Differentiable roads for real functions

by

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In 1936 Maximoff [6] proved that for a Borel measurable function f defined on a real interval I, there exists a countable set C such that for each $x \in I-C$ there exists a perfect subset P of I having x as a bilateral limit point that the restriction of f to P, $f|P$, is continuous. On the other hand, in 1966 Filipczak [4] proved that for a Borel measurable function f defined on a perfect set Q of the reals, there exists a perfect set $P \subset Q$ such that $f|P$ is monotonic.

Both of these results can be simultaneously strengthened by the following theorem:

THEOREM 1. (Bruckner, Ceder and Weiss [2]) *Let f be any real-valued Borel measurable function defined on a perfect set Q of the reals. Then there exists a countable set C such that for each $x \in Q-C$ there exists a perfect set $P \subset Q$ having x as a bilateral limit point such that $f|P$ is differentiable.*

It is unknown whether the perfect set P in the above theorem can be chosen so that $f|P$ is also monotonic. Nevertheless, the theorem does imply Filipczak's result as shown in the last paragraph of this article.

The main purpose of this paper is to establish the following natural analogue of Theorem 1 (as well as Maximoff's and Filipczak's theorems) for arbitrary real-valued functions defined on an uncountable subset of the reals.

THEOREM 2. *Let f be any real-valued function defined on an uncountable subset A of the reals. Then, there exists a countable set C such that for each $x \in A-C$ there exists a bilaterally dense-in-itself set B containing x such that $f|B$ is monotonic and differentiable.*

A set B is *bilaterally dense-in-itself* if each point of B is a bilateral limit point of B. By saying $f|B$ is differentiable we mean that $(f|B)'(x)$ exists as an extended real number for each $x \in B$.

On drawback of Theorem 2 is that the "differentiable road" B may be countable. However, because B is dense-in-itself its closure will be perfect. Nevertheless, Theorem 2 is the best possible result in the sense that it cannot be improved to assert that B is uncountable or that B is dense in some open interval. These facts are shown by the following two examples.

EXAMPLE 1. There exists a continuous function f on $[0,1]$ such that for each somewhere dense set A, $f|A$ is neither differentiable nor monotonic. For this, take f to be any continuous, nowhere differentiable function on $[0,1]$. If A is dense in some interval J then the continuity of f implies that $f|J$ is differentiable (differentiable a.e.) whenever $f|(A\cap J)$ is differentiable (respectively monotonic).

EXAMPLE 2. There exists a function g on $[0,1]$ such that for each uncountable set A, $f|A$ is neither differentiable nor monotonic. In [5], pp. 147-8 Goffman constructed a function g on $[0,1]$ such that $g|A$ fails to be continuous for each uncountable set A. Since differentiability implies continuity and monotonicity implies continuity on an uncountable subset of each uncountable set A, it follows that $g|A$ is neither differentiable nor monotonic for each uncountable set A.

Prerequisite to the proof of Theorem 2 are some terminology and two preliminary results, Lemma 1 and Theorem 3.

First of all, throughout the sequel we will regard a function as identical with its graph and we will only consider functions whose domain and range are subsets of the reals. For such a function we call $(x,f(x))$ a *bilateral condensation point of* f provided that $S\cap f$ is uncountable whenever S is an open square (with sides parallel to the coordinate axes) of side length h which has the point $\left(x\mp\frac{h}{2}, f(x)-\frac{h}{2}\right)$ as its lower-left or lower-right vertex. Next we have the following result concerning bilateral condensation points for functions, the statement and proof of which are modifications of Lemma 4 of Ceder and Pearson [3].

LEMMA 1. *Let f have domain A where A is uncountable. Let B be the domain of the bilateral condensation points of f. Then $A-B$ is countable and for each $x\in B$, $(x,f(x))$ is a bilateral condensation point for $f|B$.*

The *left-derived set*, $D_L(f,x)$, and the *right-derived set*, $D_R(f,x)$, *of a function f at the point x* are defined to be sets of all possible sequential limits (as extended real numbers) of the difference quotient $\dfrac{f(y)-f(x)}{y-x}$ as y approaches x from the left and right respectively.

The next preliminary result, Theorem 2, is possibly of independent interest, and it asserts that for any given function f with an uncountable domain, a countable number of points may be deleted from f to form a new function g whose left and right derived sets overlap at each point of the domain of g. This result extends a result of Bagemihl [1] which asserts that for a function f defined on an interval the derived sets of f overlap except possibly at a countable number of points. The proof of Theorem 3 consists of a modification of Bagemihl's proof and an application of Lemma 1.

THEOREM 3. *Suppose f has an uncountable domain A. Then, there exists a countable subset C of A such that for each $x\in A-C$*

$$D_L(f|(A-C),x)\cap D_R(f|(A-C),x)\neq\emptyset .$$

Proof: By virtue of Lemma 1 we may assume without loss of generality that each point of f is a bilateral point of condensation of f. Define $E=\{(x,y)\in A\times A\colon\ y<x\}$ and $\Delta=\{(x,x)\colon\ x\in A\}$. First we establish the following lemma.

LEMMA 2. *Suppose $M\subseteq E$. Then, there exists a $K\subseteq\Delta$ such that $\Delta-K$ is countable and for each $(x,x)\in K$ and positive integer n we have, where $L_n(x)=\{x\}\times\left(x-\frac{1}{n}, x\right)$ and $R_n(x)=\left(x, x+\frac{1}{n}\right)\times\{x\}$, that*

(1) *both $L_n(x)$ and $R_n(x)$ intersect M*

or

(2) *both $L_n(x)$ and $R_n(x)$ intersect $E-M$.*

Proof of Lemma 2: Let F_n consist of all $\langle x,x\rangle\in\Delta$ for which neither (1) nor (2) hold with respect to n. Put $K=\Delta-\bigcup_{n=1}^{\infty}F_n$. To complete the proof it suffices to show that each F_n is countable. Consider a given F_n. Then clearly for each $\langle x,x\rangle\in F_n$ we have: (a) $L_n(x)\subseteq M$ and $R_n(x)\subseteq E-M$ or (b) $L_n(x)\subseteq E-M$ and $R_n(x)\subseteq M$. Let $F_n^a(F_n^b)$ consist of all $(x,x)\in F_n$ for which (a) (resp (b)) holds. Now suppose $\langle x,x\rangle$ and $\langle y,y\rangle$ are any two points of F_n^a such that $x<y<x+\frac{1}{n}$. Then $(x,y)\in R_n(x)\cap L_n(y)$ which violates condition (a). Thus each two points of F_n^a, and, similarly, of F_n^b, are separated by at least a distance of $\frac{1}{n}$. Hence F_n is countable, which finishes the proof of the lemma.

Put $X=[-\infty,+\infty]$ and give X the two-point compactification topology. Let $\mathfrak{B}=\{B_n\}_{n=1}^{\infty}$ be a countable basis for X which separates each pair of disjoint compact (equivalently closed) subsets of X.

Define for $(x,y)\in E$,

$$D(x,y)=\frac{f(x)-f(y)}{x-y}.$$

Now for each n put $M_n=\{(x,y)\in E\colon\ D(x,y)\notin B_n\}$. Applying Lemma 2 to each set M_n we obtain a sequence of sets $\{K_n\}_{n=1}^{\infty}$ such that $K_n\subseteq\Delta$ and $\Delta-K_n$ is countable for each n and such that for any $(x,x)\in K_n$

and m (1) both $L_m(x)$ and $R_m(x)$ intersect M_n or (2) both $L_m(x)$ and $R_m(x)$ intersect $E-M_n$.

Next put $K=\bigcap_{n=1}^{\infty} K_n$ so that $\Delta - K = \Delta - \bigcup_{n=1}^{\infty} K_n$ is countable. Choose any $\langle z, z\rangle \in K$. We will show that $D_L(f, z) \cap D_R(f, z) \neq \emptyset$. Assume that this is not the case. Then since $D_L(f, z)$ and $D_R(f, z)$ are closed subsets of X there exists an n such that $D_L(f, z) \subseteq B_n$ and $D_R(f, z) \subseteq E-\overline{B_n}$.

In particular, $(z, z) \in K_n$ so that for each m (i) $R_m(z)$ and $L_m(z)$ both intersect M_n or (ii) $R_m(z)$ and $L_m(z)$ both intersect $E-M_n$. Hence we can find a sequence $\{m_k\}$ of positive integers such that either (a) for each k $R_{m_k}(z)$ and $L_{m_k}(z)$ both intersect M_n or (b) for each k $R_{m_k}(z)$ and $L_{m_k}(z)$ both intersect $E-M_n$. Supposing (a) holds, we can clearly find a sequence $\{x_k\}_{k=1}^{\infty}$ in A such that $x_k \uparrow z$, $D(z, x_k) \notin B_n$ for each k and $\lim_{k\to\infty} D(z, x_k)$ exists. Since B_n is open this implies that $\lim_{k\to\infty} D(z, x_k) \notin B_n$ and $D_L(f, z) \nsubseteq B_n$, a contradiction. Likewise supposing (b) we can contradict the fact that $D_R(f, z) \subseteq E-\overline{B_n}$.

Letting C be the domain of K we now have that C is countable and for each $x \in A-C$, $D_L(f, x) \cap D_R(f, x) \neq \emptyset$. It remains to show that the derived sets for $f|(A-C)$ also overlap.

To show this let $x \in A-C$. Then there exists a $\lambda \in X$ and sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $x_n \downarrow x$, $y_n \uparrow x$, $D(x_n, x) \to \lambda$ and $D(y_n, y) \to \lambda$. Since each point of f is a bilateral point of condensation of f, there exist for each n points x_n' and y_n' in $A-C$ such that

$$\big((x_n-x_n')^2+\big(f(x_n)-f(x_n')\big)^2\big) \leqslant \frac{1}{n}\big((x_n-x)^2+\big(f(x_n)-f(x)\big)^2\big)$$

and

$$\big((y_n-y_n')^2+\big(f(y_n)-f(y_n')\big)^2\big) \leqslant \frac{1}{n}\big((y_n-y)^2+\big(f(y_n)-f(y)\big)^2\big).$$

Then clearly $y_n' < x < x_n'$ for each n and $D(y_n', x) \to \lambda$ and $D(x_n', x) \to \lambda$. Hence, λ is common to the derived sets of $f|(A-C)$ and the theorem is proved.

Proof of Theorem 2: Suppose f is a function with an uncountable domain A. Apply Theorem 3 to obtain a countable set C such that the left and right derived sets of $f|(A-C)$ overlap at each $x \in A-C$. First we will show that for each $x \in A-C$ there exists a bilaterally dense-in-itself countable set $B \subseteq A-C$ containing x such that $f|B$ is differentiable.

The proof consists of several inductions within inductions so that a detailed proof is very cumbersome to describe. Therefore, we will describe in detail only the first "leg" of the induction and then outline how the inductive process can be reiterated to give the full induction.

Without loss of generality we may assume that $C=\emptyset$. First of all, choose $x_0 \in A$ and $\lambda_0 \in D_L(f, x_0) \cap D_R(f, x_0)$. Next choose a sequence $\{t_n\}_{n=1}^{\infty}$ in A such that $t_n \downarrow x_0$ and $D(t_n, x_0) = \dfrac{f(t_n)-f(x_0)}{t_n-x_0} \to \lambda_0$. Then for each t_k pick $\lambda_k \in D_L(f, t_k) \cap D_R(f, t_k)$ and a sequence $\{t_{k,n}\}_{n=1}^{\infty}$ in A such that

(1) $t_{k,n} \downarrow t_k$;

(2) $D(t_{k,n}, t_k) \to \lambda_k$;

(3) $|t_k - t_{k,1}| \leqslant \begin{cases} \frac{1}{3}|t_{k-1}-t_k| & \text{if} \quad k \neq 1, \\ \frac{1}{3} & \text{if} \quad k=1 \end{cases}$

and

(4) $|D(t_{k,n}, x_0) - D(x_0, t_k)| < \dfrac{1}{k}$ for each n.

Now suppose, in addition to the above, we have defined by induction the points $t_{k_1,k_2,\ldots,k_j}$ for each j-tuple $\langle k_1, k_2, \ldots, k_j\rangle$ of positive integers for $1 < j \leqslant m$ such that

(1) $t_{k_1,k_2,\ldots,k_j,n} \downarrow t_{k_1,\ldots,k_j}$ for $j < m$;

(2) for $j < m \lim_{n\to\infty} D(t_{k_1,k_2,\ldots,k_j,n}, t_{k_1,k_2,\ldots,k_j})$, denoted by $\lambda_{k_1,k_2,\ldots,k_j}$, exists and belongs to

$$D_L(f, t_{k_1,k_2,\ldots,k_j}) \cap D_R(f, t_{k_1,k_2,\ldots,k_j});$$

(3) for $j < m$ and s the first point to the right of $t_{k_1,k_2,\ldots,k_j}$ of the form $t_{l_1,l_2,\ldots l_i}$ where $i \leqslant j$, we have

$$|t_{k_1,k_2,\ldots,k_j,1} - t_{k_1,k_2,\ldots,k_j}| \leqslant \begin{cases} \frac{1}{3}|t_{k_1,k_2,\ldots,k_j} - s|, \\ \frac{1}{3}, \quad \text{if } s \text{ does not exist;} \end{cases}$$

(4) for $j < i \leqslant m$

$$|D(t_{k_1,k_2,\ldots,k_1}, t_{k_1,k_2,\ldots,k_j}) - D(t_{k_1,k_2,\ldots,k_{j+1}}, t_{k_1,\ldots,k_j})| < \frac{1}{k_{j+1}}.$$

Now we proceed to define the points $t_{k_1,k_2,\ldots,k_m,k_{m+1}}$. For each m-tuple $\langle k_1, k_2, \ldots, k_m\rangle$ we can clearly choose a sequence $\{t_{k_1,k_2,\ldots,k_m,n}\}_{n=1}^{\infty}$ such that conditions (1), (2), and (3) of the inductive hypothesis are satisfied when j is replaced by m. To satisfy condition (4) we must show that

$$|D(t_{k_1,k_2,\ldots,k_{m+1}}, t_{k_1,k_2,\ldots k_j}) - D(t_{k_1,k_2,\ldots,k_{j+1}}, t_{k_1,k_2,\ldots,k_j})| < \frac{1}{k_{j+1}}.$$

In (4) let $i=m$ and let $j < m = i$. Then there exists a neighborhood V_j of $\big(t_{k_1,k_2,\ldots,k_m}, f(t_{k_1,k_2,\ldots,k_m})\big)$ such that

$$\left|D(t_{k_1,k_2,\ldots,k_{j+1}}, t_{k_1,k_2,\ldots,k_j}) - \frac{y-f(t_{k_1,k_2,\ldots,k_j})}{x-t_{k_1,k_2,\ldots,k_j}}\right| < \frac{1}{k_{j+1}}$$

whenever $(x, y) \in V_j$. Let V be the domain of $\bigcap_{j<m} V_j$. Then if the sequence

$\{t_{k_1,\ldots,k_m,n}\}_{n=1}^{\infty}$ is chosen to lie in V and satisfy (1), (2) and (3), then it will also satisfy (4). Therefore, the points $t_{k_1,k_2,\ldots,k_j}$ for $j \leqslant m+1$ can be chosen to satisfy the conditions (1) through (4).

Let C_1 be the collection of all points $t_{k_1,k_2,\ldots,k_m}$ where $\langle k_1, k_2, \ldots, k_m\rangle$ is an arbitrary m-tuple. Clearly by condition (1) each point of C_1 is the limit from the right of a sequence in C_1. On the other hand, no point of C_1 is the limit from the left of a sequence in C_1 because it follows from condition (3) that when $c = t_{k_1,k_2,\ldots,k_m}$ and $h = \frac{1}{2}|t_{k_1,k_2,\ldots,k_m} - t_{k_1,k_2,\ldots,k_{m+1}}|$ we have $C_1 \cap (c-h, c) = \emptyset$.

We now show that $f|C_1$ is differentiable. Let $x \in C_1$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in C_1 with $x_n \downarrow x$. Then for some j we have $x_0 = t_{k_1,k_2,\ldots,k_j}$ and we may assume without loss of generality that for each n $x_n = t_{k_1,k_2,\ldots,k_j,k_{j+1}(n),\ldots,k_{i_n}(n)}$. By condition (4) we must have

$$|D(x_0, x_n) - D(x_0, t_{k_1,k_2,\ldots,k_{j+1}(n)})| < \frac{1}{k_{j+1}(n)}.$$

Since $x_n \to x_0$ we must have $\lim_{n\to\infty} k_{j+1}(n) = \infty$ so that

$$D(x_0, t_{k_1,k_2,\ldots,k_{j+1}(n)}) \to \lambda_{k_1,k_2,\ldots,k_j} \quad \text{and} \quad D(x_n, x_0) \to \lambda_{k_1,k_2,\ldots,k_j}.$$

Hence, $f|C_1$ is differentiable.

This completes the first leg of the induction.

Let us now relabel the points $t_{k_1,k_2,\ldots,k_m}$ in C_1 by t_s where s is the m-tuple $\langle k_1, \ldots, k_m\rangle$. For each $t_s \in C_1$ choose $h_s > 0$ such that $C_1 \cap (t_s - 3h_s, t_s) = \emptyset$. In the open interval $(t_s - h_s, t_s)$ we can repeat the above construction of C_1 only using limits from the left instead of limits from the right, and where t_s plays the role of x_0 and $(f|C_1)'(t_s)$ plays role the of λ_0. The set of points so obtained will be denoted by $t_{s,u}$ where u represents any finite tuple of positive integers.

Next for each fixed $t_{s,u}$ there exists a right neighborhood disjoint from all the other $t_{s,v}$. Then we can again put a copy of C_1 in this neighborhood as outlined above. In this way we obtained the points $t_{s,u,v}$ where s, u and v are finite tuples.

Continuing in this way we will obtain the set B consisting of all points $t_{s_1,s_2,\ldots,s_m}$ where each s_i is a finite tuple of positive integers. Clearly B is a bilaterally dense-in-itself set containing the original point x_0 of A.

If sufficient care is taken in picking the points of B, $f|B$ will be differentiable. To do this we must inductively define B so that the analogue of condition (4) holds namely

(4′) for each $t_{s_1,s_2,\ldots,s_n}$ where $s_n = \langle i_1, \ldots, i_l\rangle$ we have

$$|D(t_{s_1,s_2,\ldots,s_n}, t_{s_1,s_2,\ldots,s_n,\langle m\rangle}) - D(t_{s_1,s_2,\ldots,s_n}, t_{s_1,s_2,\ldots,s_n,\langle m,\ldots\rangle,\ldots})| < \frac{1}{m}$$

and

$$|D(t_{s_1,s_2,\ldots,s_n}, t_{s_1,\ldots,s_{n-1},\langle i_1,\ldots,i_l,m\rangle}) - D(t_{s_1,\ldots,s_n}, t_{s_1,\ldots,s_{n-1},\langle i_1,\ldots,i_l,m,\ldots\rangle,\ldots})| < \frac{1}{m}.$$

The proof that this can be done is similar to the corresponding proof in the construction of C_1.

By the construction of B each point $t_{s_1,s_2,\ldots,s_n}$ has a neighborhood such that on one side of $t_{s_1,s_2,\ldots,s_n}$ the points of B are all of the form $t_{s_1,s_2,\ldots,s_n s_{n+1},\ldots,s_k}$ for some $s_{n+1}, \ldots, s_k$ and on the other side all the points are of the form

$$t_{s_1,s_2,\ldots,s_{n-1},\langle i_1,\ldots,i_l,i_{l+1},\ldots,i_j\rangle, s_{n+1},\ldots,s_k}$$

for some $i_{l+1},\ldots, i_j, s_{n+1}, \ldots, s_k$ where $s_n = \langle i_1, \ldots, i_l\rangle$. Then using (4′) one can show that $f|B$ is differentiable in the same way that (4) implied the differentiability of $f|C_1$.

Hence, we have shown that for each $x \in A - C$ there exists a bilaterally dense-in-itself set $B \subseteq A - C$ containing x such that $f|B$ is differentiable. It now remains to show that C and B can be selected so that $f|B$ is also monotonic. First we need the following lemma.

LEMMA 3: *Suppose B is bilaterally dense-in-itself and $f|B$ is differentiable. Then there exists $A \subseteq B$ such that A is bilaterally dense-in-itself and $f|A$ is differentiable and monotonic.*

Proof of Lemma 3: Consider the two sets $\{x: f'(x) \geqslant 0\}$ and $\{x: f'(x) \leqslant 0\}$. It is easily shown that one of these sets is bilaterally dense-in-itself in some relative subinterval of B. Suppose then that E is bilaterally dense-in-itself and $f'(x) \geqslant 0$ for $x \in E$ (or equivalently $(f|E)'(x) \geqslant 0$).

To construct the desired set A we proceed as follows. If there is a relative subinterval on which f is constant, then it will serve as the A for which $f|A$ is monotonic. If, however, there is no relative subinterval on which f is constant we can pick $x \in E$ and sequences $\{z_n^L\}_{n=1}^{\infty}$ and $\{z_n^R\}_{n=1}^{\infty}$ in E such that $z_n^L \uparrow x$, $z_n^R \downarrow x$, $f(z_n^R) \downarrow f(x)$ and $f(z_n^L) \uparrow f(x)$. Now about each of the points $(z_n^L, f(z_n^L))$ and $(z_n^R, f(_n^R))$ we can find neighborhoods U_n^L and U_n^R such that the sequence of the domains of $\{U_n^L\}_{n=1}^{\infty}$ and $\{U_n^R\}_{n=1}^{\infty}$ are disjoint and the sequence of the ranges of $\{U_n^L\}_{n=1}^{\infty}$ and $\{U_n^R\}_{n=1}^{\infty}$ are also disjoint. In each neighborhood U_m^L (or U_m^R) we can choose sequences $\{z_{m,n}^L, f(z_{m,n}^L)\}_{n=1}^{\infty}$ and $\{z_{m,n}^R, f(z_{m,n}^R)\}_{n=1}^{\infty}$ such that $z_{m,n}^L \uparrow z_m^L$, $z_{m,n}^R \downarrow z_m^L$, $f(z_{m,n}^R) \downarrow f(z_m^L)$ and $f(z_{m,n}^L) \uparrow f(z_m^L)$. Then again choose neighborhoods of these points and continue the process. In this way we obtain the set A of the points $z_n^L, z_n^R, z_{m,n}^L, z_{m,n}^R$, etc., so that A is bilaterally dense-in-itself and $f|A$ is monotonic and differentiable. This finishes the proof of the lemma.

Returning to the proof of Theorem 2, let F consist of all $x \in A - C$ such that there does not exist a bilaterally dense-in-itself set B containing x such that $f|B$ is monotonic and differentiable. To finish proving the theorem we need only show F is countable. Suppose then that F is uncountable. Now apply the first part of this theorem to obtain a bilaterally dense-in-itself set $P \subseteq F$ such that $f|P$ is differentiable. Now apply Lemma 3 to $f|P$ to get a bilaterally dense-in-itself set $Q \subseteq P \subseteq F$ such that $f|Q$ is monotonic and differentiable. But this contradicts the definition of F.

This concludes the proof of Theorem 2.

As previously mentioned in the introduction it is unknown whether one can strengthen Theorem 1 to assert that $f|B$ is also monotonic. The proof of the second part of Theorem 2, including Lemma 3, will carry through provided the set F defined above is a Borel set. Whether this is true, however, is unknown.

In [2] it was also shown that for a Borel measurable function f defined on a perfect Q there exists a perfect $P \subseteq Q$ such that $f|Q$ is infinitely differentiable. Hence, another interesting question is whether the set P in the conclusion of Theorem 1 can be chosen so that $f|P$ is infinitely differentiable. Analogously, another question is whether Theorem 2 can be improved to assert that $f|B$ is infinitely differentiable or even twice differentiable.

As a final remark we indicate how Theorem 1 implies Filipczak's result. Using Theorem 1 we can find a perfect set P such that $f|P$ is differentiable. Secondly, apply Theorem 1 to $(f|P)'$ to find a perfect set $T \subseteq P$ such that $f|T$ is twice differentiable. Now apply Lemma 3 to $f|T$ so that the closure of A will be the desired perfect set upon which f is monotonic.

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