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On equations with several involutions of different orders and its applications to partial differential-difference equations

by

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In [8] we have shown that a differential-difference equation

$$\sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj}x^{(k)}(t-\omega_j) = y(t) \tag{1}$$

(y is a given periodic function, a_{kj}, ω_j are scalars, $\omega_0 = 0$ and $x^{(k)}$ denotes the k-th derivative of x) is equivalent in the class of periodic functions to a finite system of ordinary differential equations with constant coefficients. This permits us to find all the periodic solutions of (1).

The method used appears to be more general. This paper contains general theorems on equations with several involutions and a generalization of the result described above to partial differential-difference equations with periodic coefficients.

1. Involutions of order N. We shall enumerate here without proofs those properties on involutions of order N which will be needed later. The reader can find the respective proofs in papers [7], [8] and in book [9].

Let X be a linear space (over complex scalars). A linear operator S transforming X onto X is called an *involution of order* N if N is the smallest positive integer ($N \geqslant 2$) such that $S_N = I$, where I denotes the identity operator.

Let $\varepsilon = e^{2\pi i/N}$,

$$P_\nu = \frac{1}{N}(I+\varepsilon^{-\nu}S+\ldots+\varepsilon^{-\nu(N-1)}S^{N-1}), \qquad \nu = 1, 2, \ldots, N.$$

If S is an involution of order N, we have the following important properties of operators P_ν:

$$\sum_{\nu=1}^{N} P_\nu = I, \qquad P_\nu P_\mu = P_\mu P_\nu = \delta_{\mu\nu}P_\nu, \qquad P_\nu S = SP_\nu \tag{1.1}$$

$$(\mu, \nu = 1, 2, \ldots, N),$$

where $\delta_{\mu\nu}$ is the Kronecker symbol;

(1.2) $$SP_\nu = \varepsilon^\nu P_\nu \quad (\nu = 1, 2, \ldots, N).$$

This implies that X is a direct sum,

(1.3) $$X = \bigoplus_{\nu=1}^{N} X_{(\nu)},$$

of spaces $X_{(\nu)}$ such that

$$Sx_{(\nu)} = x_{(\nu)} \quad \text{for } x_{(\nu)} \in X_{(\nu)} \quad (\nu = 1, 2, \ldots, N).$$

Every element $x \in X$ can be written in a unique manner as a sum:

(1.4) $$x = \sum_{\nu=1}^{N} x_{(\nu)}, \quad \text{where } x_{(\nu)} = P_\nu x \in X_{(\nu)} \quad (\nu = 1, 2, \ldots, N).$$

If a linear operator A acting in X is commutative with an involution S of order N, then

(1.5) $$A(D_A \cap X_{(\nu)}) \subset X_{(\nu)} \quad \text{for } \nu = 1, 2, \ldots, N,$$

where $D_A \subset X$ denotes the domain of A. In fact, suppose that we have an arbitrary $x \in D_A$. Then $x_{(\nu)} = P_\nu x \in X_{(\nu)}$ for $\nu = 1, 2, \ldots, N$ and

$$Ax_{(\nu)} = AP_\nu x = A\Big(\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k\Big)x = \Big(\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k\Big)Ax$$
$$= P_\nu(Ax) = (Ax)_{(\nu)} \in X_{(\nu)}.$$

For any polynomial $a(t) = \sum_{k=0}^{N-1} a_k t^k$ (a_k being scalars) we have

(1.6) $$a(S) = \sum_{\nu=1}^{N} a(\varepsilon^\nu) P_\nu$$

because

$$a(S) = \sum_{k=0}^{N-1} a_k S^k = \sum_{k=0}^{N-1} a_k S^k\Big(\sum_{\nu=1}^{N} P_\nu\Big) = \sum_{\nu=1}^{N}\sum_{k=0}^{N-1} a_k S^k P_\nu$$
$$= \sum_{\nu=1}^{N}\sum_{k=0}^{N-1} a_k \varepsilon^{\nu k} P_\nu = \sum_{\nu=1}^{N} \big(a_k(\varepsilon^\nu)^k\big) P_\nu = \sum_{\nu=1}^{N} a(\varepsilon^\nu) P_\nu.$$

Then any equation with the involution of order N,

(1.7) $$a(S)x = y, \quad y \in X,$$

is equivalent to N equations

(1.8) $$a(\varepsilon^\nu)x(\nu) = y(\nu), \quad \nu = 1, 2, \ldots, N,$$

each of them being considered in the space $X_{(\nu)}$. The respective theorems on the solvability of (1.8) are given in [8].

2. Multi-involutions. Let us suppose that we have q given involutions $S_1, \ldots, S_q$ of orders $N_1, \ldots, N_q$ respectively acting in X. Let us write

$$\varepsilon_p = e^{2\pi i/N_p}, \quad P_{p,r} = \frac{1}{N_p}\sum_{k=0}^{N_p-1} \varepsilon_p^{-kr} S_p^k \quad (p = 1, 2, \ldots, q).$$

From the preceding considerations we obtain

(2.1) $$\sum_{r=1}^{N_p} P_{p,r} = I, \quad P_{p,r_1}P_{p,r} = \delta_{r_1 r}P_{p,r}, \quad S_p P_{p,r} = \varepsilon_p^r P_{p,r}$$
$$(p, r, r_1 = 1, 2, \ldots, q).$$

To simplify the theorems to be given later on, we shall now introduce multi-involutions.

Let us consider q-dimensional multi-indices $k = (k_1, \ldots, k_q)$ and $m = (m_1, \ldots, m_q)$, where k_p and m_p are non-negative integers. As usual, we assume

$$|k| = k_1 + \ldots + k_q, \quad k+m = (k_1+m_1, \ldots, k_q+m_q),$$
$$\lambda k = (\lambda k_1, \ldots, \lambda k_q) \text{ for any non-negative integer } \lambda,$$
$$km = (k_1 m_1, \ldots, k_q m_q).$$

We shall also write $(n)_q = (\underbrace{n, \ldots, n}_{q \text{ times}})$ for $n = 0, 1, 2, \ldots$ We write

$$k \leqslant m \text{ if and only if } k_p \leqslant m_p \text{ for } p = 1, 2, \ldots, q;$$
$$k = m \text{ if and only if } k_p = m_p \text{ for } p = 1, 2, \ldots, q.$$

Let $N = (N_1, \ldots, N_q)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_q)$, where $\varepsilon_p = e^{2\pi i/N_p}$ ($p = 1, 2, \ldots, q$). We write

$$\varepsilon^k = (\varepsilon_1^{k_1}, \ldots, \varepsilon_q^{k_q}),$$

where $k = (k_1, \ldots, k_q)$ is a multi-index. By definition, $\varepsilon^{-\nu k} = \varepsilon^{(N-\nu)k}$, where N, ν and k are the respective multi-indices.

A superposition $S = S_1, \ldots, S_q$ of operators $S_1, \ldots, S_q$ acting in a linear space X is called *multi-involution* of order $N = (N_1, \ldots, N_q)$ if S_p is an involution of order N_p and S_p are commutative with S_r for $p, r = 1, 2, \ldots, q$.

Let us write

$$S^k = S_1^{k_1} \ldots S_q^{k_q}, \quad \text{where } k = (k_1, \ldots, k_q);$$

then

$$S^N = S_1^{N_1} \ldots S_q^{N_q} = I.$$

If we write

$$P_\nu = P_{1,\nu_1}\dots P_{q,\nu_q}, \qquad \nu = (\nu_1, \dots, \nu_q),$$
$$x_{(\nu)} = P_\nu x, \qquad X_{(\nu)} = P_\nu X,$$

we find that the following formulae (which are analogous to (1.1) and (1.2)) are true:

$$\sum_{(1)_q\leqslant\zeta\leqslant N} P_\nu = I, \tag{2.2}$$

$$P_\nu P_\mu = \delta_{\mu\nu}P_\nu, \tag{2.3}$$
$$\nu = (\nu_1, \dots, \nu_q),\ \mu = (\mu_1, \dots, \mu_q),\ (1)_q \leqslant \nu, \mu \leqslant N,$$

$$S^\mu P_\nu = \varepsilon^{\mu\nu}P_\nu, \tag{2.4}$$

To prove (2.2) by induction we use the first of the formulae (2.1). We have further

$$P_\nu P_\mu = \Big(\prod_{1\leqslant p\leqslant q} P_{p,\nu_p}\Big)\Big(\prod_{1\leqslant p\leqslant q} P_{p,\mu_p}\Big) = \prod_{1\leqslant p\leqslant q} P_{p,\nu_p}P_{p,\mu_p} = \prod_{1\leqslant p\leqslant q} \delta_{\nu_p,\mu_p}P_{p,\nu_p}$$
$$= \begin{cases} 1 & \text{if } \nu_p = \mu_p\ (p = 1, 2, \dots, q), \\ 0 & \text{otherwise.}\end{cases}$$

Finally,

$$S^\mu P_\nu = S_1^{\mu_1}\dots S_q^{\mu_q}P_{1,\nu_1}\dots P_{q,\nu_q} = (S_1^{\mu_1}P_{1,\nu_1})\dots(S_q^{\mu_q}P_{q,\nu_q})$$
$$= (\varepsilon_1^{\mu_1\nu_1}P_{1,\nu_1})\dots(\varepsilon_q^{\mu_q\nu_q}P_{q,\nu_q}) = \varepsilon_1^{\mu_1\nu_1}\dots\varepsilon_q^{\mu_q\nu_q}P_{1,\nu_1}\dots P_{q,\nu_q} = \varepsilon^{\mu\nu}P_\nu.$$

Let us remark that the space X is a direct sum:

$$X = \bigoplus_{(1)_q\leqslant\nu\leqslant N} X_{(\nu)}, \qquad \text{where } X_{(\nu)} = P_\nu X. \tag{2.5}$$

If a linear operator A acting in X commutes with involutions $S_1, \dots, S_q$ of orders $N_1, \dots, N_q$ respectively, then A commutes also with a multi-involution $S = S_1\dots S_q$ of order $N = (N_1, \dots, N_q)$ and

$$A(D_A \cap X_{(\nu)}) \subset X_{(\nu)} \qquad \text{for } (1)_q \leqslant \nu \leqslant N, \tag{2.6}$$

where $D_A \subset X$ denotes the domain of A. In fact, let $x \in D_A$. Then $x_{(\nu)} = P_\nu x \in X_{(\nu)}$ for $(1)_q \leqslant \nu \leqslant N$ and

$$Ax_{(\nu)} = AP_\nu x = P_\nu(Ax) \in X_\nu$$

since P_ν commute with S. Hence $A(D_A \cap X_{(\nu)}) \subset X_{(\nu)}$.

3. Equations with several involutions. By $A(t)$ we denote an arbitrary polynomial of variables $t = (t_1, \dots, t_q)$, which we shall write further in one of the following manners:

$$A(t) = A(t_1, \dots, t_q) = \sum_{(0)_q\leqslant k\leqslant m} a_{k_1\dots k_q}t_1^{k_1}\dots t_q^{k_q} = \sum_{(0)_q\leqslant k\leqslant m} a_k t^k.$$

Let $S = S_1\dots S_q$ be a multi-involution of order $N = (N_1, \dots, N_q)$. Let

$$A(S) = \sum_{(0)_q\leqslant k\leqslant N-(1)_q} a_k S^k.$$

Then

$$A(S) = \sum_{(1)_q\leqslant\nu\leqslant N} A(\varepsilon^\nu)P_\nu. \tag{3.1}$$

In fact,

$$A(S) = \sum_{(1)_q\leqslant\nu\leqslant N} A(S)P_\nu = \sum_{(1)_q\leqslant\nu\leqslant N}\ \sum_{(0)_q\leqslant k\leqslant N-(1)_q} a_k S^k P_\nu$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N}\ \sum_{(0)_q\leqslant k\leqslant N-(1)_q} a_k \varepsilon^{k\nu}P_\nu = \sum_{(1)_q\leqslant\nu\leqslant N} A(\varepsilon^\nu)P_\nu.$$

This implies that any equation

$$A(S_1, \dots, S_q)x = y \tag{3.2}$$

is equivalent to a system of equations

$$A(\varepsilon^\nu)x_{(\nu)} = y_{(\nu)} \qquad \text{for } (1)_q \leqslant \nu \leqslant N.$$

The number of equations is $N_0 = N_1\dots N_q$.

THEOREM 3.1. *Let X be a linear space. Let*

$$A(S) = \sum_{0\leqslant j\leqslant m} A_j S^{n_j}, \tag{3.3}$$

where

1° *$S = S_1\dots S_q$ and $S_1, \dots, S_q, A_0, \dots, A_m$ are linear operators acting in X;*

2° *$S_pS_r - S_rS_p = 0$, $S_pA_j - A_jS_p = 0$ for $p, r = 1, 2, \dots, q$ and $j = 0, 1, \dots, m$;*

3° *$n_j = (n_{1,j}, \dots, n_{q,j})$, $n_{p,j}$ are non-negative integers and*

$$\sum_{1\leqslant p\leqslant q} n_{p,j} > 0 \qquad (j = 0, 1, \dots, m).$$

Let N_p be a common multiple of numbers $n_{p,j}$ $(j = 0, 1, \dots, m)$ and let us suppose that there is a subspace $\tilde{X} \subset X$ such that S is a multi-involution of order $N = (N_1, \dots, N_q)$ on $\tilde{X}$. Then

$$A(S) = \sum_{(1)_q\leqslant\nu\leqslant N} A(\varepsilon^\nu)P_\nu \qquad \text{on } \tilde{X}, \tag{3.4}$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_q)$, $\varepsilon_p = e^{2\pi i/N_p}$, $P_\nu = P_{1,\nu_1}, \dots, P_{q,\nu_q}$;

$$P_{p,\nu_p} = \frac{1}{N_p}\sum_{k=0}^{N_p-1} \varepsilon_p^{-k\nu_p}S_p^k.$$

Proof. Since S_p and A_j are commutative, we find

$$A(S) = A(S)\sum_{(1)_q\leqslant\nu\leqslant N} P_\nu = \sum_{0\leqslant j\leqslant m} A_j S^{n_j} \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu$$
$$= \sum_{0\leqslant j\leqslant m}\sum_{(1)_q\leqslant\nu\leqslant N} A_j S^{n_j} P_\nu = \sum_{0\leqslant j\leqslant m}\sum_{(1)_q\leqslant\nu\leqslant N} A_j \varepsilon^{\nu n_j} P_\nu$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N}\Big(\sum_{0\leqslant j\leqslant m}\varepsilon^{\nu n_j} A_j\Big) P_\nu = \sum_{(1)_q\leqslant\nu\leqslant N} A(\varepsilon^\nu) P_\nu,$$

which was to be proved.

THEOREM 3.2. *Under the assumptions of Theorem 3.1 the equation*

$$A(S)x = y, \quad y\in\tilde{X}, \tag{3.5}$$

is equivalent to $N_0 = N_1\ldots N_q$ independent equations

$$A(\varepsilon^\nu)x = y_{(\nu)}, \quad (1)_q\leqslant\nu\leqslant N, \tag{3.6}$$

where $y_{(\nu)} = P_\nu y$, and if each of the equations (3.6) has a solution x_ν, then a solution of (3.5) is given by the formula

$$x = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu x_\nu. \tag{3.7}$$

Proof. Since the operator S, as a superposition of operators $S_1, \ldots, S_q$, commutes with A_j ($j = 0, 1, \ldots, m$), every space $X_{(\nu)}$ is preserved by operator $A(\varepsilon^\nu)$. Then equation (3.6) for $y\in\tilde{X}$ can be written in the following manner:

$$0 = A(S)x - y = \sum_{(1)_q\leqslant\nu\leqslant N} A(\varepsilon^\nu)P_\nu x - \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu y$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu\big(A(\varepsilon^\nu)x\big) - \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu^2 y$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu[A(\varepsilon^\nu)x - P_\nu y]$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu[A(\varepsilon^\nu)x - y_{(\nu)}].$$

Since the space X is a direct sum of spaces $X_{(\nu)}$, we infer that equation (3.5) is equivalent to $N_0 = N_1\ldots N_q$ independent equations

$$A(\varepsilon^\nu)x = y_{(\nu)}, \quad (1)_q\leqslant\nu\leqslant N,$$

and if x is a solution of equation (3.5) in $\tilde{X}$, then each of these equations has x as a solution.

Conversely, let us suppose that each of the equations $A(\varepsilon^\nu)x = y_{(\nu)}$ has a solution. Let us denote by x_ν the solution of the ν-th equation. Then, if we write

$$x = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu x_\nu,$$

we obtain

$$A(S)x = A(S)\sum_{(1)_q\leqslant\nu\leqslant N} P_\nu x_\nu = \sum_{(1)_q\leqslant\nu\leqslant N} A(S)P_\nu x_\nu$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N}\Big(\sum_{(1)_q\leqslant\mu\leqslant N} A(\varepsilon^\mu)P_\mu\Big)P_\nu x_\nu$$
$$= \sum_{(1)_q\leqslant\mu\leqslant N} A(\varepsilon^\mu)\sum_{(1)_q\leqslant\nu\leqslant N}(P_\mu P_\nu)x_\nu = \sum_{(1)_q\leqslant\mu\leqslant N} A(\varepsilon^\mu)\Big(\sum_{(1)_q\leqslant\nu\leqslant N}\delta_{\mu\nu}P_\nu x_\nu\Big)$$
$$= \sum_{(1)_q\leqslant\mu\leqslant N} A(\varepsilon^\mu)P_\mu x_\mu = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu\big(A(\varepsilon^\nu)x_\nu\big)$$
$$= \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu y_{(\nu)} = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu^2 y = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu y = y,$$

which proves that x is a solution of equation (3.5).

Let us remark that the second part of Theorem 3.2 can be formulated more strongly:

COROLLARY 3.3. *Under the assumptions of Theorem 3.1, if each of the equations*

$$A(\varepsilon^\nu)x = y, \quad y\in\tilde{X}, \ (1)_q\leqslant\nu\leqslant N, \tag{3.8}$$

has a solution x_ν, then

$$x = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu x_\nu$$

is a solution of the equation $A(S)x = y$ on $\tilde{X}$.

Proof. In the same manner as in the proof of Theorem 3.2, we obtain

$$A(S)x = A(S)\sum_{(1)_q\leqslant\nu\leqslant N} P_\nu x_\nu = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu[A(\varepsilon^\nu)x_\nu].$$

But $A(\varepsilon^\nu)x_\nu = y$, whence

$$A(S)x = \sum_{(1)_q\leqslant\nu\leqslant N} P_\nu y = y,$$

and x is a solution of $A(S)x = y$.

4. Application to partial differential-difference equations with periodic coefficients. Let R^q be a q-dimensional real euclidean space. Let $t = (t_1, \ldots, t_q) \in R^q$, $q \geqslant 1$. As usual, we write

$$D^k x(t) = \frac{\partial^{k_1+\ldots+k_q}}{\partial t_1^{k_1} \ldots \partial t_q^{k_q}} x(t), \quad k = (k_1, \ldots, k_q).$$

Let $\omega_j = (\omega_{1,j}, \ldots, \omega_{q,j})$, $j = 0, 1, \ldots, m$. Let us consider the partial differential-difference equation

$$(4.1) \qquad \sum_{(0)_q \leqslant k \leqslant n} \sum_{0 \leqslant j \leqslant m} A_{k,j}(t) D^k x(t-\omega_j) = y(t).$$

Without loss of generality we can assume that

$$0 = \omega_{p,0} < \omega_{p,1} < \ldots < \omega_{p,m} \quad (p = 1, 2, \ldots, q).$$

We assume also that all numbers $\omega_{p,j}$ are commensurable. This implies that there is a number $r \neq 0$ and there are positive integers $n_{p,j}$ such that

$$(4.2) \qquad \omega_{p,j} = n_{p,j} r \quad \text{for } p = 1, 2, \ldots, q;\ j = 1, 2, \ldots, m;\ n_{p,0} = 0.$$

By n_j we denote the multi-index $n_j = (n_{1,j}, \ldots, n_{q,j})$.

We say that a function $x(t)$ is *ω-periodic* if x is ω_p-periodic with respect to the p-th variable t_p $(p = 1, 2, \ldots, q)$ and $\omega = (\omega_1, \ldots, \omega_q)$. The vector ω will be called the *period* of function x. Obviously, if x is ω-periodic, then for any multi-index $n = (n_1, \ldots, n_q)$

$$x(t-n\omega) = x(t_1-n_1\omega_1, \ldots, t_q-n_q\omega_q) = x(t_1, \ldots, t_q) = x(t).$$

THEOREM 4.1. *Let a real function $y(t)$ determined for $t \in R^q$ be ω_{m+1}-periodic with period $\omega_{m+1} = (\omega_{1,m+1}, \ldots, \omega_{q,m+1})$, where $\omega_{p,m+1}$ are commensurable with real commensurable numbers $\omega_{p,j}$ $(p = 1, 2, \ldots, q;\ j = 1, 2, \ldots, m)$.*

Let r be a common divisor of numbers $\omega_{p,j}$ (not necessarily the greatest one) and let $\tilde{r} = (r)_q = (\underbrace{r, \ldots, r}_{q \text{ times}})$. Let the real functions $A_{k,j}$ determined for $t \in R^q$ be $\tilde{r}$-periodic, $(0)_q \leqslant k \leqslant n$, $0 \leqslant j \leqslant m$. Then equation (4.1) *has $\tilde{\omega}$-periodic solutions belonging to the class C^n if and only if all partial differential equations*

$$(4.3) \qquad A_\nu x = \sum_{(0)_q \leqslant k \leqslant n} b_{k\nu}(t) D^k x(t) = y_{(\nu)}, \quad (1)_q \leqslant \nu \leqslant N,$$

have $\tilde{\omega}$-periodic solutions belonging to the class C^n, where

$$b_{k\nu}(t) = \sum_{0 \leqslant j \leqslant m} \varepsilon^{-\nu n_j} A_{k,j}(t),$$

$$y_{(\nu)} = \frac{1}{N_1 \ldots N_q} \sum_{(0)_q \leqslant k \leqslant N-(1)_q} \varepsilon^{-k\nu} y(t-k\tilde{r}),$$

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_q), \quad \varepsilon_p = e^{2\pi i/N_p}, \quad p = 1, 2, \ldots, q,$$

$$n_j = (n_{1,j}, \ldots, n_{q,j}), \quad \omega_j = n_j \tilde{r} \quad \text{for } j = 0, 1, \ldots, m+1,$$

N_p is a common multiple (not necessarily the smallest one) of numbers $n_{p,1}, \ldots, n_{p,m+1}$ and $N = (N_1, \ldots, N_q)$, $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_q)$, where $\tilde{\omega}_p = N_p \tilde{r}$.

The number of the equations (4.3) *is $N_0 = N_1 \ldots N_q$. The solutions are of the form*

$$x(t) = \frac{1}{N_1 \ldots N_q} \sum_{(1)_q \leqslant \nu \leqslant N} \sum_{(0)_q \leqslant k \leqslant N-(1)_q} \varepsilon^{-k\nu} x_\nu(t-k\tilde{r}),$$

where x_ν is an $\tilde{\omega}$-periodic solution of the ν-th equation (4.3).

Proof. Let us consider the space X of all $\tilde{\omega}$-periodic real functions $x(t)$ determined for $t \in R^q$ with period $\tilde{\omega}$ described above. Let

$$(4.4) \qquad S_p x = x(t_1, \ldots, t_{p-1}, t_p - r, t_{p+1}, \ldots, t_q) \quad \text{for } x \in X,\ p = 1, 2, \ldots, q.$$

Every S_p is a linear operator transforming X onto X and, moreover, S_p is an involution of order N_p. In fact,

$$\begin{aligned} S_p^{N_p} x &= x(t_1, \ldots, t_{p-1}, t_p - N_p r, t_{p+1}, \ldots, t_q) \\ &= x(t_1, \ldots, t_{p-1}, t_p - \tilde{\omega}_p, t_{p+1}, \ldots, t_q) \\ &= x(t_1, \ldots, t_{p-1}, t_p, t_{p+1}, \ldots, t_q) = x(t) \end{aligned}$$

and N_p is the smallest number satisfying (4.4). Let

$$Sx = x(t-\tilde{r}).$$

Then S is a multi-involution of order $N = (N_1, \ldots, N_q)$ because

$$\begin{aligned} S^N x &= S_1^{N_1} \ldots S_q^{N_q} x = x(t_1 - N_1 r, \ldots, t_q - N_q r) \\ &= x(t - \tilde{\omega}_1, \ldots, t_q - \tilde{\omega}_q) = x(t-\tilde{\omega}) = x(t). \end{aligned}$$

Let $a(t)$ be an arbitrary real $\tilde{r}$-periodic function determined on R^q. Then the operator a of multiplication by the function $a(t)$ acting in X is commutative with S:

$$(4.5) \qquad Sa - aS = 0.$$

Indeed,

$$(Sa-aS)x = a(t-r)x(t-\tilde{r})-a(t)x(t-\tilde{r}) = [a(t-\tilde{r})-a(t)]x(t-\tilde{r})$$
$$= [a(t)-a(t)]x(t-\tilde{r}) = 0.$$

Let $\tilde{X}$ be the subspace of all n times differentiable functions belonging to X. The operator S, as a shift-operator, is commutative with the differential operator D^k, $k \leqslant n$. Hence the superposition aD^k of the operator of multiplication by an $\tilde{r}$-periodic function $a(t)$ with the differential operator D^k is also commutative with operator S on $\tilde{X}$:

$$S(aD^k)-aD^k S = 0 \text{ on } \tilde{X}, \qquad k \leqslant n.$$

According to (2.5) we can decompose the space X into a direct sum of spaces $X_{(\nu)}$, where $X_{(\nu)} = P_\nu X$, $P_\nu = P_{1,\nu_1}\dots P_{q,\nu_q}$ and

$$x_{(p,\nu_p)} = P_{p,\nu_p}x = \frac{1}{N_p}\sum_{k_p=0}^{N_p-1} \varepsilon_p^{-k_p\nu_p} x(t_1, \dots, t_{p-1}, t_p-k_p r, t_{p+1}, \dots, t_q),$$

$$x_{(\nu)} = P_\nu x = \frac{1}{N_1\dots N_q}\sum_{(0)_q\leqslant k\leqslant N-(1)_q} \varepsilon^{-k\nu} x(t-k\tilde{r}).$$

Let us write

$$A(S)x = \sum_{(0)_q\leqslant k\leqslant n}\ \sum_{0\leqslant j\leqslant m} A_{k,j}(t)D^k x(t-\omega_j) = \sum_{(0)_q\leqslant k\leqslant n}\ \sum_{0\leqslant j\leqslant m} A_{k,j}(t)D^k S^{n_j}x$$
$$= \sum_{0\leqslant j\leqslant m} A_j S^{n_j}x, \quad \text{where } A_j = \sum_{0\leqslant k\leqslant n} A_{k,j}D^k.$$

Hence equation (4.1) can be written as follows:

(4.6) $$A(S)x = y.$$

Basing ourselves on theorem 3.1, we obtain

(4.7) $$A(S) = \sum_{(1)_q\leqslant \nu\leqslant N} A(\varepsilon^\nu)P_\nu.$$

Let us write

$$A_\nu = A(\varepsilon^\nu) = \sum_{(0)_q\leqslant k\leqslant n} b_{k\nu}D^k,$$

$$b_{k\nu}(t) = \sum_{0\leqslant j\leqslant m} \varepsilon^{-n_j\nu}A_{k,j}(t).$$

Theorem 3.2 implies that equation (4.6) (i.e. (4.1)) is equivalent to $N_0 = N_1\dots N_q$ partial differential equations,

(4.8) $$A_\nu x = y_{(\nu)}, \qquad (1)_q \leqslant \nu \leqslant N,$$

on X. If this system has an $\tilde{\omega}$-periodic solution x_ν, $(1)_q \leqslant \nu \leqslant N$, then the given equation (4.6) has an $\tilde{\omega}$-periodic solution

$$x(t) = \sum_{(1)_q\leqslant \nu\leqslant N} P_\nu x_\nu(t) = \frac{1}{N_1\dots N_q}\sum_{\substack{(0)_q\leqslant k\leqslant N-(1)\\ (1)_q\leqslant \varepsilon\leqslant N}} \varepsilon^{-k\nu}x(t-kr),$$

which was to be proved.

The assumption that $0 < \omega_{p,1} < \dots < \omega_{p,m}$ $(p = 1, \dots, q)$ is not essential. Indeed, if $\omega_{p,j} < 0$, then $\omega_{p,j} = -n_{p,j}r$, where $n_{p,j}$ is a positive integer and

$$x(t_1, \dots, t_{p-1}, t_p-\omega_{p,j}, t_{p+1}, \dots, t_q)$$
$$= x(t_1, \dots, t_{p-1}, t_p+n_{p,j}r, t_{p+1}, \dots, t_q)$$
$$= S^{-n_{p,j}}x(t_1, \dots, t_{p-1}, t_p, t_{p+1}, \dots, t_q).$$

But $S^{-n_{p,j}} = S^{N_p-n_{p,j}}$, which follows from the fact that S_p is an involution of order N_p.

The assumption that all numbers $\omega_{p,j}$ are commensurable is not essential either. It is enough to assume that for any fixed p all numbers $\omega_{p,j}$ are commensurable, and in place of vector $\tilde{r} = (r)_q$ to consider a vector $\tilde{r} = (r_1, \dots, r_q)$, where

$$\omega_{p,j} = n_{p,j}r_p \qquad (p = 1, 2, \dots, q).$$

In the same manner we can consider the case where x and y are vector-functions and $A_{k,j}(t)$ are square matrices of respective orders. This is also true without any essential changes for functions with values in a Banach space, even in a linear metric space.

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Tame singular integrals *

by

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Introduction. Let H be a real separable Hilbert space, B be a one-one Hilbert-Schmidt operator on H, and $y \to T_y$ be the regular representation of the additive group of H acting in $L^p(H)$, $1 < p < \infty$.

In [1] we studied singular integral operators

$$Z_p(f) = \lim_{\substack{\delta \downarrow 0 \\ \varrho \uparrow \infty}} \int_\delta^\varrho \Big[\int_H T_y f a(y/t) dn_{t^2} \circ B^{-1}(y) \Big] dt/t$$

acting on $L^p(H)$, where $\int_H a(y) dn \circ B^{-1}(y) = 0$ and $a(y)$ satisfies an integrability condition with respect to the Gaussian measure $n \circ B^{-1}$. In this note we shall restrict $a(y)$ to be either an absolutely integrable odd function or an r-power integrable even tame function for some $r > 1$. Under these conditions Z_p is a bounded operator on $L^p(H)$ as was shown in [1].

Extension of the results of the present note to the more general functions $a(y)$ used in [1] is a simple matter.

Singular integral operators Z_p generally map tame functions f in $L^p(H)$ to non-tame functions $Z_p(f)$. In this note we shall consider the tame singular integrals (introduced in [1]) which map tame functions to tame functions. Corresponding to each singular integral Z_p there is a net $\{(Z \circ Q^{-1}) \mid Q \in \mathcal{F}\}$ of tame singular integrals determined by the finite-dimensional orthogonal projections $Q \in \mathcal{F}$ on H and this net converges strongly to Z_p as Q tends strongly to the identity through the directed set $\mathcal{F}$. We shall prove this result in this note.

Preliminaries. We refer the reader to papers [3] and [4] of Gross and [5] of Segal for the measure theoretic preliminaries.

Definition (Segal). A *weak distribution* on a real Hilbert space H is an equivalence class F of linear maps from the conjugate space H^*

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