$\ldots, v_{i+n-1}, v_{i+2n}, v_{i+n+1}, v_{i+n+2}, \ldots, v_{i+2n-1}$. Continuing in this way, we arrive at the vertex $v_{i+qn} = v_{j-1}$, from which we proceed to $v_{j-n}, v_{j-n+1}, \ldots$ $\ldots, v_{j-2}$. The path P thus far contains all vertices of D with the exception of v_i so that D contains the arcs $v_{j-2}v_i$ and v_iv_j. Conversely, suppose v_iv_j is an arc of D and $j-i \not\equiv 1 \ (\mathrm{mod}\, n)$. We then construct a path P' which begins as follows: $v_i, v_j, v_{j+1}, \ldots, v_{i-1}, v_{i+n}, v_{i+1}, v_{i+2}, \ldots, v_{i+n-1}, v_{i+2n}$. We then continue as before until we reach the final vertex of the type v_{i+tn} which is not thus far on P'. The next vertices of P' would then be $v_{i+tn}, v_{i+(t-1)n-1}, v_{i+(t-1)n}, \ldots, v_{i+tn-1}$. Since $j \neq i+(t+1)\; n+1$, the vertex of P' following v_{i+tn-1} necessarily defines an outer transitive cycle of length less than $n+2$, and this is a contradiction. Because v_pv_1 obviously belongs to D, we have $1-p \equiv 1 \ (\mathrm{mod}\, n)$, or there exists an integer k such that $p = nk$. If for each i, $1 \leqslant i \leqslant n$, we let $V_i = \{v_s \mid s \equiv i \ (\mathrm{mod}\, n)\}$, D is seen to be the digraph $D(n, k)$. This completes the proof.

Each randomly hamiltonian graph may be considered a randomly hamiltonian digraph (obtained by replacing each edge by a symmetric pair of arcs), but among the randomly hamiltonian digraphs with p vertices, only S_p, K_p, and $D(2, p/2)$ are (ordinary) graphs. Thus, we obtain as a corollary the result presented in [1].

Corollary. *A graph is randomly hamiltonian if and only if it is a cycle, a complete graph, or a regular complete bipartite graph.*

References

[1] G. Chartrand and H. V. Kronk, *Randomly traceable graphs*, SIAM J. Appl. Math. 16 (1968), pp. 696-700.

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Extended operations and relations on the class of ordinal numbers

by

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§ 1. Introduction. This is intended as a sequel to the paper *An extended arithmetic of ordinal numbers* by John Doner and Alfred Tarski. Thus, our notation is the same as theirs. For the sake of convenience we shall repeat several of their definitions. When referring to a theorem, lemma, etc. in the Doner–Tarski paper we shall prefix the numeral by the symbol "D–T".

Lower case greek letters $\alpha, \beta, \gamma, \ldots$ represent ordinal numbers and the class of all ordinal numbers is denoted by Ω.

Definition 1. For each $\gamma \in \Omega$, O_γ is a *binary operation* from $\Omega \times \Omega$ to Ω such that for all $\alpha, \beta \in \Omega$,

(i) $\alpha O_\gamma \beta = \alpha + \beta$, if $\gamma = 0$;

(ii) $\alpha O_\gamma \beta = \bigcup\limits_{\eta<\beta, \zeta<\gamma} [(\alpha O_\gamma \eta) O_\zeta \alpha]$, if $\gamma \geqslant 1$.

Definition 2. For each $\gamma \in \Omega$, R_γ and L_γ are *relations* such that

(i) $R_\gamma, L_\gamma \subseteq \Omega \times \Omega$;

(ii) For all $\alpha, \beta \in \Omega$

$$\alpha R_\gamma \beta \quad \text{iff} \quad (\exists \delta)(\delta \neq 0 \text{ and } \alpha O_\gamma \delta = \beta),$$
$$\alpha L_\gamma \beta \quad \text{iff} \quad (\exists \delta)(\delta \neq 0 \text{ and } \delta O_\gamma \alpha = \beta).$$

(For $\gamma = 0, 1$, R_γ and L_γ have been described in Rubin [3].)

Our results include the following: If $A = \{\alpha\colon \alpha R_\gamma \beta\}$ for some $\beta, \gamma \in \Omega$, $\beta > 0$, and $\emptyset \neq X \subseteq A$ then $\bigcup X \in A$. If γ is a limit ordinal and $\Omega' = \Omega \sim \{0\}$, then $\langle \Omega', R_\gamma \rangle$ is a complete lattice. Moreover, for γ a limit ordinal we have obtained necessary and sufficient conditions for O_γ to be commutative and associative. Also, for $\alpha, \beta, \gamma \in \Omega$ we have obtained necessary and sufficient conditions on α' such that $\alpha O_\gamma \beta = \alpha' O_\gamma \beta$.

We shall assume the traditional arithmetic of ordinal numbers. (Sierpiński [5] is an excellent reference.) We frequently use the following well-known result.

LEMMA 3. *For each $a \in \Omega$, $a \neq 0$, there is a unique $n \in \omega$, $n \neq 0$, unique ordinal numbers $a_0, a_1, \ldots, a_n$ such that $a_0 > a_1 > \ldots > a_n$, and unique natural numbers $a_i \neq 0$, $i = 0, 1, 2, \ldots, n$ such that*

$$(*) \qquad a = \omega^{a_0}a_0 + \omega^{a_1}a_1 + \ldots + \omega^{a_n}a_n .$$

The form $(*)$ will be called the *normal form* of a, and a_0 is called the *degree* of a. See for example, Sierpiński [5], pp. 319–323 for a proof of Lemma 3.

Main numbers also play an important roll in what follows, so that even at the risk of being redundant, we shall restate their definition and some of their properties.

DEFINITION 4. (i) If O is a binary operation from $\Omega \times \Omega$ to Ω then $\delta \geq \omega$ is a main number of O if and only if for all $a, \beta < \delta$, $aO\beta < \delta$.

(ii) $M(O)$ denotes the class of all main numbers of O.

(iii) If γ is a limit ordinal

$$M_\gamma = \bigcap_{\eta<\gamma} M(O_\eta) .$$

The main numbers of O_γ are its fixed points. That is, δ is a main number of O_γ if and only if $\delta \geq 3$ and $aO_\gamma\delta = \delta$ for all a, $2 \leq a < \delta$ (D–T 46). In the case that $\gamma \geq 2$, we have that for all a, $2 \leq a < \delta$, $\delta \in M(O_\gamma)$ if and only if $aO_\gamma\delta = \delta$ (D–T 47). Thus, for example, the main numbers of O_0 (addition) are positive powers of ω; the main numbers of O_1 (multiplication) are all ordinals of the form ω^{ω^η}, $\eta \in \Omega$; and the main numbers of O_2 (essentially exponentiation) are ω and the epsilon numbers.

LEMMA 5. *If $\delta \in M(O_\gamma)$ then δ is a positive power of ω.*

Proof: D–T 43 (ii), D–T 52 (i), and D–T 57.

LEMMA 6. (i) *If $\gamma \geq 1$ then $2O_{2\gamma+2}[\omega(1+\eta)]$ is the η-th successive element of $M(O_{2\gamma}) = M(O_{2\gamma+1})$.*

(ii) *If $\gamma \geq 1$ and $a \geq 2$ then $aO_{2\gamma+2}[\omega(1+\eta)]$ is the η-th successive element of $M(O_{2\gamma}) = M(O_{2\gamma+1})$ exceeding a.*

(iii) *If $\gamma = \bigcup\gamma \neq 0$ then $2O_\gamma(3+\eta) = 3O_\gamma(2+\eta)$ is the η-th successive element of M_γ.*

(iv) *If $\gamma = \bigcup\gamma \neq 0$ and $a \geq 3$ then $aO_\gamma(2+\eta)$ is the η-th successive element of M_γ exceeding a.*

Proof. Part (i) follows from D–T 49 (ii); (ii) from D–T 48 (ii); (iii) from D–T 37 and D–T 55; and (iv) from D–T 54.

LEMMA 7. *$aO_\gamma\beta$ is a limit ordinal if any one of the following conditions hold,*

(i) *$\gamma \geq \omega$, $a, \beta \geq 2$, and $a = \beta = 2$ does not hold.*

(ii) *$2 \leq \gamma < \omega$, $a \geq 2$, and $\beta \geq \omega$.*

(iii) *$3 \leq \gamma < \omega$, $a \geq \omega$, and $\beta \geq 2$.*

Proof. Part (i) is the same as D–T 34. Part (ii) follows from D–T 33 (iii) and D–T 48 (ii) if β is a limit ordinal. If β is not a limit ordinal, use D–T 9 to reduce it to the case where β is a limit ordinal.

To prove (iii) note that if β is infinite (iii) follows from (ii). If β is finite use D–T 9 to reduce it to (ii).

§ 2. The equation $aO_\gamma\beta = a'O_\gamma\beta$. If $\beta < \beta'$ and $a \geq 1$ then it follows from D–T 4 (ii) that $aO_\gamma\beta < aO_\gamma\beta'$. However, we do not have strict monotonicity in the first argument of O_γ. For example, $n+\omega = \omega$ for all $n \in \omega$; $n\cdot\omega = \omega$ for all $n \in \omega$, $n \neq 0$; in fact $nO_\gamma\omega = \omega$ for all $n, \gamma \in \omega$, $n \neq 0$ because ω is a main number of O_γ for all $\gamma \in \omega$ (D–T 2 (iv)). It does follow from D–T 6 that if $a \leq a'$ then $aO_\gamma\beta \leq a'O_\gamma\beta$. It is the purpose of this section to determine for which values of a', $aO_\gamma\beta = a'O_\gamma\beta$.

First we give some negative results—values of a, β, γ, and a' for which equality does not hold.

THEOREM 8. *If $2 \leq a < a'$, $\bigcup\beta \neq \beta$, and $\bigcup\gamma \neq \gamma$ then*

$$aO_\gamma\beta < a'O_\gamma\beta .$$

Proof. D–T 11.

The cases in which $a < 2$, $a' < 2$, $\beta < 2$ are trivial (see D–T 2) so we shall omit them.

Our next two results hold for all $\gamma \geq 1$.

THEOREM 9. *If $\gamma \geq 1$, $\beta = \bigcup\beta \neq 0$, and $3 \leq a \leq a' < \omega$ then*

$$aO_\gamma\beta = a'O_\gamma\beta .$$

Proof. D–T 29.

THEOREM 10. *If $\gamma \geq 1$, $\beta = \bigcup\beta \neq 0$ and $a = \omega^{a_0}a_0 + \ldots + \omega^{a_n}a_n$, $a_0 \neq 0$ is the normal form of a, then*

$$aO_\gamma\beta = \omega^{a_0}O_\gamma\beta .$$

Proof. If $\gamma = 1$, 2, or 3, the theorem follows from the traditional arithmetic of ordinal numbers. (See for example Rubin [4], § 9.1.)

If $a = \omega^{a_0}$, the theorem clearly holds. If $a > \omega^{a_0}$ then by Lemma 5, there are no main numbers between ω^{a_0} and a. Consequently, if $\gamma = 2\zeta+2$ for some $\zeta \geq 1$, the theorem follows from Lemma 6 (ii); and if $\gamma = \bigcup\gamma \neq 0$, use Lemma 6 (iv) to get the desired result.

Suppose $\gamma = 2\zeta+1$ for some $\zeta \geq 2$. Then, by D–T 33 (iii),

$$aO_\gamma\beta = aO_{2\zeta}a\beta .$$

It follows from elementary properties of ordinal numbers that since β is a limit ordinal $a\beta = \omega^{a_0}\beta$. If $\zeta = \zeta'+1$ then by Lemma 6 (ii),

$$(*) \qquad aO_{2\zeta}\omega^{a_0}\beta = \omega^{a_0}O_{2\zeta}\omega^{a_0}\beta .$$

If $\zeta = \bigcup \zeta \neq 0$ then by Lemma 6 (iv), equation (*) also holds. Thus in either case,

$$\alpha O_\gamma \beta = \omega^{\alpha_0} O_{2\zeta} \omega^{\alpha_0} \beta = \omega^{\alpha_0} O_\gamma \beta \qquad [\text{D–T } 33 \text{ (iii)}] .$$

We now consider the case that $\bigcup \gamma \neq \gamma$. It follows from Theorem 8, that in this case we need only consider the case that β is a limit ordinal.

THEOREM 11. *If $\gamma = 2\zeta+1$ for some $\zeta \geqslant 2$, $\beta = \bigcup \beta \neq 0$ and $2 \leqslant \alpha < \omega$ then*

$$\alpha O_\gamma \beta = \omega O_\gamma \beta \quad \textit{iff} \quad \omega\beta = \beta .$$

Proof. Suppose $\gamma = 2\zeta+1$ for some $\zeta \geqslant 2$. Then by D–T 33 (iii),

$$\alpha O_\gamma \beta = \alpha O_{2\zeta} \alpha\beta = \alpha O_{2\zeta} \beta$$

and

$$\omega O_\gamma \beta = \omega O_{2\zeta} \omega\beta .$$

Since there is just one main number which is larger that α and not larger than ω (namely ω itself) and since $1+\beta = \beta$, using Lemma 6 (ii) or Lemma 6 (iv), we obtain

$$\omega O_{2\zeta} \omega\beta = \alpha O_{2\zeta} \omega\beta .$$

Therefore,

$$\begin{aligned} \alpha O_\gamma \beta = \omega O_\gamma \beta \quad &\text{iff} \quad \alpha O_{2\zeta} \beta = \alpha O_{2\zeta} \omega\beta \\ &\text{iff} \quad \omega\beta = \beta \quad [\text{D–T } 4 \text{ (ii)}] . \end{aligned}$$

THEOREM 12. *If $\gamma = 2\zeta+1$ for some $\zeta \geqslant 2$, $\beta = \bigcup \beta \neq 0$, and $\alpha \geqslant 1$ then*

$$\omega^{\alpha+\delta} O_\gamma \beta = \omega^\alpha O_\gamma \beta \quad \textit{iff} \quad \omega^\delta \beta = \beta .$$

Proof. The proof is similar to the proof of Theorem 11, but there are a few more details to worry about. By D–T 33 (iii) we have

(1) $$\omega^\alpha O_\gamma \beta = \omega^\alpha O_{2\zeta} \omega^\alpha \beta$$

and

(2) $$\omega^{\alpha+\delta} O_\gamma \beta = \omega^{\alpha+\delta} O_{2\zeta} \omega^{\alpha+\delta} \beta .$$

It is clear from (1), (2) and the monotonicity laws (D–T 4 and D–T 6) that if $\omega^\delta \beta \neq \beta$ then equality does not hold. So suppose $\omega^\delta \beta = \beta$.

We note that there are at most $\omega^{\alpha+\delta}$ main numbers of $M(O_0)$ exceeding ω^α and not exceeding $\omega^{\alpha+\delta}$.

Case 1. $\zeta = \bigcup \zeta \neq 0$. In this case it follows from Lemma 6 (iv) and the fact that

(3) $$\omega^{\alpha+\delta} + \omega^{\alpha+\delta}\beta = \omega^{\alpha+\delta}\beta ,$$

that

(4) $$\omega^{\alpha+\delta} O_{2\zeta} \omega^{\alpha+\delta}\beta = \omega^\alpha O_{2\zeta} \omega^{\alpha+\delta}\beta .$$

Case 2. $\zeta = \zeta'+1$ and $\alpha \geqslant \omega$. Then

$$\omega(1+\omega^{\alpha+\delta}\beta) = \omega^{\alpha+\delta}\beta .$$

Thus (4) also holds in this case because of (3) and Lemma 6 (ii).

Case 3. $\zeta = \zeta'+1$ and $1 \leqslant \alpha < \omega$. Let $\alpha = 1+\alpha'$ where $0 \leqslant \alpha' < \omega$. Then

$$\omega(1+\omega^{\alpha'+\delta}\beta) = \omega^{\alpha+\delta}\beta .$$

If $\delta \geqslant \omega$ then $\alpha'+\delta = \alpha+\delta$ and the proof proceeds as in Case 2. Suppose $1 \leqslant \delta < \omega$. Since β is a limit ordinal there exist ordinal numbers ξ and η, $\xi > 0$ such that $\beta = \omega^\xi(\eta+1)$. Thus, if $\xi \geqslant \omega$ then

$$\omega^{\alpha'+\delta}\beta = \omega^{\alpha+\delta}\beta$$

and the proof proceeds as in Case 2. Suppose then, that $1 \leqslant \alpha, \delta, \xi < \omega$. In this case $\omega^\delta\beta \neq \beta$ contradicting our assumption.

Thus in all 3 cases (4) holds. Therefore, it follows from (1) (2) and (4) that if $\omega^\delta\beta = \beta$ then

$$\omega^\alpha O_\gamma \beta = \omega^{\alpha+\delta} O_\gamma \beta .$$

Before proceeding it is convenient at this point to introduce some notation. It follows from D–T 2 (ii) that for each $\alpha \geqslant 1$, $\gamma \geqslant 1$ and δ there is exactly one β such that

$$\alpha O_\gamma \beta \leqslant \delta < \alpha O_\gamma (\beta+1) .$$

We denote this unique β by $\varphi_{\alpha,\gamma}(\delta)$. Thus,

DEFINITION 13. If $\alpha \geqslant 1$, $\gamma \geqslant 1$,

(i) $\varphi_{\alpha,\gamma}(\delta) = \beta$ iff $\alpha O_\gamma \beta \leqslant \delta < \alpha O_\gamma(\beta+1)$.

(ii) $\varphi_\gamma(\delta) = \beta$ iff $3 O_\gamma \beta \leqslant \delta < 3 O_\gamma(\beta+1)$.

LEMMA 14. *If $\alpha \geqslant 3$, $\beta \geqslant 2$, and $\gamma = \bigcup \gamma \neq 0$ then*

$$\alpha O_\gamma \beta = \big(\delta O_\gamma \varphi_{\delta,\gamma}(\alpha)\big) O_\gamma \beta \quad \textit{for all } \delta \geqslant 3 .$$

(*In the case $\delta = 3$ we get,*

$$\alpha O_\gamma \beta = \big(3 O_\gamma \varphi_\gamma(\alpha)\big) O_\gamma \beta .)$$

Proof. By the definition of $\varphi_{\delta,\gamma}(\alpha)$ we have

$$\delta O_\gamma \varphi_{\delta,\gamma}(\alpha) \leqslant \alpha < \delta O_\gamma\big(\varphi_{\delta,\gamma}(\alpha)+1\big) .$$

It follows from Lemma 6 (iv) that there are no main numbers of O_γ between $\delta O_\gamma \varphi_{\delta,\gamma}(\alpha)$ and α. Thus the lemma follows from Lemma 6 (iv).

LEMMA 15. *If $\alpha \geqslant 2$, $\beta = \bigcup \beta \neq 0$ and $\gamma = 2\zeta+2$ for some $\zeta \geqslant 1$ then*

$$\alpha O_\gamma \beta = \big(\delta O_\gamma \varphi_{\delta,\gamma}(\alpha)\big) O_\gamma \beta \quad \textit{for all } \delta \geqslant 3 .$$

(*In the case* $\delta = 3$ *we get*

$$\alpha O_\gamma \beta = \big(3\, O_\gamma \varphi_\gamma(\alpha)\big)\, O_\gamma \beta\,.)$$

Proof. The proof is the same as the proof of Lemma 14, using Lemma 6 (ii) instead of Lemma 6 (iv).

Before considering the general solution of the equation with γ a limit ordinal there is one annoying special case to consider.

THEOREM 16. *If* $\gamma = \bigcup \gamma \neq 0$ *and* $2 \leqslant \beta < \omega$ *then*

$$2\, O_\gamma \beta < 3\, O_\gamma \beta\,.$$

Proof. If $\beta = 2$ then $2\, O_\gamma \beta = 4$ (D–T 2 (iii)), and $3\, O_\gamma \beta \in M_\gamma$ (Lemma 6 (iii)). If $\beta > 2$ then there is a $\beta' \in \omega$ such that $\beta = 3 + \beta'$. By D–T 37,

$$\begin{aligned} 2\, O_\gamma \beta &= 3\, O_\gamma (2+\beta') \\ &< 3\, O_\gamma (3+\beta') \qquad [\text{D–T } 4\ (\text{ii})] \\ &= 3\, O_\gamma \beta\,. \end{aligned}$$

THEOREM 17. *If* $\gamma = \bigcup \gamma \neq 0$, $2 \leqslant \alpha \leqslant \alpha'$, $\omega^\delta \leqslant \beta < \omega^{\delta+1}$ *and either* $\alpha \geqslant 3$ *and* $\beta \geqslant 2$, *or* $\alpha = 2$ *and* $\beta \geqslant \omega$, *then the following three conditions are equivalent*:

(1) $$\alpha O_\gamma \beta = \alpha' O_\gamma \beta;$$

(2) $$\varphi_\gamma(\alpha) + \beta = \varphi_\gamma(\alpha') + \beta;$$

(3) *The ordinal number of* $\langle X, \leqslant\rangle$ *is less than* ω^δ *where*

$$X = \{\varrho \in M_\gamma\colon\ \alpha < \varrho \leqslant \alpha'\}\,.$$

Proof. If $\alpha = 2$ and $\beta \geqslant \omega$ then it follows from D–T 37 that

$$\alpha O_\gamma \beta = 3\, O_\gamma \beta\,.$$

Therefore, we can assume $\alpha \geqslant 3$ and $\beta \geqslant 2$.

Let $\delta = \varphi_\gamma(\alpha)$ and $\delta' = \varphi_\gamma(\alpha')$. Then by Lemma 14,

$$\alpha O_\gamma \beta = (3\, O_\gamma \delta)\, O_\gamma \beta$$

and

$$\alpha' O_\gamma \beta = (3\, O_\gamma \delta')\, O_\gamma \beta\,.$$

Let $\beta = 1 + \beta'$ and use D–T 27 (i), thereby obtaining,

$$\alpha O_\gamma \beta = 3\, O_\gamma (\delta + \beta')\,,$$
$$\alpha' O_\gamma \beta = 3\, O_\gamma (\delta' + \beta')\,.$$

Thus the equivalence of (1) and (2) follows from D–T 4 (ii).

To prove the equivalence of (3), let λ be the ordinal number of $\langle X, \leqslant\rangle$ where $X = \{\varrho \in M_\gamma\colon\ \alpha < \varrho \leqslant \alpha'\}$, and let $\beta = 2 + \beta''$. Then it follows from Lemma 6 (iv) that

$$\alpha' O_\gamma \beta = \alpha O_\gamma (2 + \lambda + \beta'')\,.$$

Thus, $\alpha O_\gamma \beta = \alpha' O_\gamma \beta$ if and only if $\lambda + \beta'' = \beta''$. The latter equation holds if and only if $\lambda < \omega^\delta$.

Next, we consider the case where γ is even but not a limit ordinal.

THEOREM 18. *If* $\gamma = 2\zeta + 2$ *for some* $\zeta \geqslant 1$, $\beta = \bigcup \beta \neq 0$, $\omega^\delta \leqslant \beta < \omega^{\delta+1}$, *and* $2 \leqslant \alpha < \alpha'$ *then the following three conditions are equivalent*

(1) $$\alpha O_\gamma \beta = \alpha' O_\gamma \beta;$$

(2) $$\varphi_\gamma(\alpha) + \beta = \varphi_\gamma(\alpha') + \beta;$$

(3) *The ordinal number of* $\langle X, \leqslant\rangle$ *is less than* ω^δ *where*

$$X = \{\varrho \in M(O_{2\zeta})\colon\ \alpha < \varrho \leqslant \alpha'\}\,.$$

Proof. The proof is similar to the proof of Theorem 17 using Lemma 15, D–T 32 (i) and Lemma 6 (ii) instead of Lemma 14, D–T 27 and Lemma 6 (iv) respectively.

Thus, theorems 8–12, 16–18 give necessary and sufficient conditions for the equation

$$\alpha O_\gamma \beta = \alpha' O_\gamma \beta$$

to hold for $\gamma > 3$. If $\gamma = 0$, O_γ is addition and necessary and sufficient conditions for equality are easy to obtain when α, α' and β are all written in normal form. If $1 \leqslant \gamma \leqslant 3$ then it follows from Theorem 8 that if $\beta \neq \bigcup \beta$ then equality does not hold and it follows from Theorem 10 that it is sufficient to consider values of α and α' which are either finite or powers of ω. Using these results and traditional properties of ordinal numbers it is an easy matter to determine whether or not $\alpha O_\gamma \beta = \alpha' O_\gamma \beta$ for $\gamma = 1, 2, 3$. We leave the details to the interested reader.

§ 3. The commutative and associative laws for O_γ, where $\gamma = \bigcup \gamma \neq 0$. In this section we shall give necessary and sufficient conditions for the commutative and associative laws to hold for O_γ when γ is a limit ordinal.

THEOREM 19. *If* $\gamma = \bigcup \gamma \neq 0$ *and* $\beta > 2$ *then* $2\, O_\gamma \beta = \beta\, O_\gamma 2$ *if and only if* $\beta = 3$ *or* $\beta = \varrho + 1$ *for some* $\varrho \in M(O_\gamma)$.

Proof. If $\beta = 3$ then D–T 37 implies that $2\, O_\gamma \beta = \beta\, O_\gamma 2$. Suppose $\beta = \varrho + 1$ for some $\varrho \in M(O_\gamma)$. Then

$$\begin{aligned} 2\, O_\gamma \beta &= 2\, O_\gamma (\varrho + 1) \\ &= (2\, O_\gamma \varrho)\, O_\gamma 2 \qquad [\text{D–T } 27] \\ &= \varrho\, O_\gamma 2 \qquad [\text{D–T } 46] \\ &= \beta\, O_\gamma 2 \qquad [\text{D–T } 26]\,. \end{aligned}$$

Conversely, suppose $2\, O_\gamma \beta = \beta\, O_\gamma 2$.

Case 1. $2 < \beta < \omega$. There is a $\beta' \in \omega$ such that $\beta = 3+\beta'$. Thus,

$$2O_\gamma\beta = 2O_\gamma(3+\beta')$$
$$= 3O_\gamma(2+\beta') . \quad \text{[D–T 37]}$$

Also, it follows from D–T 26 that

$$\beta O_\gamma 2 = 3O_\gamma 2 .$$

Therefore, if $2O_\gamma\beta = \beta O_\gamma 2$ then $\beta' = 0$ so $\beta = 3$.

Case 2. $\beta \geqslant \omega$. In this case it follows from D–T 37 that

$$2O_\gamma\beta = 3O_\gamma\beta .$$

Moreover, by Lemma 14,

$$\beta O_\gamma 2 = \big(3O_\gamma\varphi_\gamma(\beta)\big)O_\gamma 2$$
$$= 3O_\gamma\big(\varphi_\gamma(\beta)+1\big) . \quad \text{[D–T 27]}$$

Thus, if $2O_\gamma\beta = \beta O_\gamma 2$ then $\beta = \varphi_\gamma(\beta)+1$. It remains to be shown that $\varphi_\gamma(\beta) \in M(O_\gamma)$.

It follows from the monotonicity law D–T 7 and Definition 13 that

$$\varphi_\gamma(\beta) \leqslant 3O_\gamma\varphi_\gamma(\beta) \leqslant \varphi_\gamma(\beta)+1 .$$

By D–T 34, $3O_\gamma\varphi_\gamma(\beta)$ is a limit ordinal, so we must have

$$\varphi_\gamma(\beta) = 3O_\gamma\varphi_\gamma(\beta) .$$

Consequently, D–T 47 implies $\varphi_\gamma(\beta) \in M(O_\gamma)$.

To extend the preceding result to $\alpha \geqslant 3$, it is convenient first to prove a lemma.

LEMMA 20. *If $\gamma = \bigcup\gamma \neq 0$, $\alpha \geqslant 3$, and $\varphi_{\alpha,\gamma}(\beta) \geqslant 2$ then $\alpha < \varphi_{\alpha,\gamma}(\beta) \in M(O_\gamma)$ if and only if $\beta = \varphi_{\alpha,\gamma}(\beta)+\lambda$ for some $\lambda < \alpha O_\gamma\varphi_{\alpha,\gamma}(\beta)$.*

Proof. Let $\delta = \varphi_{\alpha,\gamma}(\beta)$ and suppose $\beta = \delta+\lambda$ for some $\lambda < \alpha O_\gamma\delta$. Then by Definition 13,

$$\alpha O_\gamma\delta \leqslant \delta+\lambda < \alpha O_\gamma(\delta+1) .$$

By D–T 7, $\delta \leqslant \alpha O_\gamma\delta$. If $\delta < \alpha O_\gamma\delta$ then since $\alpha O_\gamma\delta \in M_\gamma \subseteq M(O_0)$,

$$\delta+(\alpha O_\gamma\delta) = \alpha O_\gamma\delta .$$

Since $\lambda < \alpha O_\gamma\delta$, we obtain

$$\delta+\lambda < \alpha O_\gamma\delta$$

which is a contradiction. Hence, $\delta = \alpha O_\gamma\delta$. Therefore, it follows from D–T 5 (ii) and D–T 47 that

$$\alpha < \delta \in M(O_\gamma) .$$

Conversely, suppose $\alpha < \delta \in M(O_\gamma)$. Then

$$\alpha O_\gamma(\delta+1) = (\alpha O_\gamma\delta)O_\gamma 2 \quad \text{[D–T 27]}$$
$$= \delta O_\gamma 2 \quad \text{[D–T 47]}$$
$$\geqslant \delta\cdot 2 \quad \text{[D–T 8]}$$
$$= \delta+\delta .$$

Therefore, by D–T 47 and Definition 13,

$$\alpha O_\gamma\delta = \delta \leqslant \beta < \delta+\delta .$$

This implies that there is a $\lambda < \delta = \alpha O_\gamma\delta$ such that $\beta = \delta+\lambda$, which completes the proof of the lemma.

THEOREM 21. *If $\gamma = \bigcup\gamma \neq 0$, $3 \leqslant \alpha < \beta$, and $\alpha = 1+\alpha'$ then the following conditions are equivalent*

(1) $\alpha O_\gamma\beta = \beta O_\gamma\alpha$;

(2) $\beta = \varphi_{\alpha,\gamma}(\beta)+\alpha'$;

(3) $\beta = \varrho+\alpha'$ *for some ϱ such that $\alpha < \varrho \in M(O_\gamma)$.*

Proof. By Lemma 14,

$$\beta O_\gamma\alpha = \big(\alpha O_\gamma\varphi_{\alpha,\gamma}(\beta)\big)O_\gamma\alpha$$
$$= \alpha O_\gamma\big(\varphi_{\alpha,\gamma}(\beta)+\alpha'\big) . \quad \text{[D–T 27]}$$

Therefore, by D–T 4 (ii), $\alpha O_\gamma\beta = \beta O_\gamma\alpha$ if and only if $\beta = \varphi_{\alpha,\gamma}(\beta)+\alpha'$, which proves the equivalence of (1) and (2).

By hypothesis, $3 \leqslant \alpha < \beta$. Therefore, if (2) holds it follows from Definition 13 that $\varphi_{\alpha,\gamma}(\beta) \geqslant 2$. Moreover, by D–T 5 (ii), $\alpha' < \alpha O_\gamma\varphi_{\alpha,\gamma}(\beta)$. Thus, Lemma 20 applies and we obtain (2) implies (3).

Suppose (3) holds. Then

$$\alpha O_\gamma\beta = \alpha O_\gamma(\varrho+\alpha')$$
$$= (\alpha O_\gamma\varrho)O_\gamma\alpha \quad \text{[D–T 27]}$$
$$= \varrho O_\gamma\alpha . \quad \text{[D–T 47]}$$

On the other hand since $\alpha' \leqslant \alpha < \varrho$, and there are no main numbers between ϱ and $\varrho+\alpha'$, it follows from Lemma 6 (iv) that

$$\beta O_\gamma\alpha = (\varrho+\alpha')O_\gamma\alpha = \varrho O_\gamma\alpha .$$

Therefore, (3) implies (1) and the proof is complete.

Thus, we have shown that $\gamma = \bigcup\gamma \neq 0$, $2 \leqslant \alpha < \beta$, and $\alpha = 1+\alpha'$ then $\alpha O_\gamma\beta = \beta O_\gamma\alpha$ if and only if either

$$\alpha = 2 \quad \text{and} \quad \beta = 3$$

or

$$\beta = \varrho + \alpha' \quad \text{for some } \varrho \text{ such that } \alpha < \varrho \in M(O_\gamma).$$

The next theorem describes necessary and sufficient conditions for O_γ to be associative, if γ is a limit ordinal.

THEOREM 22. *If* $\gamma = \bigcup \gamma \neq 0$ *and* $\alpha, \beta, \delta \geqslant 2$ *then*

$$(\alpha O_\gamma \beta) O_\gamma \delta = \alpha O_\gamma (\beta O_\gamma \delta)$$

if and only if $\beta < \delta \in M(O_\gamma)$.

Proof. Let $\delta = 1 + \delta'$. Then by D–T 27,

$$(\alpha O_\gamma \beta) O_\gamma \delta = \alpha O_\gamma (\beta + \delta').$$

Thus, by D–T 4 (ii),

$$(\alpha O_\gamma \beta) O_\gamma \delta = \alpha O_\gamma (\beta O_\gamma \delta)$$

if and only if

$$(\ast) \qquad \beta + \delta' = \beta O_\gamma \delta.$$

If $\beta < \delta \in M(O_\gamma)$ then it follows from D–T 47 and elementary properties of ordinal arithmetic that

$$\beta + \delta' = \beta O_\gamma \delta = \delta.$$

So $(\ast)$ holds.

Conversely, suppose $(\ast)$ holds. Then one of β or δ must be larger than 2. (For if $\beta = \delta = 2$ then $\beta + \delta' = 3$ and by D–T 2 (iii), $\beta O_\gamma \delta = 4$.) If $\beta > 2$ then it follows from D–T 54 that $\beta O_\gamma \delta \in M_\gamma$. If $\delta > 2$ and $\beta = 2$ then

$$\beta O_\gamma \delta = 3 O_\gamma \delta' \quad [\text{D–T } 37]$$
$$\in M_\gamma. \quad [\text{D–T } 54]$$

Thus, $\beta + \delta' \in M_\gamma \subseteq M(O_0)$. So $\beta + \delta'$ is a power of ω. This implies

$$\beta < \delta' = \delta \in M(O_0)$$

and

$$\beta + \delta' = \delta = \beta O_\gamma \delta.$$

Then using D–T 47, we obtain that $\delta \in M(O_\gamma)$ thus completing the proof of the theorem.

§ 4. Properties of R_γ and L_γ. The first few theorems describe how Ω is ordered by R_γ when γ is a limit ordinal.

THEOREM 23. *If* $\gamma = \bigcup \gamma \neq 0$ *then*:

(i) *If* $\alpha \geqslant 3$ *then* $\{\beta\colon \alpha R_\gamma \beta\} = \{\alpha\} \cup \{\beta \in M_\gamma\colon \beta > \alpha\}$.

(ii) *If* $\alpha = 2$ *then* $\{\beta\colon \alpha R_\gamma \beta\} = \{2, 4\} \cup M_\gamma$.

(iii) *If* $\beta \neq 0, 4$ *and* $\beta \notin M_\gamma$ *then* $\{\alpha\colon \alpha R_\gamma \beta\} = \{1, \beta\}$.

(iv) *If* $\beta = 4$ *then* $\{\alpha\colon \alpha R_\gamma \beta\} = \{1, 2, 4\}$.

(v) *If* $\beta \in M_\gamma$ *then* $\{\alpha\colon \alpha R_\gamma \beta\} = \{\alpha\colon 1 \leqslant \alpha \leqslant \beta\}$.

Proof. Part (i) follows from D–T 2 (ii) and D–T 54; (ii) follows from D–T 2, D–T 37 and D–T 54; and (iii)–(v) follow from (i), (ii) and elementary properties of O_γ.

THEOREM 24. *If* $\gamma = \bigcup \gamma \neq 0$ *and* $\Omega' = \Omega \sim \{0\}$ *then* $\langle \Omega', R_\gamma \rangle$ *is a complete lattice.* (*That is, a lattice in which each subset of* Ω' *has an* R_γ*-least upper bound and an* R_γ*-greatest lower bound.*)

Proof. R_γ is reflexive because of D–T 2 (ii); anti-symmetric D–T 5 (i) and D–T 7; and transitive, D–T 27.

Suppose $\emptyset \neq X \subseteq \Omega'$ and X is a set. Then $\bigcup X \in \Omega'$ and $\bigcup X$ is the $\leqslant$-least upper bound of X. Let β be an element of M_γ larger than $\bigcup X$. (It follows from D–T 39 that there is an element of M_γ with the required property.) Then it follows from Theorem 23 (v) that β is an R_γ-upper bound of X and that the smallest R_γ-upper bound is the R_γ-least upper bound.

Suppose again that $\emptyset \neq X \subseteq \Omega'$. 1 is an R_γ-lower bound of X. Moreover, it follows from Theorem 23 (iii)–(v) that the set of R_γ-lower bounds of X is an intersection of closed sets and is therefore closed. Therefore, X has an R_γ-greatest lower bound. This completes the proof.

In the case that $\gamma \neq \bigcup \gamma$ the explicit description of $\{\alpha\colon \alpha R_\gamma \beta\}$ and $\{\beta\colon \alpha R_\gamma \beta\}$ is rather complicated and not very instructive. However, we did obtain some results for the case that γ is not a limit ordinal, the most important of which is that $\{\alpha\colon \alpha R_\gamma \beta\}$ is a closed set for all β and γ.

THEOREM 25. *If* $A = \{\alpha\colon \alpha R_\gamma \beta\}$ *and* $\emptyset \neq X \subseteq A$ *then* $\bigcup X \in A$.

Proof. Suppose $A = \{\alpha\colon \alpha R_\gamma \beta\}$ and $\emptyset \neq X \subseteq A$. If X is finite the theorem is trivial, thus, let us suppose X is infinite. Let

$$X = \{\alpha_\varkappa\colon \varkappa \in B\}$$

where $B \subseteq \Omega$, B is infinite, and $\alpha_{\varkappa_1} < \alpha_{\varkappa_2}$ if $\varkappa_1 < \varkappa_2$. Therefore, for each $\varkappa \in B$ there is a $\delta_\varkappa \in \Omega$, $\delta_\varkappa \neq 0$, such that

$$\alpha_\varkappa O_\gamma \delta_\varkappa = \beta.$$

Consequently, it follows from the monotonicity laws, that if $\varkappa_1 < \varkappa_2$, $\delta_{\varkappa_1} \geqslant \delta_{\varkappa_2}$. Thus the δ's form a decreasing sequence, $\delta_{\varkappa_1} > \delta_{\varkappa_2} > \ldots > \delta_{\varkappa_n} = \delta$, with $n \in \omega$. Let

$$X' = \{\alpha \in X\colon \alpha O_\gamma \delta = \beta\}.$$

Then $\bigcup X = \bigcup X'$. We need only consider the case where X' is infinite and show that $\bigcup X' R_\gamma \beta$. Let

$$\theta = \bigcup X'.$$

Case 1. $\gamma = \bigcup \gamma \neq 0$. By Theorem 23 (iii)–(v), A is closed for each $\beta \in \Omega$, so $\theta \in A$.

It follows from Theorem 8, that if $\gamma \neq \bigcup \gamma$ then δ is a limit ordinal, otherwise X' would not be infinite.

Case 2. $\gamma = 2\zeta + 2$ for some $\zeta \geqslant 1$. Suppose that $\omega^{\varkappa} \leqslant \delta < \omega^{\varkappa+1}$. By Theorem 18, if $\alpha, \alpha' \in X'$, $\alpha < \alpha'$, then there are less than $\omega^{\varkappa}$ elements in the set

$$Y_{\alpha,\alpha'} = \{\varrho \in M(O_{2\zeta})\colon\ \alpha < \varrho \leqslant \alpha'\}\,.$$

Therefore, if $\theta \notin X'$ then each of the sets $Y_{\alpha,\theta}$, $\alpha \in X'$ has at least $\omega^{\varkappa}$ elements. Since $\theta = \bigcup X'$, this implies $\theta \in M(O_{2\zeta})$. Also, it follows from Lemma 6 (ii) that $\beta \in M(O_{2\zeta})$. Clearly $\theta \leqslant \beta$. If $\theta = \beta$ then $\theta O_\gamma 1 = \beta$ so $\theta \in A$. If $\theta < \beta$, then by Lemma 6 (ii) there exist ordinal numbers η and η', $\eta < \eta'$ such that

$$\theta = 2\,O_\gamma\,\omega(1+\eta)$$

and

$$\beta = 2\,O_\gamma\,\omega(1+\eta')\,.$$

Since $\eta < \eta'$, there is a $\xi > 0$ such that

$$\eta + \xi = \eta'\,.$$

Thus,

$$\begin{aligned}
\beta &= 2\,O_\gamma\,\omega(1+\eta+\xi)\\
&= 2\,O_\gamma[\omega(1+\eta)+\omega\xi]\\
&= [2\,O_\gamma\,\omega(1+\eta)]\,O_\gamma\,\omega\xi \qquad \text{[D–T 32 (i)]}\,.\\
&= \theta\,O_\gamma\,\omega\xi\,.
\end{aligned}$$

Therefore, $\theta R_\gamma \beta$ and $\theta \in A$.

Case 3. $\gamma = 2\zeta + 1$ for some ζ such that $\bigcup \zeta = \zeta \neq 0$. Since δ is a limit ordinal there is an η such that

$$\delta = \omega(1+\eta)\,.$$

Suppose $\theta \notin X'$. If all the elements of X' are finite ordinals then $\theta = \omega$ and for each $\alpha \in X'$, $\alpha \geqslant 2$,

$$\begin{aligned}
\beta &= \alpha\,O_\gamma\,\delta\\
&= \alpha\,O_{2\zeta}\,\alpha\delta && \text{[D–T 33 (iii)]}\\
&= \alpha\,O_{2\zeta}\,\omega(1+\eta)\\
&= \omega\,O_{2\zeta}\,\omega(1+\eta) && \text{[Lemma 6 (iv)]}\\
&= \omega\,O_\gamma(2+\eta)\,. && \text{[D–T 33 (iii)]}
\end{aligned}$$

This implies that $\theta \in A$.

If X' contains an infinite ordinal and $\theta \notin X'$ then X' contains an infinite number of infinite ordinals. So we might as well assume that all elements of X' are infinite.

If $\theta \notin X'$, then it follows from Theorem 10 that either θ is a limit of elements of X' each of which has the same degree or each of which is a non-zero power of ω. If the former alternative occurs then

$$\theta = \omega^{\varkappa+1}, \qquad \varkappa \geqslant 1$$

where

$$\beta = \omega^{\varkappa}\,O_\gamma\,\delta\,, \qquad \delta = \omega(1+\eta)\,.$$

Then

$$\begin{aligned}
\beta &= \omega^{\varkappa}\,O_{2\zeta}\,\omega^{\varkappa+1}(1+\eta) && \text{[D–T 33 (iii)]}\\
&= \omega^{\varkappa+1}\,O_{2\zeta}\,\omega^{\varkappa+1}(1+\eta) && \text{[Lemma 6 (iv)]}\\
&= \omega^{\varkappa+1}\,O_\gamma(2+\eta)\,. && \text{[D–T 33 (iii)]}
\end{aligned}$$

Consequently, $\theta R_\gamma \beta$.

Now suppose

$$\theta = \lim_{\varkappa \in C} \omega^{\varkappa} = \omega^{\sigma}$$

where

$$\beta = \omega^{\varkappa}\,O_\gamma\,\delta\,, \qquad \text{for each } \varkappa \in C\,.$$

If $\varkappa \in C$ then $\varkappa < \sigma$ so there is a $\varkappa' > 0$ such that

$$\sigma = \varkappa + \varkappa'\,.$$

Since δ is a limit ordinal, there is a $\mu \geqslant 1$ and an η' such that

$$\delta = \omega^{\mu}(\eta'+1)\,.$$

Now, we have

$$\begin{aligned}
\beta &= \omega^{\varkappa}\,O_\gamma\,\delta\\
&= \omega^{\varkappa}\,O_{2\zeta}\,\omega^{\varkappa+\mu}(\eta'+1)\,. \qquad \text{[D–T 33 (iii)]}
\end{aligned}$$

Suppose $\sigma > \varkappa + \mu$. Then, since $\sigma = \varkappa + \varkappa'$ there must be a $\varkappa''$ such that $\varkappa + \varkappa'' \in C$ and

$$\varkappa + \varkappa'' \geqslant \varkappa + \mu\,,$$

which implies

$$\varkappa'' \geqslant \mu\,. \tag{1}$$

But, $\varkappa + \varkappa'' \in C$, so

$$\beta = \omega^{\varkappa+\varkappa''}\,O_\gamma\,\delta\,.$$

Therefore, by Theorem 12, $\omega^{\varkappa''}\delta = \delta$, or equivalently, $\varkappa'' + \mu = \mu$. This latter equation implies $\varkappa'' < \mu$ which contradicts (1). Thus, we must have,

$$\sigma \leqslant \varkappa + \mu\,.$$

This implies there is a σ' such that

$$\sigma+\sigma' = \varkappa+\mu\,.$$

Now we have

$$\beta = \omega^{\varkappa} O_{2\zeta}\, \omega^{\sigma+\sigma'}(\eta'+1)\,.$$

If

$$\begin{aligned}\beta &= \omega^{\sigma} O_{2\zeta}\, \omega^{\sigma+\sigma'}(\eta'+1)\\ &= \omega^{\sigma} O_{\gamma}[1+\omega^{\sigma'}(\eta'+1)] \qquad \text{[D–T 33 (iii)]}\end{aligned}$$

then $\theta R_\gamma \beta$. Otherwise, it follows from Theorem 17 that $\omega^{\sigma} \in M_{2\zeta}$ and the proof follows along the same lines as the end of the proof of case 2, using Lemma 6 (iv) instead of Lemma 6 (ii).

Case 4. $\gamma = 2\zeta+3$ for some $\zeta \geqslant 1$. If $\zeta \geqslant \omega$ or X' contains an infinite ordinal number the proof is similar to the proof of case 3, using Lemma 6 (ii) instead of Lemma 6 (iv) and Theorem 18 instead of Theorem 17. If $\zeta < \omega$ and $X' \subseteq \omega$ the proof is modified as follows.

First, we have as before that δ is a limit ordinal so there is an η such that $\delta = \omega(1+\eta)$. Also, since X' is an infinite subset of ω, $\theta = \omega$ and for each $a \in X'$, $a \geqslant 2$,

$$\begin{aligned}\beta &= a O_\gamma \delta\\ &= a O_{2\zeta+2}\, a\delta \qquad \text{[D–T 33 (iii)]}\\ &= a O_{2\zeta+2}\, \omega(1+\eta)\,.\end{aligned}$$

If $\eta = 0$ then since $a, \zeta \in \omega$, it follows from Definition 1 that $\beta = \omega = \theta$.

If $\eta > 0$ then since $\omega \in M(O_{2\zeta})$ for all finite ζ, it follows from Lemma 6 (ii), that

$$\begin{aligned}\beta &= \omega\, O_{2\zeta+2}\, \omega\eta\\ &= \omega\, O_\gamma(1+\eta)\,. \qquad \text{[D–T 33 (iii)]}\end{aligned}$$

Thus, in either case $\theta \in A$.

The proof of the theorem for the remaining cases, $\gamma = 0, 1, 2, 3$, is an exercise in the traditional arithmetic of ordinal numbers. We leave the details for the interested reader. (For $\gamma = 1, 2$, see Carruth [2].)

Before stating the results for L_γ it is convenient to introduce some notation.

Definition 26. *If*

$$0 \neq \delta = \omega^{\delta_0} d_0 + \ldots + \omega^{\delta_n} d_n$$

is the normal form of δ, then for each $m, e_m \in \omega$ such that $0 \leqslant m \leqslant n$ and $0 \leqslant e_m \leqslant d_m$,

$$\tau_\delta(m, e_m) \underset{\mathrm{Df}}{=} \omega^{\delta_m} e_m + \omega^{\delta_{m+1}} d_{m+1} + \ldots + \omega^{\delta_n} d_n\,.$$

Lemma 27. *If*

$$0 \neq \delta = \omega^{\delta_0} d_0 + \ldots + \omega^{\delta_n} d_n$$

is the normal form of δ then $aL_0\delta$ if and only if there exist $m, e_m \in \omega$ such that $0 \leqslant m \leqslant n$, $0 < e_m \leqslant d_m$ and

$$a = \tau_\delta(m, e_m)\,.$$

Proof. The lemma follows from elementary properties of ordinal arithmetic.

Theorem 28. *If $\gamma = \bigcup \gamma \neq 0$ then*

(i) $\{\beta\colon 2L_\gamma\beta\} = \{2, 4\} \cup \{\beta\colon (\exists \xi)[\xi \geqslant 2,\ \beta = 3\,O_\gamma \xi,$ *and* $1L_0\xi]\}$.
$= \{2, 4\} \cup \{\beta\colon (\exists \delta)[\beta$ *is the smallest element of* M_γ *exceeding* $\delta]\}$.

(ii) *If* $a > 0$ *then*
$\{\beta\colon 2+aL_\gamma\beta\} = \{2+a\} \cup \{\beta\colon (\exists \xi)[\beta = 3\,O_\gamma\xi$ *and* $(1+a)L_0\xi]\}$.
$= \{2+a\} \cup \{\beta\colon (\exists \delta)[\beta$ *is the ath successive element of* M_γ *exceeding* $\delta]\}$.

(iii) *If* $\beta \neq 0, 4$ *and* $\beta \notin M_\gamma$ *then* $\{a\colon aL_\gamma\beta\} = \{1, \beta\}$.

(iv) *If* $\beta = 4$ *then* $\{a\colon aL_\gamma\beta\} = \{1, 2, 4\}$.

(v) *If* $\beta \in M_\gamma$ *then* $\beta = 3\,O_\gamma\delta$ *for some* $\delta \geqslant 2$ *and*
$\{a\colon (1+a)L_\gamma\beta\} = \{\beta\} \cup \{a\colon aL_0\delta\}$.

Proof. To prove (i) we have $2L_\gamma\beta$ if and only if there is a $\delta > 0$ such that, $\delta\, O_\gamma 2 = \beta$. If $\delta = 1$, $\beta = 2$ and if $\delta = 2$, $\beta = 4$. Suppose $\delta \geqslant 3$. Then, by Lemma 6 (iv), β is the smallest element of M_γ exceeding δ. By Lemma 14,

$$\begin{aligned}\beta &= \delta\, O_\gamma 2\\ &= \big(3\,O_\gamma \varphi_\gamma(\delta)\big)\, O_\gamma 2\\ &= 3\,O_\gamma\big(\varphi_\gamma(\delta)+1\big)\,. \qquad \text{[D–T 27]}\end{aligned}$$

Thus, part (i) is true.

The proof of (ii) is similar. The proofs of (iii) and (iv) follow from (i) and (ii). To prove (v) use Lemma 14 and D–T 27.

Theorem 24 does not hold if "R_γ" is replaced by "L_γ", but we do have the following result for L_γ.

Theorem 29. *If $\gamma = \bigcup \gamma \neq 0$ and $a < a'$ then $1+a$ and $1+a'$ have an L_γ-upper bound if and only if aL_0a' or $(1+a)L_\gamma a'$.*

Proof. The proof follows from Theorem 28 (v).

It is clear that 1 is an L_γ-lower bound for every subset of $\Omega' = \Omega \sim \{0\}$, if $\gamma > 0$ (D–T 2 (ii)). Since the set of all L_γ-lower bounds of an ordinal number is finite (D–T 14), it follows that every non-empty subclass of Ω' has an L_γ-greatest lower bound.

It is also clear that L_γ is reflexive and anti-symmetric for all γ and that L_γ is transitive for $\gamma = 0, 1, 2, 3$. But we know that L_4, for example, is not transitive. We do not know whether or not L_γ is transitive even when γ is a limit ordinal. Our results for R_γ and L_γ are incomplete and it is probable that much more could be learned about these relations by additional study.

Our bibliography just includes those books and articles explicitly referred to in the paper. Additional references may be found in the bibliography of the Doner–Tarski paper [1].

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