simpler.

Let us arrange the components of E^2-X in a sequence $G_0, G_1, \ldots$, where $G_i \neq G_j$ for $i \neq j$ and G_0 is the unbounded component. Since no G_i is contained in X', we can select a point

$$a_i \in G_i - X' \quad \text{for every } i = 1, 2, \ldots$$

It is clear that for every $k = 1, 2, \ldots$ there exists in E^2 a sequence of disks $D_{k,0}, D_{k,1}, \ldots$ with interiors $\mathring{D}_{k,0}, \mathring{D}_{k,1}, \ldots$ satisfying the following conditions

(1) $X \subset D_{k,0}$, $a_i \in \mathring{D}_{k,i} \subset D_{k,i} \subset G_i$ *for* $i = 1, 2, \ldots$;

(2) *if* $x \in G_0 \cap D_{k,0}$, *then* $\varrho(x, X) < 1/k$;

(3) *if* $x \in G_i - D_{k,i}$, *then* $\varrho(x, X) < 1/k$ *for* $i = 1, 2, \ldots$;

(4) $D_{k,0} \supset D_{k+1,0}$, $D_{k,i} \subset D_{k+1,i}$ *for* $i = 1, 2, \ldots$

By (1) we can assign to every $k, i = 1, 2, \ldots$, an open disk $U_{k,i} \subset \mathring{D}_{k,i} - X'$ such that

$$a_i \in U_{k,i} \quad \text{and} \quad U_{k,i} \supset U_{k+1,i} \quad \text{for} \quad k, i = 1, 2, \ldots$$

Now lest us consider a disk $D \subset E^2$ such that

$$X' \cup D_{1,0} \subset \mathring{D}$$

and let us set

$$A_k = D - \bigcup_{i=1}^{k} U_{k,i}, \quad B_k = D_{k,0} - \bigcup_{i=1}^{k} D_{k,i}.$$

It is clear that A_k, B_k are compacta (even curvilinear polyhedra) such that

$$X \subset B_k \subset A_k \subset E^2, \quad X' \subset A_k, \quad B_{k+1} \subset B_k \quad \text{for} \quad k = 1, 2, \ldots,$$

and that for k sufficiently large the distance of every point of B_k from X is arbitrarily small.

Moreover, there exist for $k = 1, 2, \ldots$ a retraction

$$s_k\colon A_k \to B_k$$

such that

$$s_k(D_{k,i} - U_{k,i}) = D_{k,i} - \mathring{D}_{k,i} \quad \text{for} \quad i = 1, 2, \ldots, k,$$

and a map $\bar{s}_k\colon E^2 \to H$ such that

$$\bar{s}_k(x) = s_k(x) \quad \text{for every point } x \in A_k,$$

$$\bar{s}_k(E^2 - D_{k,0}) \subset D_{k,0} - \mathring{D}_{k,0} \quad \text{and} \quad \bar{s}_k(U_{k,i}) \subset D_{k,i} \quad \text{for} \quad i = 1, 2, \ldots, k.$$

One can easily see that there exists a homotopy $\varphi_k\colon A_k \times \langle 0, 1\rangle \to B_k$ such that

$$\varphi_k(x, 0) = \bar{s}_k(x), \quad \varphi_k(x, 1) = \bar{s}_{k+1}(x) \quad \text{for every point } x \in A_k$$

and

$$\varphi_k(x, t) = x \quad \text{for every } (x, t) \in X \times \langle 0, 1\rangle.$$

Let us show that setting

$$r_k = \bar{s}_k p \quad \text{for} \quad k = 1, 2, \ldots,$$

we get a fundamental retraction $\underline{r} = \{r_k, X', X\}$. First let us observe that for every neighborhood U of X (in H) the inclusion $B_k \subset U$ holds for almost all k. Moreover, it is clear that the set

$$V_k = p^{-1}(A_k) \subset r_k^{-1}(B_k)$$

is a neighborhood of the set X' (in H) for every $k = 1, 2, \ldots$ Setting

$$\psi_k(x, t) = \varphi_k(p(x), t) \quad \text{for every } (x, t) \in V_k \times \langle 0, 1\rangle,$$

we get a homotopy

$$\psi_k\colon V_k \times \langle 0, 1\rangle \to B_k$$

joining the map r_k/V_k with the map r_{k+1}/V_k and satisfying the condition

$$\psi_k(x, t) = x \quad \text{for every point } x \in X.$$

Hence $\underline{r}$ is a fundamental retraction of X' to X and the proof of Theorem (3.1) is complete.

4. A special kind of fundamental retraction. Let us prove the following

(4.1) THEOREM. *If $\underline{r}$ is a fundamental retraction of a compactum $Z \subset H$, then for every sequence $\{V_k\}$ of neighborhoods of Y such that each neighborhood of Y contains V_k for almost all k, there exists a fundamental retraction $\underline{r}' = \{r'_k, Z, Y\} \simeq \underline{r}$ satisfying the condition $r'_k(y) = y$ for every point $y \in V_k$, $k = 1, 2, \ldots$*

Proof. Let $\underline{r} = \{r_k, Z, Y\}$ be a fundamental retraction of Z to Y. First let us prove that

(4.2) *There exists a sequence of indices $n_1 \leqslant n_2 \leqslant \ldots$ such that $\lim_{i=\infty} n_i = \infty$ and that $(n_i - 1)\cdot\varrho(y, r_{n_i}(y)) \leqslant 1$ for every point $y \in \overline{V}_i$.*

Since $r_i(y) = y$ for every point $y \in Y$, and since for each neighborhood V of Y the inclusion $V_k \subset V$ holds for almost all k, one can easily see that there exists a sequence of indices $1 < k_1 < k_2 < \ldots$ such that

(4.3) $$\varrho(y, r_{m+1}(y)) \leqslant \frac{1}{m} \quad \text{if} \quad y \in \overline{V}_j \text{ with } j > k_m.$$

Setting

$$n_i = 1 \quad \text{for} \quad i \leqslant k_1,$$

$$n_i = m+1 \quad \text{for} \quad k_m < i \leqslant k_{m+1},\ m = 1, 2, \ldots,$$

we get a sequence of indices $n_1 \leqslant n_2 \leqslant \ldots$ such that $\lim_{i=\infty} n_i = \infty$. Moreover, the inequality $(n_i - 1)\cdot \varrho\big(y, r_{n_i}(y)\big) \leqslant 1$ is obvious for $i \leqslant k_1$, because then $n_i = 1$. If $k_m < i \leqslant k_{m+1}$, then $n_i = m+1$ and consequently (4.3) implies that for every point $y \in \overline{V}_i$, the relation

$$(n_i - 1)\cdot \varrho\big(y, r_{n_i}(y)\big) = m\cdot \varrho\big(y, r_{m+1}(y)\big) \leqslant 1$$

holds. Thus the proof of (4.2) is finished.

Now let us set

$$\lambda_k(y) = y - r_{n_k}(y) \quad \text{for every point } y \in \overline{V}_k.$$

In particular, $\lambda_k(y) = 0$ for every point $y \in Y$. Thus one gets a map $\lambda_k\colon \overline{V}_k \to H$ such that $(n_k - 1)\cdot|\lambda_k(y)| \leqslant 1$ for every point $y \in \overline{V}_k$. It is clear that λ_k can be extended to a map $\lambda_k'\colon H \to H$ satisfying the condition

$$(n_k - 1)\cdot|\lambda_k'(y)| \leqslant 1 \quad \text{for every point } y \in H.$$

Setting

$$r_k'(y) = r_{n_k}(y) + \lambda_k'(y) \quad \text{for every point } y \in H,$$

one gets a sequence of maps $r_k'\colon H \to H$ such that

(4.4) $$r_k'(y) = r_{n_k}(y) + \lambda_k(y) = y \quad \text{for every point } y \in V_k.$$

Since $\lim_{k=\infty} n_k = \infty$, we infer that $\{r_{n_k}, Z, Y\}$ is a fundamental sequence homotopic to $\underline{r}$, actually, a fundamental retraction of Z to Y. Moreover,

$$\varrho\big(r_k'(y), r_{n_k}(y)\big) = |\lambda_k'(y)| \leqslant \frac{1}{n_k - 1} \quad \text{for every point } y \in H \text{ and } k > k_1.$$

It follows that the sequence $\{r_k'\}$ is obtained from the sequence $\{r_{n_k}\}$ by an infinitely small translation. Hence

$$\underline{r}' = \{r_k', Z, Y\} \simeq \{r_{n_k}, Z, Y\} \simeq \underline{r}.$$

Thus we have shown that $\underline{r}'$ is a fundamental retraction of Z to Y satisfying by (4.4) the condition of Theorem (4.1).

5. Fundamental retracts and extension of fundamental sequences. Now let us prove the following

(5.1) THEOREM. *If X is a fundamental retract of X', then for every fundamental sequence $\underline{f} = \{f_k, X, Y\}$ there exists a fundamental sequence $\underline{f}'$ from X' to Y which is an extension of $\underline{f}$.*

Proof. Let $\underline{r} = \{r_k, X', X\}$ be a fundamental retraction of X' to X. It suffices to set $\underline{f}' = \underline{f}\underline{r}$ in order to obtain the required extension of $\underline{f}$.

(5.2) THEOREM. *Let X, X' be compacta such that $X \subset X' \subset H$, and let Y be a fundamental retract of a compactum $Y' \subset H$. If for a fundamental sequence $\underline{f} = \{f_k, X, Y\}$ there exists a fundamental sequence $\underline{f}' = \{f_k', X', Y'\}$ such that $f_k'(x) = f_k(x)$ for every point $x \in X$, then there exists also a fundamental sequence $\hat{\underline{f}}$ from X' to Y which is an extension of $\underline{f}$.*

Proof. It is clear that there is a sequence $\{V_k\}$ of neighborhoods of Y satisfying both of the following conditions:

(1) *If V is a neighborhood of Y, then $V_k \subset V$ for almost all k.*

(2) $f_k(X) \subset V_k$ *for every* $k = 1, 2, \ldots$

By Theorem (4.1), there exists a fundamental retraction $\underline{r}' = \{r_k', Y', Y\}$ such that

$$r_k'(y) = y \quad \text{for every point } y \in V_k,\ k = 1, 2, \ldots$$

Setting $\hat{\underline{f}} = \underline{r}'\underline{f}' = \{r_k' f_k', X', Y\}$, we get a fundamental sequence such that

$$r_k' f_k'(x) = r_k' f_k(x) = f_k(x) \quad \text{for every point } x \in X,$$

because $f_k'(x) = f_k(x) \in f_k(X) \subset V_k$. Hence $\hat{\underline{f}}$ is an extension of $\underline{f}$.

6. Fundamental absolute retracts and absolute neighborhood retracts. A compactum $X \subset H$ is said to be a *fundamental absolute retract* (shortly $X \in$ FAR) if it is a fundamental retract of every compactum $X' \subset H$ containing X. If for every compactum X' such that $X \subset X' \subset H$ the set X is a fundamental neighborhood retract of X', then X is said to be a *fundamental absolute neighborhood retract* (shortly $X \in$ FANR). Evidently every FAR is an FANR, and every ANR is an FANR.

It is clear that

(6.1) *Every* AR-*set lying in H is an* FAR *and every* ANR-*set lying in H is an* FANR.

Now let us prove the following

(6.2) THEOREM. *Every fundamental retract of an* FAR-*set is an* FAR-*set.*

Proof. If Y_0 is a fundamental retract of $Y \in$ FAR, then there exists a fundamental retraction $\underline{r} = \{r_k, Y, Y_0\}$. Let Y' be a compactum such that $Y_0 \subset Y' \subset H$. Since $Y \in$ FAR, there exists a fundamental retraction $\hat{\underline{r}} = \{\hat{r}_k, Y \cup Y', Y\}$. Setting $\underline{r}' = \underline{r}\hat{\underline{r}} = \{r_k\hat{r}_k, Y \cup Y', Y_0\}$ we get (by (2.2)) a fundamental retraction of the set $Y \cup Y'$ to Y_0. It follows by (2.9) that $\{r_k \hat{r}_k, Y', Y_0\}$ is a fundamental retraction, whence $Y_0 \in$ FAR.

(6.3) Corollary. *FAR-sets are the same as fundamental retracts of the AR-sets lying in H.*

In order to obtain (6.3) from (6.2), it suffices to observe that

(6.4) *For every compactum $X \subset H$ there exists an AR-sets X' such that $X \subset X' \subset H$.*

In fact, the convex hull X' of X is a compactum ([4], p. 7) containing X. The convexity of X' implies ([3], p. 358; also [1], p. 85) that $X' \in$ AR.

If we recall that a disk is an AR-set, we get from (6.2) and (3.1) the following

(6.5) Corollary. *Every non-empty continuum $X \subset E^2$ which does not decompose E^2 is an FAR-set.*

The following proposition is useful for the sequel:

(6.6) *Every compactum $X \subset H$ is the intersection of a decreasing sequence of ANR-sets lying in the convex hull X' of X.*

Proof. Let $K(x, \varepsilon)$ denote, for every point $x \in X$ and every $\varepsilon > 0$, the set of all points $y \in H$ such that $\varrho(x, y) \leqslant \varepsilon$. Since X is compact, there exists for every $n = 1, 2, \ldots$, a finite system of points $a_{n,1}, a_{n,2}, \ldots$ $\ldots, a_{n,k_n} \in X$ such that X is contained in the set

$$Y_n = \bigcup_{i=1}^{k_n} \left[K\left((a_{n,i}, \frac{1}{n}\right) \cap X'\right].$$

Setting

$$X_n = Y_1 \cap Y_2 \cap \ldots \cap Y_n,$$

we obtain a decreasing sequence of compacta $X_1, X_2, \ldots$ lying in X' such that $X = \bigcap_{n=1}^{\infty} X_n$. It is evident that X_n may be represented as a finite union of sets which are common parts of the set X' and of a finite system of closed balls in H. We infer, by easy induction, that each X_n is an ANR-set. Thus the proof of (6.6) is finished.

(6.7) Theorem. *Every fundamental retract of an FANR-set is an FANR-set.*

Proof. If Y_0 is a fundamental retract of $Y \in$ FANR, then there exists a fundamental retraction $\underline{r} = \{r_k, \underline{Y}, Y_0\}$. Let Y' be a compactum such that $Y_0 \subset Y' \subset H$. Since $Y \in$ FANR, there exists a closed neighborhood V of Y (whence also of Y_0) such that Y is a fundamental retract of the set $V \cap (Y \cup Y')$. Since Y_0 is a fundamental retract of Y, we infer by (2.2) that Y_0 is a fundamental retract of the set $V \cap (Y \cup Y')$, whence also (by (2.9)) a fundamental retract of the set $V \cap Y'$, which is a neighborhood of Y_0 in the space Y'. Thus the proof of the first part of Theorem (6.7) is finished.

Combining Theorem (6.7) with proposition (6.6), we get:

(6.8) Corollary. *FANR-sets are the same as fundamental retracts of ANR-sets lying in H.*

(6.9) Corollary. *For every $X \in$ FANR there exists a polyhedron P such that every group $H_n(X, \mathfrak{A})$ is an r-image of the group $H_n(P, \mathfrak{A})$.*

Proof. By (6.8), there exists an ANR-set Z such that X is a fundamental retract of Z. By Theorem (2.13), each group $H_n(X, \mathfrak{A})$ is an r-image of the group $H_n(Z, \mathfrak{A})$. Moreover, $Z \in$ ANR implies ([1],)p. 106) that there is a polyhedron P such that every group $H_n(Z, \mathfrak{A})$ is an r-image of the group $H_n(P, \mathfrak{A})$. It remains to recall that every r-image of an r-image of a group is an r-image of this group.

Applying Theorem (2.15) and (6.8), we get by an analogous argument the following

(6.10) Corollary. *For every $X \in$ FANR and for every point $x_0 \in X$, there exist a polyhedron P and a point $a_0 \in P$ such that every group $\underline{\pi}_n(X, x_0)$ is an r-image of the group $\underline{\pi}_n(P, a_0)$.*

(6.11) Corollary. *All Betti groups of an FANR-set are finitely generated and almost all are trivial.*

(6.12) Corollary. *All fundamental groups of an FANR-set are finitely generated.*

It follows by Corollary (6.11) that every connected FANR-set lying in the plane $E^2 \subset H$ decomposes E^2 into a finite number of regions. On the other hand, it is clear that every continuum X lying in E^2 and decomposing E^2 into a finite number of regions is contained in a polyhedron $P \subset E^2$ such that no component of the set $E^2 - X$ is contained in P. Hence, by Theorem (3.1), we obtain the following

(6.13) Corollary. *A continuum $X \subset E^2$ is an FANR-set, if and only if $E^2 - X$ has a finite number of components.*

(6.14) Theorem. *If Y is an ANR-set lying in the Hilbert space, then every fundamental neighborhood retract of Y is an FANR-set.*

Proof. Since $Y \in$ ANR, there exists a neighborhood V of Y and a map $\varphi\colon H \to H$ such that

$$\varphi(V) = Y \quad \text{and} \quad \varphi(y) = y \quad \text{for every point } y \in Y.$$

Let Y_0 be a fundamental neighborhood retract of Y. Then there are a closed neighborhood V_0 of Y_0 (in H) and a fundamental retraction $\underline{r} = \{r_k, V_0 \cap Y, Y_0\}$. To this neighborhood V_0 there exists a closed neighborhood $V_1 \subset V$ of Y_0 such that the set $\varphi(V_1)$ lies in the interior of V_0.

Consider now a compactum Y' such that $Y_0 \subset Y' \subset H$, and let us set $r'_k = r_k\varphi$: $H \to H$ for $k = 1, 2, \ldots$ In order to finish the proof of (6.14), it remains to show that

$$\underline{r}' = \{r'_k, V_1 \cap Y', Y_0\}$$

is a fundamental retraction.

Since $Y_0 \subset Y$, the equality $\varphi(y) = y$ holds for every point $y \in Y_0$. We have $r'_k(y) = y$ for every point $y \in Y_0$ and $k = 1, 2, \ldots$ In order to show that $\underline{r}'$ is a fundamental retraction of $V_1 \cap Y'$ into Y_0, consider an arbitrary neighborhood W if Y_0. Since $\underline{r}$ is a fundamental retraction, there exist a neighborhood U of the set $V_0 \cap Y$ and a homotopy

$$\psi_k\colon\ U \times \langle 0, 1\rangle \to W$$

such that $\psi_k(x, 0) = r_k(x)$, $\psi_k(x, 1) = r_{k+1}(x)$ for every point $x \in U$.

Since $\varphi(V_1)$ lies in the interior of V_0, there exists a neighborhood U_1 of the set $V_1 \cap Y'$ such that

$$\varphi(U_1) \subset V_0 \cap Y \subset U .$$

It follows that the formula

$$\vartheta_k(x, t) = \psi_k[\varphi(x), t] \quad \text{for} \quad (x, t) \in U_1 \times \langle 0, 1\rangle$$

defines a homotopy ϑ_k: $U \times \langle 0, 1\rangle \to W$ joining r'_k with r'_{k+1}.

Thus we have shown that $\underline{r}'$ is a fundamental retraction and the proof of Theorem (6.14) is finished.

(6.15) Remark. Let us observe that not every fundamental neighborhood retract of an FAR-set is an FANR-set. In fact, let A be the set consisting of points $a_n = (1/n, 0, 0, \ldots)$ with $n = 1, 2, \ldots$ and of the point $a_0 = (0, 0, \ldots)$. Let L_n denote the segment (in H) with endpoints a_n and $b = (0, 1, 0, \ldots)$ for $n = 0, 1, 2, \ldots$ One can easily see (by (3.1)) that the set $X = \bigcup_{i=0}^{\infty} L_i$ is an FAR-set and A is its fundamental neighborhood retract (even a neighborhood retract). However, A is not an FANR-set, because its 0-dimensional Betti number is infinite.

7. FAR-sets and FANR-sets and extension of fundamental sequences.

By Theorem (1.5), in general not every fundamental sequence from X to Y has an extension onto a given compactum $X' \subset H$ containing X. Now let us consider some cases in which the existence of an extension is ensured.

It follows by Theorem (5.1) that

(7.1) *If $X \in$ FAR and X' is a compactum such that $X \subset X' \subset H$, then every fundamental sequence from X to Y can be extended to a fundamental sequence from X' to Y.*

and

(7.2) *If $X \in$ FANR and X' is a compactum such that $X \subset X' \subset H$, then there is a closed neighborhood Z of X such that every fundamental sequence from X to Y can be extended to a fundamental sequence from the set $Z \cap X'$ to Y.*

Now let us prove the following proposition:

(7.3) *If X and $X' \supset X$ are compacta in H and if $Y \in$ FAR, then every fundamental sequence from X to Y can be extended to a fundamental sequence from X' to Y.*

Proof. Let $\underline{f} = \{f_k, X, Y\}$ be a fundamental sequence. It is clear that the set

$$Z = Y \cap \bigcup_{k=1}^{\infty} f_k(x)$$

is a compactum lying in H. By (6.4) there exists in H and AR-set $Y' \supset Z$. Then there is a map φ: $H \to H$ such that

$$\varphi(H) \subset Y' \quad \text{and} \quad \varphi(y) = y \quad \text{for every point } y \in Y' .$$

Setting $f'_k = \varphi f_k$ for $k = 1, 2, \ldots$, we get a sequence of maps f'_k: $H \to H$ with values in Y'. Since $Y' \in$ AR, all these maps are homotopic in Y', and we infer that $\underline{f}' = \{f'_k, X', Y'\}$ is a fundamental sequence. Moreover, if $x \in X$, then $f_k(x) \in f_k(X) \subset Z \subset Y'$ and consequently $f'_k(x) = \varphi f_k(x) = f_k(x)$.

Since Y, as an FAR-set lying in Y', is a fundamental retract of Y', we see that all the hypotheses of Theorem (5.2) are satisfied and we infer that there is a fundamental sequence $\underline{\hat{f}}$ from X' to Y which is an extension of $\underline{f}$. Moreover, let us show that

(7.4) *If X and $X' \supset X$ are compacta in H and if $Y \in$ FANR, then for every fundamental sequence $\underline{f} = \{f_k, X, Y\}$ there is a closed neighborhood M of X such that $\underline{f}$ can be extended to a fundamental sequence from $M \cap X'$ to Y.*

Proof. Let Y' denote the convex hull of the set $Z = Y \cup \bigcup_{k=1}^{\infty} f_k(X)$. Since $Y \in$ FANR, there is a closed neighborhood N of Y such that Y is a fundamental retract of the set $N \cap Y'$. Evidently N contains a neighborhood N_0 of Y which is the union of a finite number of balls (in the space H). Since Y' is a convex compactum, we can easily see that the set $Y'' = N_0 \cap Y'$ is an ANR-set which is a neighborhood of Y in the space Y'. It follows that there exists an index k_0 such that

$$f_k(X) \subset Y'' \quad \text{for every } k > k_0 .$$

Since $Y'' \in$ ANR, there exist a neighborhood V_0 of Y'' and a map $\varphi\colon H \to H$ such that

$$\varphi(V_0) \subset Y'' \quad \text{and} \quad \varphi(y) = y \quad \text{for every point } y \in Y'' .$$

Let us show that setting

$$f'_k = f_k \text{ for } k = 1, 2, \ldots, k_0 \quad \text{and} \quad f'_k = \varphi f_k \text{ for } k > k_0 ,$$

one gets a sequence of maps $f'_k\colon H \to H$ such that there is a closed neighborhood M of X such that $\{f'_k, M \cap X', Y''\}$ is a fundamental sequence. In fact, since V_0 is a neighborhood of $Y'' \supset Y$ and since $\{f_k, X, Y\}$ is a fundamental sequence, there exists a closed neighborhood M_0 of X such that

$$f_k/M_0 \simeq f_{k+1}/M_0 \text{ in } V_0 \quad \text{for almost all } k .$$

If we recall that $\varphi(V_0) \subset Y''$, we infer that $\varphi f_k/M_0 \simeq \varphi f_{k+1}/M_0$ in Y'' for almost all k, and consequently the homotopy

$$f'_k/M_0 \simeq f'_{k+1}/M_0 \text{ in } V$$

holds for every neighborhood V of Y'' for almost all k. It follows that if M is a closed neighborhood of X contained in the interior of M_0, then $\{f'_k, M \cap X', Y''\}$ is a fundamental sequence. Moreover, if $x \in X$ then

$$f'_k(x) = f_k(x) \quad \text{for every point } x \in X \text{ and } k = 1, 2, \ldots ,$$

because $f'_k = f_k$ for $k \leqslant k_0$ and $f_k(x) \in f_k(X) \subset Y''$, whence $f'_k(x) = \varphi f_k(x) = f_k(x)$ for $k > k_0$.

The inclusion $N_0 \subset N$ implies that $Y'' \subset N \cap Y'$. As we have already shown, $\{f'_k, M \cap X', Y''\}$ is a fundamental sequence, and consequently $[f'_k, M \cap X', N \cap Y'\}$ is also a fundamental sequence, being an extension of the fundamental sequence $\{f_k, X, N \cap Y'\}$. Since Y is a fundamental retract of $N \cap Y'$, we infer by Theorem (5.2) that there exists a fundamental sequence $\underline{\hat{f}}$ from $M \cap X'$ to Y which is an extension of $\underline{f}$. Thus the proof of (7.4) is achieved.

8. Topological invariance of FAR and of FANR. Let us prove the following

(8.1) THEOREM. *If X is an FAR-set, then every set $Y \subset H$ homeomorphic to X is also an FAR-set.*

Proof. Let $h\colon X \to Y$ be a homeomorphism. It is clear that there exist two maps $f\colon H \to H$ and $g\colon H \to H$ such that

$$f(x) = h(x) \text{ for every point } x \in X \text{ and } g(y) = h^{-1}(y) \text{ for every point } y \in Y .$$

Consider a set $\hat{Y} \in$ AR such that $Y \subset \hat{Y} \subset H$ and let us denote the set $X \cup g(\hat{Y})$ by $\hat{X}$. Since $X \in$ FAR, there exists a fundamental retraction $\underline{r} = \{r_k, \hat{X}, X\}$. Let us show that $\{fr_k g, \hat{Y}, Y\}$ is a fundamental retraction.

Let V be a neighborhood of Y. Then $U = f^{-1}(V)$ is a neighborhood of X and since $\underline{r}$ is a fundamental retraction, there exists a neighborhood $\hat{U}$ of the set $\hat{X}$ such that for almost all k there is a homotopy

$$\varphi_k\colon \hat{U} \times \langle 0, 1\rangle \to U$$

satisfying the condition:

$$\varphi_k(x, 0) = r_k(x) , \quad \varphi_k(x, 1) = r_{k+1}(x) \quad \text{for every point } x \in \hat{U} .$$

Since $g(\hat{Y}) \subset \hat{X}$, we infer that $\hat{V} = g^{-1}(\hat{U})$ is a neighborhood of the set $\hat{Y}$. It follows that setting

$$\vartheta_k(y, t) = \varphi_k(g(y), t) \quad \text{for every } (y, t) \in \hat{V} \times \langle 0, 1\rangle ,$$

we get a homotopy $\vartheta_k\colon \hat{V} \times \langle 0, 1\rangle \to U$ satisfying the condition

$$\vartheta_k(y, 0) = r_k g(y); \quad \vartheta_k(y, 1) = r_{k+1} g(y) \quad \text{for every point } y \in \hat{V} .$$

Since $f(U) \subset V$, we infer that $f\vartheta_k\colon \hat{V} \times \langle 0, 1\rangle \to V$ is a homotopy satisfying the condition

$$f\vartheta_k(y, 0) = fr_k g(y); \quad f\vartheta_k(y, 1) = fr_{k+1} g(y) \quad \text{for every point } y \in \hat{V} .$$

Thus we have shown that $\{fr_k g, \hat{Y}, Y\}$ is a fundamental retraction. Hence Y is a fundamental retract of the set $\hat{Y} \in AR$, and we infer by Theorem (6.2) that $y \in$ FAR.

(8.2) THEOREM. *If X is an FANR-set, then every set $Y \subset H$ homeomorphic to X is also an FANR-set.*

Proof. Let $\hat{X}, \hat{Y}, f, g, h, h^{-1}$ be as in the proof of Theorem (8.1). Since $X \in$ FANR, there exist a closed neighborhood M of X and a fundamental retraction $\underline{r} = \{r_k, M \cap \hat{X}, X\}$.

Let V be a neighborhood of Y. Then $U = f^{-1}(V)$ is a neighborhood of X and since $\underline{r}$ is a fundamental retraction, there exists a neighborhood U' of the set $X' = M \cap \hat{X}$ such that for almost all k there is a homotopy $\varphi_k\colon U' \times \langle 0, 1\rangle \to U$ satisfying the following condition

$$\varphi_k(x, 0) = r_k(x) , \quad \varphi_k(x, 1) = r_{k+1}(x) \quad \text{for every point } x \in U' .$$

Now let us set $Y' = g^{-1}(M) \cap \hat{Y}$. Evidently Y' is a neighborhood of Y in the space $\hat{Y}$. Since $g(\hat{Y}) \subset \hat{X}$, we infer that $g(Y') \subset X'$ and consequently the set $V' = g^{-1}(U')$ is a neighborhood of the set Y'. It follows that setting

$$\vartheta_k(y, t) = \varphi_k(g(y), t) \quad \text{for every} \quad (y, t) \in V' \times \langle 0, 1\rangle,$$

we get a homotopy ϑ_k: $V' \times \langle 0, 1\rangle \to U$. By an analogous argument to that used in the proof of (8.1) we show that $f\vartheta_k$: $V' \times \langle 0, 1\rangle \to V$ is a homotopy joining the maps $fr_k g/V'$ and $fr_{k+1}g/V'$.

Hence $\{fr_k g, Y', Y\}$ is a fundamental retraction. Since Y' is a neighborhood of Y in the space $\hat{Y} \in \mathrm{AR}$, we infer by Theorem (6.14) that $Y \in \mathrm{FANR}$.

Remark. Theorems (8.1) and (8.2), allow to generalize the concepts of the FAR- and FANR-sets as follows:

A compactum X (not necessarily lying in H) is an FAR (or an FANR, respectively) if there exists a subset Y of H homeomorphic to X and being an FAR (or an FANR, respectively) in the previous sense.

9. Two conditions characterizing FAR**-sets.** The first of these conditions appears in the following

(9.1) THEOREM. *A compactum $X \subset H$ is an* FAR*-set if and only if every neighborhood U of X contains a neighborhood U_0 of X which is contractible in U.*

Proof. If $X \in \mathrm{FAR}$, then there exists, by (6.4), an AR-set $\hat{X}$ such that $X \subset \hat{X} \subset H$. Consider a map φ: $H \to H$ such that

$$\varphi(H) \subset \hat{X} \quad \text{and} \quad \varphi(x) = x \quad \text{for every point } x \in \hat{X}.$$

Let $\{V_k\}$ be a sequence of neighborhoods of X such that for each neighborhood V of X the inclusion $V_k \subset V$ holds for almost all k. By Theorem (4.1) there exists a fundamental retraction $\underline{r} = \{r_k, \hat{X}, X\}$ such that $r_k(x) = x$ for every point $x \in V_k$.

Consider now a neighborhood U of X. It is clear that there exists an index k_1 such that

$$V_{k_1} \subset U \quad \text{and} \quad r_{k_1}(\hat{X}) \subset U.$$

For this index k_1, there exists an index k_0 such that for every point $x \in V_{k_0}$ the segment (in H) with endpoints x and $\varphi(x)$ lies in V_{k_1}. Now let us set

(9.2) $f(x, t) = 2t\cdot\varphi(x) + (1-2t)\cdot x$ for every point $x \in V_{k_0}$ and $0 \leqslant t \leqslant \frac{1}{2}$,

(9.3) $f(x, t) = r_{k_1}\varphi[(2-2t)\cdot\varphi(x)]$ for every point $x \in V_{k_0}$ and $\frac{1}{2} \leqslant t \leqslant 1$.

The formulas (9.2) and (9.3) are compatible, because for $t = \frac{1}{2}$ the value of $f(x, t)$ given by (9.2) is $\varphi(x)$, and (9.3) gives $f(x, \frac{1}{2}) = r_{k_1}\varphi\varphi(x) = r_{k_1}\varphi(x) = \varphi(x)$, since φ satisfies the condition $\varphi\varphi(x) = \varphi(x)$ for every point $x \in H$, and $\varphi(x)$ belongs, for $x \in V_{k_0}$, to V_{k_1}, whence $r_{k_1}\varphi(x) = \varphi(x)$.

Thus formulas (9.2) and (9.3) both define a map of $V_{k_0} \times \langle 0, 1\rangle$ into H. It follows by (9.2) that for $0 \leqslant t \leqslant \frac{1}{2}$ the point $f(x, t)$ belongs to the segment with endpoints x and $\varphi(x)$, whence $f(x, t) \in V_{k_1} \subset U$ for $(x, t) \in V_{k_0} \times \langle 0, \frac{1}{2}\rangle$. On the other hand, the formula (9.3) implies that for $(x, t) \in V_{k_0} \times \langle \frac{1}{2}, 1\rangle$ the point $f(x, t)$ lies in the set $r_{k_1}(\hat{X}) \subset U$. Moreover, $f(x, 0) = x$ and $f(x, 1) = r_{k_1}\varphi(0) = \text{const}$. Hence f is a homotopy contracting the set $U_0 = V_{k_0}$ in the set U to the point $r_{k_1}\varphi(0)$. Thus the necessity of the condition is proved.

Now let us assume that every neighborhood U of a compactum $X \subset H$ contains a neighborhood U_0 contractible to a point in U. As before, let us consider an AR-set $\hat{X}$ such that $X \subset \hat{X} \subset H$. In order to prove that $X \in \mathrm{FAR}$, it suffices to show—by Theorem (6.2)—that there exists a fundamental retraction of X to X.

Consider a decreasing sequence $\{V_k\}$ of open neighborhoods of X such that for each neighborhood V of X the inclusion $V_k \subset V$ holds for almost all k. By our hypothesis, there exists a sequence $\{W_k\}$ of closed neighborhoods of X such that W_k is contractible in V_k to a point $a \in X$. Since V_k, as an open subset of H, is an absolute neighborhood retract for metric spaces, we infer by the theorem on the extension of a homotopy ([1], p. 94) that there is a map r_k: $H \to H$ such that $r_k(H) \subset V_k$ and $r_k(x) = x$ for every point $x \in W_k$, and that r_k is homotopic in V_k to the constant map a. Let us show that $\{r_k, \hat{X}, X\}$ is a fundamental retraction. In fact, since $X \subset W_k$ we have $r_k(x) = x$ for every point $x \in X$. Moreover, if U is a neighborhood of X, then there exists an index k_0 such that $V_k \subset U$ for every $k \geqslant k_0$. Hence $r_k \simeq a$ in $V_k \subset U$ for every $k \geqslant k_0$ and consequently $r_k \simeq r_{k+1}$ in U for every $k \geqslant k_0$. Thus we have shown that $\{r_k, \hat{X}, X\}$ is a fundamental retraction and the proof of Theorem (9.1) is finished.

(9.4) COROLLARY. *The intersection of every decreasing sequence of* FAR*-sets is an* FAR*-set.*

In fact, if $X_1 \supset X_2 \supset \ldots$ is a sequence of FAR-sets and $X = \bigcap_{k=1}^{\infty} X_k$, then for every neighborhood U of X there exists an index k_0 such that U is a neighborhood of X_{k_0}. By Theorem (9.1), there is a neighborhood U_0 of X_{k_0} contractible in U. Since U_0 is also a neighborhood of X, the set X satisfies the condition characterizing FAR-sets.

(9.5) COROLLARY. *A compactum $X \subset H$ is an* FAR*-set if and only if it is contractible in each of its neighborhoods (in H).*

Proof. The necessity of the condition is an immediate consequence of Theorem (9.1). In order to prove its sufficiency, consider an open neighborhood U of X (in H). Then there are a point $a \in X$ and a map

$f\colon X\times\langle 0,1\rangle\to U$ such that $f(x,0)=x$ and $f(x,1)=a$ for every point $x\in X$. Setting

$$\bar f(x,t)=f(x,t)\quad \text{for every } (x,t)\in X\times\langle 0,1\rangle\,,$$

$$\bar f(x,0)=x\,,\quad \bar f(x,1)=a\quad \text{for every point } x\in H\,,$$

we get a map $\bar f$ of the set $Z=\big(H\times(0)\big)\cup\big(X\times\langle 0,1\rangle\big)\cup\big(H\times(1)\big)$ into H. By Dugundji's extension theorem ([3], p. 357), the map $\bar f$ can be extended to a map $\hat f\colon H\times\langle 0,1\rangle\to H$.

Since $\hat f$ maps the closed subset $X\times\langle 0,1\rangle$ of Z into U, we infer that the set $V=\hat f^{-1}(U)$ is a neighborhood of the set $X\times\langle 0,1\rangle$ in the space $H\times\langle 0,1\rangle$. Consequently there exists a neighborhood U_0 of X such that $U_0\times\langle 0,1\rangle\subset V$. Setting

$$\varphi(x,t)=\hat f(x,t)\ \text{for every } (x,t)\in U_0\times\langle 0,1\rangle\,,$$

we get a map $\varphi\colon U_0\times\langle 0,1\rangle\to U$ contracting U_0 in U to the point a. It follows, by Theorem (9.1), that $X\in$ FAR.

(9.6) PROBLEM. *Is it true that every* FAR*-set is the intersection of a decreasing sequence of* AR*-sets?*

(9.7) PROBLEM. *Let $\{X_k\}$ be a decreasing sequence of fundamental retracts of a compactum $X\subset H$. Is it true that the set $\bigcap_{k=1}^{\infty} X_k$ is a fundamental retract of X?*

(9.8) THEOREM. *A compactum $X\subset H$ is an* FAR*-set if and only if the fundamental identity sequence $\underline{i}_X=\{i,X,X\}$ is homotopic to a constant fundamental sequence $\underline{c}_X=\{c,X,X\}$, where $c\in X$.*

First let us prove the following two lemmas:

(9.9) LEMMA. *If $X\subset H$ is a compactum which is contractible in itself to a point $c\in X$, then $\underline{i}_X\simeq\underline{c}_X=\{c,X,X\}$.*

Proof. Let $\varphi\colon X\times\langle 0,1\rangle\to X$ be a homotopy contracting X in itself to c, i.e. $\varphi(x,0)=x$ and $\varphi(x,1)=c$ for every point $x\in X$. It is clear that there exists a homotopy $\bar\varphi\colon H\times\langle 0,1\rangle\to H$ such that

$$\bar\varphi(x,0)=x\,,\quad \bar\varphi(x,1)=c\quad \text{for every point } x\in H\,,$$

and

$$\bar\varphi(x,t)=\varphi(x,t)\quad \text{for every } (x,t)\in X\times\langle 0,1\rangle\,.$$

If U is a neighborhood of X, then we infer that there is a neighborhood U_0 of X such that $\bar\varphi(U_0\times\langle 0,1\rangle)\subset U$. It follows $i/U_0\simeq c/U_0$ in U, whence $\underline{i}_X\simeq\underline{c}=\{c,X,X\}$.

(9.10) LEMMA. *If c is a point of a fundamental retract Y of a compactum $X\subset H$ satisfying the condition $\underline{i}_X\simeq\underline{c}_X=\{c,X,X\}$, then $\underline{i}_Y\simeq\{c,Y,Y\}$.*

Proof. Let $\underline{r}=\{r_k,X,Y\}$ be a fundamental retraction. Setting $\underline{r}'=\{r_k,Y,Y\}$, we get a fundamental sequence $\underline{r}'$. Since $r_k(y)=i(y)$ for every point $y\in Y$, we infer by (1.1) that

$$\underline{r}'\simeq\underline{i}_Y\,. \tag{9.11}$$

Now let V be a neighborhood of Y. Since $\underline{r}$ is a fundamental sequence, there exists a neighborhood U of X and an index k_0 such that

$$r_k(U)\subset V\quad \text{for every } k\geqslant k_0\,. \tag{9.12}$$

Since $\underline{i}_X\simeq\underline{c}_X$, there exists a neighborhood U_0 of X such that

$$i/U_0\simeq c \text{ in } U\,. \tag{9.13}$$

It follows by (9.12) and (9.13) that

$$r_k/U_0=r_ki/U_0\simeq r_kc=c \text{ in } V\quad \text{for every } k\geqslant k_0\,.$$

Since $Y\subset X$, the set U_0 is a neighborhood of Y. Thus we have shown that for every neighborhood V of Y there exists a neighborhood U_0 of Y and an index k_0 such that

$$r_k/U_0\simeq c \text{ in } V\quad \text{for every } k\geqslant k_0\,.$$

Hence $\underline{r}'\simeq\underline{c}_Y=\{c,Y,Y\}$. It follows by (9.11) that $\underline{i}_Y\simeq\underline{c}_Y$, and the proof of Lemma (9.10) is finished.

Proof of Theorem (9.8). If $X\in$ FAR, then (6.3) implies that X is a fundamental retract of an AR-set $\hat X$. Since $\hat X$ is contractible in itself to every point $c\in X$, we infer by (9.9) and (9.10) that $\underline{i}_X\simeq\underline{c}_X=\{c,X,X\}$. On the other hand, if $X\subset H$ is a compactum such that $\underline{i}_X\simeq\underline{c}_X=\{c,X,X\}$, where c is a point of X, then for every neighborhood U of X there exists a neighborhood U_0 of X such that $i/U_0\simeq c/U_0$ in U. This means that U_0 is contractible in U and we infer by Theorem (9.1) that $X\in$ FAR.

10. A property of FANR-sets. The following theorem gives a condition for FANR-sets which is to some extent similar to the condition for FAR-sets appearing in Theorem (9.1):

(10.1) THEOREM. *Every* FANR*-set X satisfies the following condition:* Condition (*). *For every neighborhood U of X there is a neighborhood U_0 of X such that for every neighborhood V of X there exists a homotopy $\varphi\colon U_0\times\langle 0,1\rangle\to U$ such that $\varphi(x,0)=x$ and $\varphi(x,1)\in V$ for every point $x\in U_0$, and $\varphi(x,1)=x$ for every point $x\in X$.*

Proof. By (6.4) there exists an AR-set $\hat X\supset X$. Let $\alpha\colon H\to\hat X$ be a retraction. Consider a sequence $V_1\supset V_2\supset\ldots$ of neighborhoods of X such that every neighborhood of X contains V_k for almost all k. Since $X\in$ FANR, we infer by Theorem (4.1) that there exist a closed neighborhood W of X and a fundamental retraction $\underline{r} = \{r_k, W \cap \hat{X}, X\}$ such that

(10.2) $$r_k/V_k = i/V_k \quad \text{for every } k = 1, 2, \ldots$$

Now let U be a neighborhood of X. Then there is an index k_0 such that

(10.3) $$V_{k_0} \subset U \cap W,$$

(10.4) $$r_{k_0}/(W \cap \hat{X}) \simeq r_{k_0+m}(W \cap \hat{X}) \text{ in } U \quad \text{for every } m = 1, 2, \ldots$$

Moreover, there exists a neighborhood U_0 of X such that

(10.5) *For every point $x \in U_0$ the segment (in H) with endpoints x and $\alpha(x)$ lies in V_{k_0}.*

It follows by (10.3) and (10.5) that

(10.6) $$\alpha(U_0) \subset V_{k_0} \cap \hat{X} \subset W \cap \hat{X}.$$

Moreover, (10.5) implies that

(10.7) $$i/U_0 \simeq \alpha/U_0 \text{ in } V_{k_0}.$$

Applying (10.2) and (10.7), we infer that

(10.8) $$\alpha/U_0 = r_{k_0}\alpha/U_0.$$

Formulas (10.4) and (10.6) imply that

(10.9) $$r_{k_0}\alpha/U_0 \simeq r_{k_0+m}\alpha/U_0 \text{ in } U \quad \text{for every } m = 1, 2, \ldots$$

It follows by (10.3), (10.7), (10.8) and (10.9) that

(10.10) $$i/U_0 \simeq r_{k_0+m}\alpha/U_0 \text{ in } U \quad \text{for every } m = 1, 2, \ldots$$

But $\underline{r}$ is a fundamental retraction of the set $W \cap \hat{X} \supset \alpha(U_0)$ to X. Hence, for every neighborhood V of X the inclusion

$$r_{k_0+m}\alpha(U_0) \subset V$$

holds for almost all m. Moreover, if $x \in X$ then $\alpha(x) = x$ and $r_k(x) = x$, whence $r_{k_0+m}\alpha(x) = x$. It follows by (10.10) that condition (*) is satisfied.

11. A sufficient condition for a set to be an FANR. Let us prove the following

(11.1) THEOREM. *If $X_1, X_2, \ldots$ are* ANR*-sets lying in H and if X_{k+1} is a deformation retract of X_k for every $k = 1, 2, \ldots$, then the set $X = \bigcap_{k=1}^{\infty} X_k$ is an* FANR*-set.*

Proof. Since $X_1 \in$ ANR, there exist a neighborhood U_0 of X_1 (in H) and a map $s_0\colon H \to H$ such that $s_0(U_0) = X_1$ and $s_0(x) = x$ for every point $x \in X_1$.

Since X_{k+1} is a deformation retract of the set $X_k \in$ ANR, there exists ([5], p. 448) a homotopy

$$\psi_k\colon X_k \times \langle 0, 1\rangle \to X_k$$

such that

$$\psi_k(x, 0) = x \quad \text{for every point } x \in X_k,$$
$$\psi_k(x, t) = x \quad \text{for every } (x, t) \in X_{k+1} \times \langle 0, 1\rangle,$$
$$\psi_k(X_k, 1) = X_{k+1}.$$

Hence the formula $r_k(x) = \psi_k(x, 1)$ defines a retraction

$$r_k\colon X_k \to X_{k+1}.$$

Consider now a map $s_k\colon H \to H$ such that $s_k(x) = r_k(x)$ for every point $x \in X_k$. Setting

$$f_k = s_k s_{k-1} \ldots s_1 s_0,$$

we get a map $f_k\colon H \to H$ such that the formula

$$r'_k(x) = f_k(x) \quad \text{for every point } x \in X_1$$

defines a retraction $r'_k\colon X_1 \to X_{k+1}$.

Let us show that $\{f_k, X_1, X\}$ is a fundamental retraction. Consider a neighborhood V of X (in the space H). Then there exists an index k_0 such that the set $U_0 \cap V$ is a neighborhood of X_{k_0}, whence also a neighborhood of X_k for every $k \geqslant k_0$. But

$$f_{k_0}(X_{k_0}) = s_{k_0}s_{k_0-1}\ldots s_1 s_0(X_{k_0}) \subset s_{k_0}s_{k_0-1}\ldots s_1 s_0(U_0) = X_{k_0+1}.$$

It follows that there exists a neighborhood U of the set X_{k_0} (in H) such that $U \subset U_0 \cap V$ and that $f_{k_0}(U) \subset U_0 \cap V$. Then, for every $k \geqslant k_0$, the set U is a neighborhood of the set X_k such that

$$f_k(U) \subset s_k s_{k-1} \ldots s_1 s_0(U_0) = X_{k+1} \subset V$$

and

$$f_k(x) = x \quad \text{for every point } x \in X.$$

Since the values of the homotopy ψ_{k+1} belong to the set $X_{k+1} \subset V$, we infer that setting

$$\varphi_k(x, t) = \psi_{k+1}(f_k(x), t) \quad \text{for every } (x, t) \in U \times \langle 0, 1\rangle,$$

we get a homotopy $\varphi_k\colon U\times\langle 0,1\rangle\to V$ satisfying the following conditions:

$$\varphi_k(x,0)=f_k(x) \quad \text{for every point } x\in U,$$
$$\varphi_k(x,1)=r_{k+1}f_k(x)=s_{k+1}f_k(x)=f_{k+1}(x) \quad \text{for every point } x\in U,$$
$$\varphi_k(x,t)=x \quad \text{for every } (x,t)\in X\times\langle 0,1\rangle,$$

because $x\in X$ implies that $f_k(x)=x\in X\subset X_{k+1}$.

Thus we have shown that the homotopy $\varphi_k\colon U\times\langle 0,1\rangle\to V$ joins the map f_k/U with the map f_{k+1}/U and satisfies the condition $\varphi_k(x,t)=x$ for every $(x,t)\in X\times\langle 0,1\rangle$. Hence $\{f_k, X_1, X\}$ is a fundamental retraction. It follows that

(11.2) *X is a fundamental retract of X_1.*

Since $X_1\in$ ANR, we infer by Theorem (6.14) that $X\in$ FANR.

As a consequence of (11.2) and of Theorem (6.2) we get the following

(11.3) COROLLARY. *The intersection of a decreasing sequence of* AR*-sets lying in H is an* FAR*-set.*

In fact, it suffices to observe that for $Y,Z\in$ AR the inclusion $Z\subset Y$ implies that Z is a deformation retract of Y.

(11.4) PROBLEM. *Does Theorem* (11.1) *remain true if we replace the hypothesis that X_{k+1} is a deformation retract of X_k by a weaker one, namely that X_{k+1} is a retract of X_k?*

(11.5) PROBLEM. *Is it true that for every sequence $A_1, A_2, \ldots$ of* ANR*-sets such that A_{k+1} is a retract of A_k for $k=1,2,\ldots$, there exists an index k_0 such that A_{k+1} is a deformation retract of A_k for every $k\geqslant k_0$?*

12. Cartesian product of FAR-sets. First let us prove the following

(12.1) LEMMA. *Let A be an* AR*-set in H and let X and X' be closed subsets of A such that $X\subset X'$. In order that X be a fundamental retract of X' it is necessary and sufficient that there exists a sequence $\{\alpha_k\}$ of maps of A into itself such that for every neighborhood U of X in the space A there exists a neighborhood U' of X' in A such that for almost all k there is a homotopy $\varphi_k\colon U'\times\langle 0,1\rangle\to U$ such that*

(12.2) $\varphi_k(x,0)=\alpha_k(x)$, $\varphi_k(x,1)=\alpha_{k+1}(x)$ *for every point* $x\in U'$.

Proof. Since $A\in$ AR, there exists a map

$$s\colon H\to H$$

such that

$$s(H)=A,\quad s(x)=x \quad \text{for every point } x\in A.$$

If X is a fundamental retract of X', then there exists a fundamental retraction

$$\underline{r}=\{r_k, X', X\}.$$

Setting $\alpha_k(x)=sr_k(x)$ for every point $x\in A$ and $k=1,2,\ldots$, we get a sequence $\{\alpha_k\}$ of maps of A into itself. Moreover, if U is a neighborhood of X in the space A, then $V=s^{-1}(U)$ is a neighborhood of X in the space H. Since $\underline{r}$ is a fundamental retraction, there is a neighborhood V' of X' (in H) such that for almost all k there exists a homotopy

$$\psi_k\colon V'\times\langle 0,1\rangle\to V$$

such that

$$\psi_k(x,0)=r_k(x),\quad \psi_k(x,1)=r_{k+1}(x) \quad \text{for every point } x\in V'.$$

It suffices to set $U'=A\cap V'$ and

$$\varphi_k(x,t)=s\psi_k(x,t) \quad \text{for every } (x,t)\in U'\times\langle 0,1\rangle$$

for almost all k, in order to obtain a neighborhood U' of X' in A and a homotopy $\varphi_k\colon U'\times\langle 0,1\rangle\to U$ satisfying (12.2).

Now let us assume that there is a sequence $\{\alpha_k\}$ of maps of A into itself with the required properties. Setting

$$r_k(x)=\alpha_k s(x) \quad \text{for every point } x\in H,$$

one gets a sequence of maps $r_k\colon H\to H$. Let us prove that $\{r_k, X', X\}$ is a fundamental retraction.

Let V be a neighborhood of X in the space H. Then the set $U=A\cap V$ is a neighborhood of X in A and we infer that there exists a neighborhood U' of X' in A and a homotopy φ_k satisfying (12.2). Setting $V_0=s^{-1}(U')$, we get a neighborhood V_0 of X' in the space H. It is clear that for almost all k the formula

$$\psi_k(x,t)=\varphi_k(s(x),t)$$

defines a homotopy $\psi_k\colon V_0\times\langle 0,1\rangle\to V$ such that

$$\psi_k(x,0)=\alpha_k s(x)=r_k(x);\quad \psi_k(x,1)=\alpha_{k+1}s(x)=r_{k+1}(x) \quad \text{for every point } x\in V_0.$$

Hence $\{r_k, X', X\}$ is a fundamental retraction and the proof of Lemma (12.1) is finished.

Now let us prove the following

(12.3) THEOREM. *Let X be a compactum in H homeomorphic to the Cartesian product $\mathop{P}\limits_{i=1}^{\infty} X_i$, where $X_i\subset H$ for $i=1,2,\ldots$ Then $X\in$* FAR *if and only if $X_i\in$* FAR *for every $i=1,2,\ldots$*

Proof. Let us order all natural numbers in a double sequence $m_{i,j}$ such that

$$m_{i,j} = m_{i',j'} \quad \text{implies} \quad i = i' \text{ and } j = j',$$
$$m_{i,j} < m_{i,j+1} \quad \text{for every } i, j.$$

Then the Hilbert cube Q^ω can be represented as a product

$$(12.4) \qquad Q^\omega = \mathop{P}_{i=1}^{\infty} Q_i^\omega,$$

where Q_i^ω denotes the set of all points $x = (x_1, x_2, \ldots)$ such that for $n = m_{i,j}$, the coordinate x_n runs through the interval $\langle 0, 1/m_{i,j}\rangle$, and for $n \neq m_{i,j}$ with $j = 1, 2, \ldots$, $x_n = 0$. It is obvious that Q_i^ω is homeomorphic to Q^ω for $i = 1, 2, \ldots$ Since FAR are topologically invariant (Theorem (8.1), and since every compactum is homeomorphic to a subset of Q^ω, we can assume that $X_i \subset Q_i^\omega$ for $i = 1, 2, \ldots$ and that $X = \mathop{P}_{i=1}^{\infty} X_i \subset Q^\omega$.

By formula (12.4), every point $x \in Q^\omega$ can be represented in the form $x = [x_1, x_2, \ldots]$, with $x_i \in Q_i^\omega$. In particular $x \in X$ if and only if $x_i \in X_i$ for $i = 1, 2, \ldots$ Since X_i is homeomorphic to a retract of X, we infer by Theorems (6.2) and (8.1) that $X \in$ FAR implies $X_i \in$ FAR for $i = 1, 2, \ldots$

Now let us assume that $X_i \in$ FAR for every $i = 1, 2, \ldots$ By Lemma (12.1), there exists a sequence of maps $a_k^i\colon Q_i^\omega \to Q_i^\omega$ such that for every neighborhood U_i of X_i in Q_i^ω there is an index k_i such that for every $k > k_i$ there is a homotopy $\varphi_k^i\colon Q_i^\omega \times \langle 0, 1\rangle \to U_i$ satisfying the condition

$$(12.5) \qquad \varphi_k^i(x, 0) = a_k^i(x), \quad \varphi_k^i(x, 1) = a_{k+1}^i(x) \quad \text{for every point } x \in Q_i^\omega.$$

Let us notice that in the case $U_i = Q_i^\omega$ the homotopy φ_k^i satisfying this condition exists for every $k = 1, 2, \ldots$ Hence in the case $U_i = Q_i^\omega$, we can set $k_i = 0$.

Setting

$$a_k(x) = [a_k^1(x_1), a_k^2(x_2), \ldots] \quad \text{for every point } x = [x_1, x_2, \ldots] \in Q^\omega,$$

we get a sequence of maps $a_k\colon Q^\omega \to Q^\omega$.

Consider now a neighborhood U of X in Q^ω. Then there exists for every $i = 1, 2, \ldots$ a neighborhood U_i of X_i in Q_i^ω and an index i_0 such that

(1) $U_i = Q_i^\omega$ for every $i > i_0$,

(2) $\mathop{P}_{i=1}^{\infty} U_i \subset U$.

It is clear that $U_0 = \mathop{P}_{i=1}^{\infty} U_i$ is a neighborhood of X in Q^ω.

Consider the homotopies $\varphi_k^1, \varphi_k^2, \ldots$ and the indices $k_1, k_2, \ldots$ as defined above. Then $k_i = 0$ for $i > i_0$ and we infer that there is an index k_0 such that (12.5) holds for every $k > k_0$.

Setting

$$\varphi_k(x, t) = [\varphi_k^1(x_1, t), \varphi_k^2(x_2, t), \ldots]$$

for every $x = [x_1, x_2, \ldots] \in Q^\omega$ and $0 \leqslant t \leqslant 1$, we get a homotopy

$$\varphi_k\colon Q^\omega \times \langle 0, 1\rangle \to U$$

such that

$$\varphi_k(x, 0) = a_k(x), \quad \varphi_k(x, 1) = a_{k+1}(x) \quad \text{for every point } x \in Q.$$

It follows by Lemma (12.1) that X is a fundamental retract of Q^ω, and consequently (by Theorem (6.2)), $X \in$ FAR.

13. Cartesian product of FANR-spaces. We now pass to the proof of the following

(13.1) THEOREM. *Let X be a compactum lying in H, homeomorphic to the Cartesian product $\mathop{P}_{i=1}^{\infty} X_i$, where $X_i \subset H$. Then $X \in$ FANR if and only if $X_i \in$ FANR for $i = 1, 2, \ldots$ and $X_i \in$ FAR for almost all i.*

Proof. First let us assume that $X \in$ FANR. Then X_i is homeomorphic to a retract of X and we infer by (6.7) and (8.2) that $X \in$ FANR for $i = 1, 2, \ldots$

In order to prove that $X_i \in$ FAR for almost all i, let us consider a decomposition of Q^ω into the Cartesian product $\mathop{P}_{i=1}^{\infty} Q_i^\omega$ with factors Q_i^ω homeomorphic to Q^ω and let us assume that $X_i \subset Q_i^\omega$ for $i = 1, 2, \ldots$ and that $X = \mathop{P}_{i=1}^{\infty} X_i$. Thus every point $x \in Q^\omega$ can be represented in the form $x = [x_1, x_2, \ldots]$ with $x_i \in Q_i^\omega$ for $i = 1, 2, \ldots$ Let us select a point $a = [a_1, a_2, \ldots] \in X$. Since $X \in$ FANR, there is a closed neighborhood V of X in Q^ω such that X is a fundamental retract of V. It follows by Lemma (12.1) that there exists a sequence of maps

$$a_k\colon Q^\omega \to Q^\omega$$

such that for every neighborhood U of X (in Q^ω) there exists an index $k(U)$ such that for every $k > k(U)$ there exists a homotopy

$$\varphi_k\colon V \times \langle 0, 1\rangle \to U$$

satisfying the condition

(13.2) $\varphi_k(x, 0) = \alpha_k(x)$, $\varphi_k(x, 1) = \alpha_{k+1}(x)$ for every point $x \in V$.

Since V is a neighborhood of the compactum X, there is an index i_0 such that V is a neighborhood of the set

$$V_0 = X_1 \times X_2 \times \ldots \times X_{i_0} \times Q^{\omega}_{i_0+1} \times Q^{\omega}_{i_0+2} \times \ldots$$

It is evident that the maps α_k and φ_k can be represented in the form

$$\alpha_k = [\alpha^1_k, \alpha^2_k, \ldots], \quad \text{where} \quad \alpha^i_k\colon Q^{\omega} \to Q^{\omega}_i,$$
$$\varphi_k = [\varphi^1_k, \varphi^2_k, \ldots], \quad \text{where} \quad \varphi^i_k\colon V \times \langle 0, 1\rangle \to Q^{\omega}.$$

Consider now an index $i > i_0$ and an arbitrary neighborhood U_0 of X_i in Q^{ω}_i. Then the set

$$U = Q^{\omega}_1 \times Q^{\omega}_2 \times \ldots \times Q^{\omega}_{i-1} \times U_0 \times Q^{\omega}_{i+1} \times \ldots$$

is a neighborhood of X and thus (13.2) holds for every $k > k(U)$. Now let us set, for every $i > i_0$:

(13.3) $\beta^i_k(x_i) = \alpha^i_k([a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots])$ for every $x \in Q^{\omega}_i$

and

(13.4) $\psi^i_k(x_i, t) = \varphi^i_k([a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots], t)$
for every $(x, t) \in Q^{\omega}_i \times \langle 0, 1\rangle$.

It follows by (13.2), (13.3) and (13.4) that the maps $\beta^i_k\colon Q^{\omega}_i \to Q^{\omega}_i$ and the homotopy $\psi^i_k\colon Q^{\omega}_i \times \langle 0, 1\rangle \to U_0$ satisfy the following conditions:

$$\begin{aligned}\psi^i_k(x_i, 0) &= \varphi^i_k([a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots], 0)\\ &= \alpha^i_k([a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots]) = \beta^i_k(x_i),\\ \psi^i_k(x_i, 1) &= \varphi^i_k([a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots], 1)\\ &= \alpha^i_{k+1}([a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots]) = \beta^i_{k+1}(x_i).\end{aligned}$$

It follows by Lemma (12.1) that X_i is a fundamental retract of Q^{ω}_i. We infer by Theorem (6.2) that $X_i \in$ FAR for every $i > i_0$.

Before we prove the converse, let us show that

(13.5) *Let X be a compactum in H homeomorphic to the Cartesian product of two* FANR*-sets X', X''. Then $X \in$ FANR.*

Proof. Let us represent H as the Cartesian product $H' \times H''$, where H', H'' are homeomorphic to H. We can assume that $X' \subset H'$, $X'' \subset H''$ and $X = X' \times X''$. By (6.8) there exist two ANR-sets $A' \subset H'$ and $A'' \subset H''$ and two fundamental retractions $\underline{r}' = \{r'_k, A', X'\}$ and $\underline{r}'' = \{r''_k, A'', X''\}$. Since $H = H' \times H''$, every point $x \in H$ is of the form $x = (x', x'')$, where $x' \in H'$, $x'' \in H''$. Setting

(13.6) $r_k(x) = \big(r'_k(x'), r''_k(x'')\big)$ for every point $x = (x', x'') \in H$,

we get a sequence of maps $r_k\colon H \to H$.

Consider now a neighborhood V of X (in H). Then there exist a neighborhood V' of X' (in H') and a neighborhood V'' of X'' (in H'') such that

(13.7) $$V' \times V'' \subset V.$$

Since $\underline{r}', \underline{r}''$ are fundamental sequences, there exists a neighborhood U' of A' in H' and a neighborhood U'' of A'' in H'' and two homotopies

$$\varphi'_k\colon U' \times \langle 0, 1\rangle \to V', \quad \varphi''_k\colon U'' \times \langle 0, 1\rangle \to V''$$

such that for almost all k

(13.8) $\varphi'_k(x', 0) = r'_k(x')$; $\varphi'_k(x', 1) = r'_{k+1}(x')$
for every point $x' \in U'$,

(13.9) $\varphi''_k(x'', 0) = r''_k(x'')$; $\varphi''_k(x'', 1) = r''_{k+1}(x'')$
for every point $x'' \in U''$.

Then $U = U' \times U''$ is a neighborhood (in H) of the set $A = A' \times A''$ being an ANR-set and we infer by (13.6), (13.7), (13.8), and (13.9) that the formula

$$\varphi_k(x, t) = \big(\varphi'_k(x', t), \varphi''_k(x'', t)\big) \quad \text{for every } (x, t) \in U \times \langle 0, 1\rangle$$

defines a homotopy $\varphi_k\colon U \times \langle 0, 1\rangle \to V$ satisfying the condition

$$\varphi_k(x, 0) = r_k(x); \quad \varphi_k(x, 1) = r_{k+1}(x) \quad \text{for every point } x \in U.$$

It follows that $\underline{r} = \{r_k, A, X\}$ is a fundamental retraction of the set $A \in$ ANR to X. Hence $X \in$ FANR and proposition (13.5) is proved.

In order to finish the proof of Theorem (13.1), let us assume that $X = \mathop{P}\limits_{i=1}^{\infty} X_i$, where $X_i \in$ FANR for $i = 1, 2, \ldots$ and that there exists an index i_0 such that $X_i \in$ FAR for every $i > i_0$. Then X is homeomorphic to $X' \times X''$, where $X' = \mathop{P}\limits_{i=1}^{i_0} X_i$, $X'' = \mathop{P}\limits_{k=1}^{\infty} X_{i_0+k}$. It follows by (13.5) that $X' \in$ FANR, and by (13.4)—that $X'' \in$ FAR. Again applying (13.5), we nfer that $X \in$ FANR.

14. Union of two FAR-sets. We now pass to the proof of a theorem on FAR-sets, analogous to a well-known theorem on AR-sets:

(14.1) THEOREM. *If X_1, X_2 and $X_0 = X_1 \cap X_2$ are* FAR*-sets, then the set $X = X_1 \cup X_2$ is also an* FAR*-set.*

First let us establish the following

(14.2) LEMMA. *Let A_0 be a closed subset of a metric space A and let a be a point of a space M which is an absolute neighborhood retract for metric spaces. Suppose $f\colon A \to M$ and $\varphi\colon A_0 \times \langle 0, 1\rangle \to M$ be two maps such that $\varphi(x, 0) = f(x)$, $\varphi(x, 1) = a$ for every point $x \in A_0$. If the set $f(A)$ is contractible in M to a, then there exists a homotopy*

$$\psi\colon A \times \langle 0, 1\rangle \to M$$

such that $\psi(x, 0) = f(x)$, $\psi(x, 1) = a$ for every point $x \in A$ and $\psi(x, t) = \varphi(x, t)$ for every $(x, t) \in A_0 \times \langle 0, 1\rangle$.

Proof. Consider the set

$$Z = [A \times (0)] \cup [A_0 \times \langle 0, 1\rangle] \cup [A \times (1)],$$

and the map $g\colon Z \to M$ given by the formula

$$g(x, 0) = f(x), \quad g(x, 1) = a \quad \text{for every point } x \in A,$$
$$g(x, t) = \varphi(x, t) \quad \text{for every } (x, t) \in A_0 \times \langle 0, 1\rangle.$$

Setting $\vartheta(x, t, u) = g(x, t(1-u))$ for every $(x, t) \in Z$ and $0 \leqslant u \leqslant 1$, we get a homotopy $\vartheta\colon Z \times \langle 0, 1\rangle \to M$ joining the map g with a map having all values in $f(A)$. Since the set $f(A)$ is contractible in M, we infer that g is homotopic in M to the constant map a. If we observe that the set Z is closed in $A \times \langle 0, 1\rangle$ and recall that M is an absolute neighborhood retract for metric spaces, we infer by the homotopy extension theorem ([1], p. 94) that g can be extended to a map $\psi\colon A \times \langle 0, 1\rangle \to M$ satisfying the lemma.

Proof of Theorem (14.1). By Theorem (9.1), for every open neighborhood U of X there exists a closed neighborhood U_i of X_i $(i = 1, 2)$ contractible in U. Since the set $U_1 \cap U_2$ is a neighborhood of X_0, we infer by (9.1) that there is a closed neighborhood U_0 of X_0 contractible in $U_1 \cap U_2$. Let $\varphi_0\colon U_0 \times \langle 0, 1\rangle \to U_1 \cap U_2$ be a homotopy contracting U_0 to a point $a \in U_0$. Consider, for $i = 1, 2$, a closed neighborhood V_i of the set X_i such that $V_i \subset U_i$ and that $V_0 = V_1 \cap V_2 \subset U_0$.

Now let us consider the map $f_i\colon V_i \to U$ given by the formula

$$f_i(x) = x \quad \text{for every point } x \in V_i.$$

Since U (as an open subset of H) is an absolute neighborhood retract for metric spaces ([1], p. 96), we infer by Lemma (14.2) (where we set $A_0 = V_0$, $A = V_i$, $M = U$, $\varphi = \varphi_0/[V_0 \times \langle 0, 1\rangle]$ and $f = f_i$) that there exists a homotopy $\psi_i\colon V_i \times \langle 0, 1\rangle \to U$ such that

$$\psi_i(x, 0) = x, \quad \psi_i(x, 1) = a \quad \text{for every point } x \in V_i,$$
$$\psi_i(x, t) = \varphi_0(x, t) \quad \text{for every } (x, t) \in V_0 \times \langle 0, 1\rangle.$$

It remains to set

$$\psi(x, t) = \psi_i(x, t) \quad \text{for every } (x, t) \in V_i \times \langle 0, 1\rangle,\ i = 1, 2,$$

in order to obtain a homotopy ψ contracting the neighborhood $V = V_1 \cup V_2$ of the set X in the neighborhood U. It follows by Theorem (9.1) that $X \in$ FAR.

(14.3) PROBLEM. *Let X denote the union and X_0 the common part of two compacta X_1, X_2. Is it true that $X_0, X_1, X_2 \in$ FANR implies $X \in$ FANR?*

14.4) PROBLEM. *Is it true that for every two compacta $X_1, X_2 \subset H$ such that the set $X_1 \cap X_2$ is a fundamental retract of X_2, the set X_1 is a fundamental retract of $X_1 \cup X_2$?*

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Reçu par la Rédaction le 15. 8. 1967