Hence taking into account (7) we get the continuity equation

$$\sum_i \underset{Y_i}{£}\langle j, \omega^i\rangle \underset{\varnothing}{\Delta} = 0.$$

References

[1] N. Bourbaki, *Algèbre*, Paris 1958, Chapter III, p. 45.

[2] J. Komorowski, *A modern version of the E. Noether's theorems in the calculus of variations, I*, Studia Math. 29 (1968), p. 261-273.

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Reçu par la Rédaction le 9. 4. 1968

STUDIA MATHEMATICA, T. XXXII. (1969)

A differentiable structure in the set of all bundle sections over compact subsets

by

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In several branches of mathematics (e.g. in the calculus of variations, mathematical physics etc.) we have to deal with sets of maps, e.g. with families of parametrized curves or, more generally, with k-cubes in a finite-dimensional differentiable manifold.

The case of the set of C^k-maps from a compact Banach C^∞-manifold into a separable Banach C^∞-manifold has been investigated by Eells [1, 2]. He has shown that this set has the structure of a C^∞-manifold modelled on a separable Banach space.

A particular case was worked out by Palais [5]. The construction of the Hilbert manifold of parametrized curves was one of the main items of his general Morse theory.

The above-mentioned results are inadequate for many important problems. For example, in the modern formulation of the classical field theory the states are described by sections of the respective bundles; besides, the compact sections play a fundamental role.

In the present note we prove that the set of compact sections of finite-dimensional differentiable bundle can be naturally equipped with the structure of a differentiable manifold modelled on a Fréchet space. Some other problems of this kind are solved, e.g. a differentiable structure in a set of non-parametrized curves or, more generally, in a set of compact submanifolds; the results will be published in this journal.

We want to emphasize that in the construction of a differentiable structure in such sets there are difficulties which do not occur in "parametrized" cases. The set of homotopic C^k-submanifolds which are boundaries of relatively compact domains in a given finite-dimensional C^∞-manifold has a canonical structure of a topological manifold modelled on a Banach space $C^k(\Omega)$, where Ω is one of those C^k-submanifolds, but the coordinate maps are not differentiable (the formally calculated derivative of a coordinate map contains differential operators; cf. Remark on p. 200). To overcome this difficulty, in the present paper we consider C^∞-submanifolds and we take as a model space the space $C^\infty(\Omega)$ in which differential operators are continuous and which is not a Banach space. We prove that a manifold constructed in this way is of class C^∞ in the sense of the theory of differentiation in Fréchet spaces which are Schwartz spaces [4].

We should like to express our gratitude to Dr. W. Tulczyjew for many profitable conversations. We are also indebted to Professor K. Maurin for his lively interest in our work.

A MANIFOLD OF BORDERS

1. Preliminaries. Let X be a real $(n+1)$-dimensional differentiable C^∞-manifold. An n-dimensional imbedded C^∞-submanifold is a *border* if it is the boundary of a relatively compact domain in X.

The set of all borders splits into the equivalence classes of diffeomorphic borders.

Let $\mathscr{P}$ be one of those classes. In the set $\mathscr{P}$ we shall introduce the topology and the structure of a differentiable manifold modelled on the space $\mathscr{E} \cong \mathscr{E}(\Omega)$, where Ω is any border from $\mathscr{P}$. $\mathscr{E}(\Omega)$ is the space of C^∞-functions on Ω with the topology of uniform convergence of all derivatives.

In the following we shall identify the isomorphic spaces $\mathscr{E}(\Omega)$, where $\Omega \in \mathscr{P}$, by means of a fixed but arbitrarily chosen isomorphism.

2. Topology in $\mathscr{P}$. Let $\Omega \in P$; then a C^∞-(tangent) vector field u defined in a neighbourhood of the set $\Omega \subset X$ is called *transversal* to Ω if $u(p) \notin T_p(\Omega)$ for $p \in \Omega$ ($T_p(\Omega)$ is the tangent space at p).

LEMMA 1. *If $\Omega \in \mathscr{P}$, then there exist a neighbourhood of $\Omega \subset X$ and a vector field on this neighbourhood, transversal to Ω.*

Proof. For every $p \in \Omega$ there exist a neighbourhood $\omega_p \in X$ and a coordinate chart $(\omega_p, \varkappa_p)$ such that

$$(q \in \omega_p \cap \Omega) \Leftrightarrow (\varkappa_p(q) = (q^1, \ldots, q^n, 0) \in \boldsymbol{R}^{n+1}),$$

and that points of ω_p with the negative n-th coordinate belong to the domain of which Ω is the boundary.

Let $\{\omega_{p_i}\}_1^k$ be a finite set of neighbourhoods with the above properties and $\Omega \subset \bigcup_{i=1}^{k} \omega_{p_i} =: \mathcal{O}$.

Let $u_{p_i} \in T(\omega_{p_i})$ be a vector field such that $(0, \ldots, 0, 1)$ are its coordinates in the coordinate chart $(\omega_{p_i}, \varkappa_{p_i})$.

Let $\{\varphi_i\}_1^k$ be a partition of unity, of class C^∞, on $\mathcal{O}$, subordinated to the covering $\{\omega_{p_i}\}_1^k$. We define $u := \sum_{i=1}^{k} \varphi_i u_{p_i}$. If $q \in \Omega$, then $u(q) \in T_q(X)$ is a convex linear combination of vectors $u_{p_i}(q) \notin T_q(\Omega)$ which belong to one half-space of the space $T_q(X)$. Thus $u(q) \notin T_q(\Omega)$ and so the vector field u is transversal to Ω, q.e.d.

Definition. By a *transversal homotopy H through a border $\Omega \in \mathscr{P}$* we mean a C^∞-diffeomorphism

$$\Omega \times]-r, r[\ni (p, t) \to H(p, t) \in X, \quad H(p, 0) = p, \quad r > 0.$$

The set of all transversal homotopies through a border $\Omega \in \mathscr{P}$ is denoted by $\mathfrak{H}(\Omega)$

LEMMA 2. *For every border $\Omega \in \mathscr{P}$ there exists a transversal homotopy through it.*

Proof. Let u be a vector field transversal to Ω and let $t \to H(p, t)$ be the solution of the dynamical system $dx/dt = u$ with the initial condition $x(0) = p$. Since Ω is compact, there exists an $\varepsilon_1 > 0$ such that $H(p, t)$ is defined for $|t| < \varepsilon_1$. It follows from the uniqueness of the solution of the Cauchy problem for dynamical systems that the map H is injective. Besides, we infer from the theory of ordinary differential equations that H is a C^∞-map. For every $p \in \Omega$ the jacobian of H is non-zero. Then the map H is invertible in a neighbourhood $\Omega \times]-\varepsilon_2, \varepsilon_2[\subset \Omega \times]-\varepsilon_1, \varepsilon_1[$ and the inverse map is of class C^∞, q.e.d.

LEMMA 3. *If $\Omega \in \mathscr{P}$, $H \in \mathfrak{H}(\Omega)$ is defined for $|t| < r$, $\varphi \in \mathscr{E}$ and $|\varphi(p)| < r$ for $p \in \Omega$, then*

$$\mathscr{H}(\varphi) := \{H(p, \varphi(p)) \in X : p \in \Omega\} \in \mathscr{P}.$$

Proof. Since the map $\Omega \ni p \to H(p, \varphi(p)) \in \mathscr{H}(\varphi)$ transfers the differentiable structure from Ω to $\mathscr{H}(\varphi)$, $\mathscr{H}(\varphi)$ is an n-dimensional imbedded C^∞-submanifold in X. It can be seen that $\mathscr{H}(\varphi)$ is the boundary of a domain in X, q.e.d.

Thus, if $H \in \mathfrak{H}(\Omega)$, then there exists a domain $\mathscr{U}_H \subset \mathscr{E}(\Omega)$ on which the map H defines the map $\mathscr{H}$ as

$$(1) \qquad U_H \ni \varphi \to \mathscr{H}(\varphi) \in \mathscr{P}.$$

It is easily seen that $\mathscr{H}$ is injective.

THEOREM 1. *If $\Omega_i \in \mathscr{P}$, $H_i \in \mathfrak{H}(\Omega_i)$, $\mathscr{U}_i \subset \mathscr{E}$, $i = 1, 2$, are such that $V := \mathscr{H}_1(\mathscr{U}_1) \cap \mathscr{H}_2(\mathscr{U}_2) \neq \emptyset$, then $\mathscr{H}_1^{-1} \circ \mathscr{H}_2$ is a homeomorphism from $\mathscr{H}_2^{-1}(V) \subset \mathscr{E}$ onto $\mathscr{H}_1^{-1}(V) \subset \mathscr{E}$.*

It follows directly from this theorem that if the sets $\mathscr{H}_i(\mathscr{U}_i)$, $i = 1, 2$, are equipped with the topologies transfered from $\mathscr{E}$ by the injections $\mathscr{H}_i$, $i = 1, 2$, then those topologies are compatible on V. Thus the set $\mathscr{P}$ has the canonical topology given by transversal homotopies and the map $\mathscr{H}$, for $H \in \mathfrak{H}(\Omega)$, $\Omega \in \mathscr{P}$, is a local homeomorphism from $\mathscr{E}$ to $\mathscr{P}$. In other words, $\mathscr{P}$ is a topological manifold modelled on $\mathscr{E}$ (see Remark on p. 200).

Proof of Theorem 1. Let $\Omega := \mathscr{H}_1(\varphi_1) = \mathscr{H}_2(\varphi_2)$; then the homotopies H_i, $i = 1, 2$, define transversal homotopies H_i', $i = 1, 2$, through Ω as follows:

$$H_i'\big(H_i(p, \varphi_i(p)), t\big) := H_i(p, \varphi_i(p)+t), \quad p \in \Omega_i.$$

It is obvious that the topology given in a neighbourhood of the point $\Omega \in \mathscr{P}$ by the injection $\mathscr{H}_i'$ coincides with one given by $\mathscr{H}_i$. Now it is enough to show that the topologies given in a neighbourhood of the point $\Omega \in \mathscr{P}$ by injections $\mathscr{H}_i'$, $i = 1, 2$, coincide. Then we get the theorem directly from the following

LEMMA 4. *If $\Omega \in \mathscr{P}$, $H, G \in \mathfrak{H}(\Omega)$, then there exists a domain $\mathscr{V} \subset \mathscr{E}(\Omega)$ such that the map $\mathscr{H}^{-1} \circ \mathscr{G}$ is a homeomorphism from $\mathscr{V}$ onto $\mathscr{H}^{-1} \circ \mathscr{G}(\mathscr{V}) \subset \mathscr{E}$ ($\mathscr{H}$ and $\mathscr{G}$ are defined by* (1)).

Before the proof of lemma 4 we state a few definitions.

Let $\Omega \in \mathscr{P}$, $H \in \mathfrak{H}(\Omega)$. We define maps

$$H(\Omega \times]-r, r[) \ni x \to \pi_H(x) \in \Omega,$$

$$H(\Omega \times]-r, r[) \ni x \to a_H(x) \in \mathbf{R}^1$$

such that $\big(\pi_H(x), a_H(x)\big) = H^{-1}(x)$, and

$$\Omega \times]-r, r[\ni (p, t) \to \tau(p, t) := \pi_H \circ G(p, t) \in \Omega,$$

$$\Omega \times]-r, r[\ni (p, t) \to \gamma(p, t) := a_H \circ G(p, t) \in \mathbf{R}^1.$$

Obviously, τ and γ are C^∞-maps.

Let $\varphi \in \mathscr{E}(\Omega)$ and $|\varphi(p)| < r$ for $p \in \Omega$; then we have the maps

$$\Omega \ni p \to \tau_\varphi(p) := \tau\big(p, \varphi(p)\big) \in \Omega,$$

$$\Omega \ni p \to \gamma_\varphi(p) := \gamma\big(p, \varphi(p)\big) \in \mathbf{R}^1.$$

The last map will be denoted by two symbols, γ_φ or $\bar{\gamma}(\varphi)$. It can be seen that $\bar{\gamma}(\varphi) \in \mathscr{E}(\Omega)$. Besides we define the map

$$\varphi \to L(\varphi) \in \mathscr{L}\big(\mathscr{E}(\Omega), \mathscr{E}(\Omega)\big)$$

as

$$\mathscr{E}(\Omega) \ni \psi \to L(\varphi)\psi := \psi \circ \tau_\varphi \in \mathscr{E}(\Omega);$$

$\mathscr{L}\big(\mathscr{E}(\Omega), \mathscr{E}(\Omega)\big)$ is the set of linear continuous operators from $\mathscr{E}(\Omega)$ to $\mathscr{E}(\Omega)$.

The continuity of the operator $L(\varphi)$ follows from the fact that τ_φ is a C^∞-map and Ω is compact.

Proof of Lemma 4. The domain $\mathscr{V}$ must be such that $\mathscr{G}(\mathscr{V}) \subset \mathscr{H}(U_H)$. Then the map $S := \mathscr{H}^{-1} \circ \mathscr{G}$ is well defined on $\mathscr{V}$ because of $\mathscr{G}$ and $\mathscr{H}$ are injections. $S(\varphi)$ will also be denoted by S_φ. Let us notice that

$$H\big(\tau_\varphi(p), \gamma_\varphi(p)\big) = G\big(p, \varphi(p)\big), \quad p \in \Omega.$$

S_φ must satisfy the condition

$$\mathscr{H}(S_\varphi) = \mathscr{G}(\varphi). \tag{2}$$

Obviously, if for a given φ there exists a function S_φ satisfying (2), then it is unique. Condition (2) is equivalent to

$$H\big(\tau_\varphi(p), S_\varphi \circ \tau_\varphi(p)\big) = G\big(p, \varphi(p)\big), \quad p \in \Omega,$$

i.e. $S_\varphi \circ \tau_\varphi = \gamma_\varphi$ or

$$L(\varphi) S_\varphi = \gamma_\varphi.$$

Now it is sufficient to show that there exists a domain $\mathscr{V} \subset \mathscr{E}(\Omega)$ such that for every $\varphi \in \mathscr{V}$ the operator $L(\varphi)$ has the inverse operator $L(\varphi)^{-1} \in \mathscr{L}\big(\mathscr{E}(\Omega), \mathscr{E}(\Omega)\big)$, i.e. for every $\varphi \in \mathscr{V}$ the map τ_φ is a diffeomorphism Ω onto Ω. But every point $p \in \Omega$ has neighbourhoods $\mathscr{O}, \mathscr{O}' \subset \Omega$ diffeomorphic to some subsets of $\mathbf{R}^n$ and such that $\tau(\mathscr{O}, t) \subset \mathscr{O}'$ for $|t| < r$. In the following part of the proof we denote by the same symbols the neighbourhoods $\mathscr{O}, \mathscr{O}'$ and their diffeomorphic images in $\mathbf{R}^n$. Analogically, the transfer of τ by the above-mentioned diffeomorphism will also be denoted by τ; in this case $\tau: \mathbf{R}^n \times]-r, r[\to \mathbf{R}^n$. By $t \in \mathbf{R}^1$ we denote the function on Ω identically equal to t. The jacobian of τ_0 is bounded away from zero ($\tau_0 = id$). Then there exists an $\varepsilon > 0$ such that if $|t| < \varepsilon$, then the jacobian of the map τ_t is bounded away from zero. But

$$\tau_\varphi'(p) = \tau_{\varphi(p)}'(p) + \frac{\partial \tau}{\partial t}\big(p, \varphi(p)\big) \cdot \varphi'(p).$$

Let us notice that $\partial\tau/\partial t$ is bounded on $\mathscr{O} \times]-\varepsilon, \varepsilon[$.

Let φ be such that $|\varphi(p)| < \varepsilon$ and

$$\|\varphi'(p)\| \sup_{\substack{q \in \mathscr{O} \\ |t| < \varepsilon}} \left\| \frac{\partial \tau}{\partial t}(q, t) \right\| < \inf_{\substack{q \in \mathscr{O} \\ |t| < \varepsilon}} \|\tau_t'(q)\|,$$

where the norms are taken in $\mathscr{L}(\mathbf{R}^n, \mathbf{R}^1)$, $\mathbf{R}^n$, $\mathscr{L}(\mathbf{R}^n, \mathbf{R}^n)$ respectively. The set of functions $\varphi \in \mathscr{E}(\Omega)$ satisfying the above conditions is denoted by $\mathscr{V}_\mathscr{O}$. Obviously $\mathscr{V}_\mathscr{O}$ is a domain in $\mathscr{E}(\Omega)$.

Since for every $\varphi \in \mathscr{V}_\mathscr{O}$ the jacobian of τ_φ is bounded away from zero, then it follows from the Inverse Function Theorem that τ_φ is a local diffeomorphism $\mathscr{O}$ onto $\tau_\varphi(\mathscr{O})$. We can follow this procedure for a finite covering $\{\mathscr{O}_i\}_1^k$ of the border Ω. Let $\mathscr{V} := \bigcap_{i=1}^{k} \mathscr{V}_{\mathscr{O}_i}$; then for every $\varphi \in \mathscr{V}$ the map τ_φ is a local diffeomorphism Ω into Ω. We shall show that τ_φ is a global diffeomorphism Ω onto Ω.

Let

$$\Omega \times]-\delta, 1+\delta[\ni (p, t) \to \beta(p, t) := \big(\tau_{t\varphi}(p), t\big) \in \Omega \times]-\delta, 1+\delta[.$$

The map β is a local diffeomorphism.

Let us define the function

$$\Omega \times]-\delta, 1+\delta[\ni (p, t) \to m(p, t) \in \boldsymbol{R}^1,$$

where $m(p, t)$ is equal to the number of points $a_i \in \Omega$ such that $\beta(a_i, t) = (p, t)$. Since Ω is compact, the number $m(p, t)$ is finite.

To prove the continuity of m let us take a sequence $(p_k, t_k) \underset{k\to\infty}{\to} (p_0, t_0)$.

Let $m_0 = m(p_0, t_0)$ and let points a_i, $i = 1, \ldots, m_0$, be such that $\beta(a_i, t_0) = (p_0, t_0)$.

There exist neighbourhoods $\mathcal{O}_0 \ni (p_0, t_0)$ and $\mathcal{O}_i \ni (a_i, t_0)$, $i = 1, \ldots, m_0$, such that $\mathcal{O}_0$ is β-diffeomorphic with every $\mathcal{O}_i$, $i = 1, \ldots, m_0$, and $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ for $i \neq j$.

Then for every point $(p, t) \in \mathcal{O}_0$ there are at least m_0 points in Ω which are β-mapped onto (p, t), thus $\lim\limits_{k\to\infty} m(p_k, t_k) \geqslant m_0$.

Let us assume that $\lim\limits_{k\to\infty} m(p_k, t_k) > m_0$; then for every point $(p_k, t_k) \in \mathcal{O}_0$ there exists at least one point (q_k, t_k) such that $(q_k, t_k) \notin \bigcup\limits_{i=1}^{m_0} \mathcal{O}_i$ and $\beta(q_k, t_k) = (p_k, t_k)$. Since Ω is compact, we take a subsequence $(q_{k_l}, t_{k_l}) \underset{l\to\infty}{\to} (q_0, t_0)$. Obviously $(q_0, t_0) \neq (p_i, t_0)$ for $i = 1, \ldots, m_0$. But

$$\beta(q_0, t_0) = \beta\big(\lim_{l\to\infty}(q_{k_l}, t_{k_l})\big)$$

$$= \lim_{l\to\infty} \beta(q_{k_l}, t_{k_l}) = \lim_{l\to\infty}(p_k, t_k) = (p_0, t_0),$$

whence $m(p_0, t_0) \geqslant m_0+1$ (a contradiction). Thus we have proved that $\lim\limits_{k\to\infty} m(p_k, t_k) = m_0$. Hence the function m is continuous.

Since $m(p, 0) \underset{p}{\equiv} 1$, $m(p, t) \underset{(p,t)}{\equiv} 1$ and τ_φ is a bijection. Hence

$$S(\varphi) = L(\varphi)^{-1}\bar{\gamma}(\varphi), \quad \varphi \in \mathscr{V}.$$

The continuity of the map S follows from Theorem 2, q.e.d.

3. Differentiable structure in $\mathscr{P}$. Let E be a Fréchet space of type S (Schwartz space; cf. [3]). The differentiability of a map from E to E and the continuity of its derivative is understood in the sense of [4].

We say that a topological manifold M is a *C^k-manifold modelled on E, with the atlas* $\{(\mathcal{O}_i, \varkappa_i)\}_{i\in I}$, where $\mathcal{O}_i \subset M$, $\bigcup\limits_{i\in I} \mathcal{O}_i = M$, $\varkappa_i: \mathcal{O}_i \to E$ if $\varkappa_i \circ \varkappa_j^{-1}$ is a C^k-diffeomorphism from $\varkappa_j(\mathcal{O}_j \cap \mathcal{O}_i)$ into E.

We shall show that the topological manifold $\mathscr{P}$ is a C^∞-manifold modelled on $\mathscr{E}$.

It is sufficient to prove the following

THEOREM 2. *The map*

$$\mathscr{E} \supset \mathscr{V} \ni \varphi \to S(\varphi) = L(\varphi)^{-1}\bar{\gamma}(\varphi) \in \mathscr{E}$$

is of class C^∞ ($\mathscr{V}$ is the same as in Lemma 4).

Before the proof we state a few lemmas.

LEMMA 5. *Let $\mathscr{L}\big(C^k(\Omega), C^k(\Omega)\big)$ be the Banach space of linear continuous maps from $C^k(\Omega)$ to $C^k(\Omega)$; then the injection*

$$\mathscr{E} \ni \varphi \to A(\varphi) \in \mathscr{L}\big(C^k(\Omega), C^k(\Omega)\big)$$

defined as

$$C^k(\Omega) \ni \psi \to A(\varphi)\psi := \varphi\psi \in C^k(\Omega)$$

is continuous and (because of linearity) differentiable.

The proof is trivial.

Thus we can say the same about the map

$$\mathscr{E} \ni \varphi \to A(\varphi) \in \mathscr{L}_s(\mathscr{E}, \mathscr{E}),$$

defined similarly, where $\mathscr{L}_s(\mathscr{E}, \mathscr{E})$ is the space of linear continuous maps with a simple topology (weak topology).

Let $f: \Omega \times]-r, r[\to \boldsymbol{R}^1$ be a C^∞-function; then we define the map

$$\mathscr{V} \ni \varphi \to \bar{f}(\varphi) \in \mathscr{E}(\Omega)$$

as

$$\Omega \ni p \to \big(\bar{f}(\varphi)\big)(p) := f\big(p, \varphi(p)\big) \in \boldsymbol{R}^1.$$

LEMMA 6. *The map $\bar{\gamma}$ is a C^∞-map and its derivative is given by*

$$\bar{\gamma}'(\varphi) = A\left(\overline{\frac{\partial\gamma}{\partial t}}(\varphi)\right),$$

i.e.

$$\big(\bar{\gamma}'(\varphi)\psi\big)(p) = \psi(p)\frac{\partial\gamma}{\partial t}\big(p, \varphi(p)\big),$$

where $\psi \in \mathscr{E}(\Omega)$, $p \in \Omega$.

For every $\varphi \in \mathscr{V}$ we define the vector field u_φ as

$$\Omega \ni p \to u_\varphi(p) := \frac{\partial\tau}{\partial t}\big(\tau_\varphi^{-1}(p), \varphi\circ\tau_\varphi^{-1}(p)\big) \in T_p(\Omega),$$

i.e. if $\psi \in \mathscr{E}(\Omega)$, then

$$\langle\psi, u_\varphi\rangle\big(\tau_\varphi(p)\big) = \frac{\partial\psi\circ\tau}{\partial t}\big(p, \varphi(p)\big).$$

LEMMA 7. *For every $\varphi_0 \in \mathscr{V}$ there exist a neighbourhood $\mathscr{W} \subset \mathscr{V}$, a finite set of first order differential operators $D_i \in \mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big)$ and C^∞-functions*

$f^i: \Omega\times \mathbf{R}^1 \to \mathbf{R}^1$, $i = 1, \ldots, m$, *such that*

$$\langle \psi, u_\varphi\rangle = \sum_{i=1}^{m} (D_i\psi) L(\varphi)^{-1}\overline{f^i}(\varphi),$$

where $\varphi \in \mathscr{W}$, $\psi \in C^{k+1}(\Omega)$.

LEMMA 8. *The map* $\mathscr{V} \ni \varphi \to K(\varphi) := L(\varphi)^{-1} \in \mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big)$ *is continuous.*

COROLLARY 1. *The map* $\mathscr{V} \ni \varphi \to K(\varphi) \in \mathscr{L}_s(\mathscr{E}, \mathscr{E})$ *is continuous.*

COROLLARY 2. *The map*

$$\mathscr{V}\times\mathscr{V} \ni (\varphi, \psi) \to K(\varphi)\bar{f}(\psi) \in \mathscr{E}$$

is continuous for every $f \in \mathscr{E}(\Omega\times]-r, r[)$.

Proof. By virtue of Lemmas 6 and 8 we have

$$\begin{aligned}\|K(\varphi+\Delta\varphi)\bar{f}(\psi+\Delta\psi)-K(\varphi)\bar{f}(\psi)\|_k \\ \leqslant \|K(\varphi+\Delta\varphi)\|_{k,k+1}\|\bar{f}(\psi+\Delta\psi)-\bar{f}(\psi)\|_{k+1}+ \\ +\|K(\varphi+\Delta\varphi)-K(\varphi)\|_{k,k+1}\|\bar{f}(\psi)\|_{k+1} \underset{\substack{\Delta\varphi\to 0\\ \Delta\psi\to 0}}{\longrightarrow} 0.\end{aligned}$$

where $\|\ \|_j$ denotes one of the equivalent norms in the space $C^j(\Omega)$, q.e.d.

LEMMA 9. *The map K defined in Lemma 8 is differentiable and*

$$\big(K'(\varphi)\chi\big)\psi = -\big(K(\varphi)\chi\big)\langle K(\varphi)\psi, u_\varphi\rangle,$$

where $\chi, \psi \in \mathscr{E}$ (cf. (5)).

Proof of Theorem 2. We divide the proof into three parts:

1° the map S is differentiable;

2° the map $K: \mathscr{V} \to \mathscr{L}_s(\mathscr{E}, \mathscr{E})$ is of class C^1;

3° the map S is of class C^∞.

Ad 1°. Let us define the map

$$\mathscr{V}\times\mathscr{V} \ni (\varphi, \psi) \to F(\varphi, \psi) := L(\varphi)^{-1}\bar{\gamma}(\psi) \in \mathscr{E}.$$

It follows from lemmas 6 and 9 that F is partially differentiable. If we take $f = \partial\gamma/\partial t$ in Corollary 2, then we infer that the partial derivative of F with respect to the second variable is continuous. Thus F is differentiable on $\mathscr{V}\times\mathscr{V}$ (cf. [4]). Let

$$\mathscr{E} \ni \varphi \to P(\varphi) := (\varphi, \varphi) \in \mathscr{E}\times\mathscr{E}.$$

Then $S = F\circ P$ is differentiable.

Ad 2°. In the same way as in the first part of the present proof we can show that the map $\mathscr{V} \ni \varphi \to L(\varphi)^{-1}\overline{f^i}(\varphi) \in \mathscr{E}$ (cf. Lemma 7) is differentiable, and thus it is continuous.

From Lemma 5 we infer that the map

$$\mathscr{V} \ni \varphi \to A\big(K(\varphi)\overline{f^i}(\varphi)\big) \in \mathscr{L}\big(C^k(\Omega), C^k(\Omega)\big) \tag{3}$$

is continuous. Thus from Lemma 7 follows the continuity of the map

$$\mathscr{V} \ni \varphi \to u_\varphi \in \mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big).$$

Besides, from Lemma 8, we see that the map

$$\mathscr{V} \ni \varphi \to K(\varphi)\psi \in C^k(\Omega) \tag{4}$$

is continuous for $k \geqslant 1$, $\psi \in C^{k+1}(\Omega)$.

Now from (3) and (4) we get the continuity of the map

$$\mathscr{V} \ni \varphi \to \Phi(\varphi) := \big(K(\varphi)\chi, K(\varphi)\psi, u_\varphi\big) \in C^k(\Omega)\times C^{k+1}(\Omega)\times\mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big)$$

for any given $\chi, \psi \in \mathscr{E}$.

Moreover, the map

$$\begin{aligned}C^k(\Omega)\times C^{k+1}(\Omega)\times\mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big) \ni (g, h, u) \\ \to \Psi(g, h, u) := A(g)uh \in C^k(\Omega)\end{aligned}$$

is linear and partially continuous, and thus continuous (cf. Mazur-Orlicz Theorem). Thus, recalling the form of the derivative of K (see Lemma 9), we conclude that for any given $\chi, \psi \in \mathscr{E}$, $k \geqslant 1$ the map

$$\mathscr{V} \ni \varphi \to \Psi\circ\Phi(\varphi) = \big(K'(\varphi)\chi\big)\psi \in C^k(\Omega)$$

is continuous.

Thus we have obtained the continuity of the map

$$\mathscr{V} \ni \varphi \to \big(K'(\varphi)\chi\big)\psi \in \mathscr{E},$$

where $\chi, \psi \in \mathscr{E}$.

Hence, at last, we have proved that the map

$$\mathscr{V} \ni \varphi \to K'(\varphi) \in \mathscr{L}_s\big(\mathscr{E}, \mathscr{L}_s(\mathscr{E}, \mathscr{E})\big) \tag{5}$$

is continuous. $\mathscr{L}_s\big(\mathscr{E}, \mathscr{L}_s(\mathscr{E}, \mathscr{E})\big) \cong \mathscr{L}_s(\mathscr{E}, \mathscr{E}; \mathscr{E})$.

Ad 3°. Let $\mathscr{A}_r$, $r = 0, 1, \ldots$, be the set of all maps from $\mathscr{E}$ into $\mathscr{L}_s(\underbrace{\mathscr{E}, \ldots, \mathscr{E}}_{r\ \text{times}}; \mathscr{E})$ ($\mathscr{A}_0$ is the set of maps from $\mathscr{E}$ into $\mathscr{E}$) which are superpositions of the map K, any maps of the form $\bar{f}$ (where $f \in \mathscr{E}(\Omega\times]-r, r[)$) and any linear continuous maps. It follows from Lemma 6, Lemma 9 and the first and second parts of the present proof that if $a \in \mathscr{A}_r$, then a is a C^1-map and $a' \in \mathscr{A}_{r+1}$. Thus for every $r \geqslant 0$ if $a \in \mathscr{A}_r$, then a is a C^∞-map.

Since $S \in \mathscr{A}_0$, S is a C^∞-map, q.e.d.

It immediately follows from 1° that

$$S'(\varphi)\chi = \big(F'(P(\varphi))\circ P'(\varphi)\big)\chi$$
$$= -\big(K(\varphi)\chi\big)\langle K(\varphi)\bar{\gamma}(\varphi), u_\varphi\rangle + K(\varphi)A\left(\overline{\frac{\partial\gamma}{\partial t}}(\varphi)\right)\chi,$$

where $\varphi\in\mathscr{V}$, $\chi\in\mathscr{E}$.

Remark. One can notice that the map $K:\mathscr{E}\to\mathscr{L}\big(C^k(\Omega), C^k(\Omega)\big)$ is not continuous. However, the map $S: C^k(\Omega)\to C^k(\Omega)$ is continuous. The formally calculated derivative of S contains differential operators which do not preserve $C^k(\Omega)$. Thus the above construction of the manifold $\mathscr{P}$ of borders does not work if X is a C^k-manifold, $k<\infty$. In this case $\mathscr{P}$ is not even a C^1-manifold; however, it is a topological manifold.

4. Proofs of the lemmas.

Proof of Lemma 6. Let

$$r_\varphi(\psi) := \bar{\gamma}(\varphi+\psi) - \bar{\gamma}(\varphi) - \psi\overline{\frac{\partial\gamma}{\partial t}}(\varphi).$$

From Taylor's formula for a function of one real variable we have

$$\gamma\big(p, (\varphi+\psi)(p)\big) = \gamma\big(p, \varphi(p)\big) + \psi(p)\frac{\partial\gamma}{\partial t}\big(p, \varphi(p)\big) + \psi^2(p)\varrho\big(p, \psi(p)\big),$$

where

$$\varrho(p, t) := \int_0^1 (1-s)\frac{\partial^2\gamma}{\partial t^2}\big(p, \varphi(p)+st\big)\,ds.$$

It follows from the theorem on the differentiability of integrals with a parameter that ϱ is a C^∞-function. Hence $r_\varphi(\psi) = \psi^2\bar{\varrho}(\psi)$. We shall show that

$$\bigwedge_{i\geqslant 0}\bigvee_{j\geqslant 0}\frac{\|r_\varphi(\psi)\|_i}{\|\psi\|_j}\xrightarrow[\psi\to 0]{} 0, \tag{6}$$

where by $\|\psi\|_k$ we denote the supremum of moduli of all partial derivatives, at most of order k, counted with help of some fixed covering of Ω by coordinate charts.

But any partial derivative of order k of the function $r_\varphi(\psi)$ can be expressed as a sum of products of: (1) at least two derivatives (of order less than k) of the function ψ; (2) derivatives of the function ϱ. Since all derivatives of the function ϱ are bounded on the set $\{(p,t)\in\Omega\times\boldsymbol{R}^1: |\varphi(p)+t|<r-\varepsilon\}$, we have

$$\|r_\varphi(\psi)\|_k \leqslant \text{const}\,\|\psi\|_k^2.$$

Thus if $j=i$, then (6) occurs. Hence $\bar{\gamma}$ is differentiable. $\partial\gamma/\partial t$ is a C^∞-map, and so the map

$$\mathscr{E}\ni\varphi\to\overline{\frac{\partial\gamma}{\partial t}}(\varphi)\in\mathscr{E}$$

is differentiable. Hence it is continuous.

We have

$$\bar{\gamma}' = A\circ\overline{\frac{\partial\gamma}{\partial t}}$$

and therefore $\bar{\gamma}$ is a C^1-map.

By induction we can prove that $\bar{\gamma}$ is a C^∞-map, q.e.d.

Proof of Lemma 7. Let $\varphi_0\in\mathscr{V}$ be given. Since Ω is compact, there exist: (1) $\delta>0$; (2) a covering $\{(\mathscr{O}'_\mu, \varkappa_\mu)\}_1^s$ by coordinate charts; (3) a compact covering $\{K_\mu\}_1^s$; (4) an open covering $\{\mathscr{O}_\mu\}_1^s$ such that $\tau_\varphi(\mathscr{O}_\mu)\subset K_\mu\subset\mathscr{O}'_\mu$ if $\sup|(\varphi-\varphi_0)(\Omega)|<\delta$. Let $\{\eta_\mu\}_1^s$ be a partition of unity subordinated to the covering $\{\mathscr{O}_\mu\}_1^s$. If $p\in\mathscr{O}_\mu$, $|t-\varphi_0(p)|<\delta$, then $\tau(p,t)\in\mathscr{O}'_\mu$. Let $\tau^j(p,t)$, $j=1,\ldots,n$, be coordinates of the point $\tau(p,t)$ in the coordinate chart $(\mathscr{O}'_\mu, \varkappa_\mu)$.

Let us state

$$f_\mu^j(p,t) := \begin{cases}\eta^\mu(p)\dfrac{\partial\tau^j}{\partial t}(p,t), & p\in\mathscr{O}_\mu,\\ 0, & p\notin\mathscr{O}_\mu.\end{cases}$$

Let $\zeta_\mu\in\mathscr{E}$ be such that $\zeta_\mu(p)\underset{K_\mu}{\equiv}1$ and $\operatorname{supp}\zeta_\mu\subset\mathscr{O}'_\mu$. Let $\psi\in\mathscr{E}$. We define

$$(D_\mu^j\psi)(q) := \begin{cases}\zeta_\mu(q)\dfrac{\partial\psi}{\partial p^j}(q), & q\in\mathscr{O}'_\mu,\\ 0, & q\notin\mathscr{O}'_\mu.\end{cases}$$

Now let $\varphi\in\mathscr{V}$ be such that $\sup|(\varphi-\varphi_0)(\Omega)|<\delta$. Moreover, let $p\in\mathscr{O}_\mu$ and $q=\tau_\varphi(p)$. Then

$$\langle\psi, u_\varphi\rangle(q) = \sum_{j=1}^n\frac{\partial\psi}{\partial p^j}(q)\frac{\partial\tau^j}{\partial t}\big(p,\varphi(p)\big)$$
$$= \sum_{j=1}^n\sum_{\mu=1}^s\zeta_\mu(q)\frac{\partial\psi}{\partial p^j}(q)\eta_\mu(p)\frac{\partial\tau^j}{\partial t}\big(p,\varphi(p)\big) = \sum_{j=1}^n\sum_{\mu=1}^s\big((D_\mu^j\psi)L(\varphi)^{-1}\bar{f}_\mu^j(\varphi)\big)(q),$$

q.e.d.

Proof of Lemma 8. Since τ_φ is a C^∞-diffeomorphism,

$$L(\varphi), K(\varphi)\in\mathscr{L}\big(C^k(\Omega), C^k(\Omega)\big)\subset\mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big),\quad \text{where } k\geqslant 0.$$

Let $\varphi_0 \in \mathscr{E}$; then we can see, as in the previous lemma, that there exist: (1) a neighbourhood $\mathscr{W}$ of φ_0; (2) a covering $\{(\mathcal{O}'_\mu, \varkappa_\mu)\}_1^s$ of Ω by coordinate charts; (3) an open covering $\{\mathcal{O}_\mu\}_1^s$ such that if $\varphi \in \mathscr{W}$, then $\tau_\varphi^{-1}(\bar{\mathcal{O}}_\mu) \subset \mathcal{O}'_\mu$ and the jacobian of the map τ_φ is bounded away from zero. If $q \in \mathcal{O}_\mu$, then by $h^j_\varphi(q)$ we denote the coordinates of the point $\tau_\varphi^{-1}(q) \in \mathcal{O}'_\mu$ in the coordinate chart $(\mathcal{O}'_\mu, \varkappa_\mu)$. It is obvious that if $\mathscr{E} \ni \varphi \to \to \varphi_0 \in \mathscr{E}$, then the functions h^j_φ converge to $h^j_{\varphi_0}$ uniformly with all derivatives. But $\big(K(\varphi)\psi\big)(q) = \psi \circ \tau_\varphi^{-1}(q)$, and so the k'-th derivative of the function $K(\varphi)\psi$, $k' \leqslant k$, can be expressed by the product of: (1) terms of the form $(D^\alpha \psi)\big(\tau_\varphi^{-1}(\cdot)\big)$, $|\alpha| \leqslant k'$; (2) derivatives of functions h^j_φ.

The elementary Mean Value Theorem implies that for $|\alpha| \leqslant k'$

$$\sup_{\|\psi\|_{k+1} \leqslant 1} \sup_{q \in \bar{\mathcal{O}}_\mu} |(D^\alpha \psi)\big(\tau_\varphi^{-1}(q)\big) - (D^\alpha \psi)\big(\tau_{\varphi_0}^{-1}(q)\big)| \underset{\varphi \to \varphi_0}{\longrightarrow} 0. \tag{7}$$

Let us notice that the above difference can be expressed by: (1) the first order derivatives of the function $D^\alpha \psi$ which are bounded if $\|\psi\|_{k+1} \leqslant 1$; (2) the terms $h^j_\varphi(q) - h^j_{\varphi_0}(q)$ which uniformly converge to zero. It immediately follows from (7) that if $\|\psi\|_{k+1} \leqslant 1$ then all k'-th derivatives of the function $K(\varphi)\psi$, $k' \leqslant k$, converge: (1) uniformly with respect to ψ; (2) uniformly on $\bar{\mathcal{O}}_\mu$. In other words,

$$\mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big) \ni K(\varphi) \underset{\varphi \to \varphi_0}{\longrightarrow} K(\varphi_0) \in \mathscr{L}\big(C^{k+1}(\Omega), C^k(\Omega)\big), \quad \text{q.e.d.}$$

Proof of Lemma 9. At first we prove that the map

$$\mathscr{V} \ni \varphi \to L(\varphi) \in \mathscr{L}_s(\mathscr{E}, \mathscr{E})$$

is differentiable and

$$\big(L'(\varphi)\chi\big)\psi = \chi L(\varphi)\langle \psi, u_\varphi \rangle. \tag{8}$$

Let

$$r_\varphi(\chi) := L(\varphi + \chi) - L(\varphi) - L'(\varphi)\chi,$$

where $L'(\varphi)$ is defined by (8). Taylor's formula gives

$$\big(r_\varphi(\chi)\psi\big)(p) = \psi \circ \tau\big(p, \varphi(p) + \chi(p)\big) - \psi \circ \tau\big(p, \varphi(p)\big) - \chi(p)\, \frac{\partial(\psi \circ \tau)}{\partial t}\big(p, \varphi(p)\big)$$

$$= \chi^2(p) \int_0^1 (1-s) \frac{\partial^2(\psi \circ \tau)}{\partial t^2}\big(p, \varphi(p) + s\chi(p)\big)\, ds.$$

For any fixed $\psi \in \mathscr{E}$ the partial derivatives of the function given by the above integral are bounded. We can show, as in the proof of Lemma 6, that

$$\frac{\|r_\varphi(\chi)\psi\|_k}{\|\chi\|_k} \underset{\|\chi\|_k \to 0}{\longrightarrow} 0.$$

Thus we have proved that for any given $\psi \in \mathscr{E}$ the map

$$\mathscr{V} \ni \varphi \to L(\varphi)\psi \in \mathscr{E}$$

is differentiable. Obviously it is equivalent to the differentiability of the map

$$\mathscr{V} \ni \varphi \to L(\varphi) \in \mathscr{L}_s(\mathscr{E}, \mathscr{E}).$$

Now we pass on to the gist of the proof;

$$\begin{aligned} L(\varphi+\chi)^{-1}\psi - L(\varphi)^{-1}\psi &= L(\varphi+\chi)^{-1} \circ \big(L(\varphi) - L(\varphi+\chi)\big) \circ L(\varphi)^{-1}\psi \\ &= -L(\varphi+\chi)^{-1} \circ \big(L'(\varphi)\chi + r_\varphi(\chi)\big) \circ L(\varphi)^{-1}\psi \\ &= -\big(L(\varphi+\chi)^{-1} - L(\varphi)^{-1}\big) \circ \big(L'(\varphi)\chi\big) \circ L(\varphi)^{-1}\psi - \qquad (9) \\ &\quad -\big(L(\varphi+\chi)^{-1} - L(\varphi)^{-1}\big) \circ r_\varphi(\chi) \circ L(\varphi)^{-1}\psi - \\ &\quad -L(\varphi)^{-1} \circ \big(L'(\varphi)\chi\big) \circ L(\varphi)^{-1}\psi - L(\varphi)^{-1} \circ r_\varphi(\chi) \circ L(\varphi)^{-1}\psi. \qquad (10) \end{aligned}$$

Thus we have got a decomposition into two parts: one linear with respect to χ (the first summand in (10)) and the rest which is a remainder in the sense of the definition of differentiability.

For example, let us consider the term (9)

$$\begin{aligned} &\frac{\big\|\big(L(\varphi+\chi)^{-1} - L(\varphi)^{-1}\big) \circ \big(L'(\varphi)\chi\big) \circ L(\varphi)^{-1}\psi\big\|_k}{\|\chi\|_{k+1}} \\ &\quad \leqslant \|L(\varphi+\chi)^{-1} - L(\varphi)^{-1}\|_{k,k+1} \|L(\varphi)\langle L(\varphi)^{-1}\psi, u_\varphi\rangle\|_{k+1} \\ &\quad \leqslant C\|L(\varphi+\chi)^{-1} - L(\varphi)^{-1}\|_{k,k+1} \underset{\chi \to 0}{\longrightarrow} 0. \end{aligned}$$

Applying (8) to (10) we get the lemma, q.e.d.

A MANIFOLD OF SECTIONS OF A BUNDLE

1. Topology in the set of sections. Let M be a finite-dimensional differentiable bundle with a base $\boldsymbol{X}$ ($\dim \boldsymbol{X} = m$). Let Γ be a set of C^∞-sections of the bundle M such that if $v \in \Gamma$, then the domain of v is a compact m-dimensional submanifold with the boundary and this boundary is a border. From now on by "domain" we mean a submanifold of $\boldsymbol{X}$ with the above properties. As was done for the set $\mathscr{P}$, we shall introduce in Γ the structure of a differentiable manifold.

At first we will introduce a differentiable structure in the set of domains. Since a deformation of the boundary of a domain does not contain full information about the deformation of the domain, we state the following

Definition. By a *dragging of a domain $D \subset \boldsymbol{X}$ along a transversal homotopy $H \in \mathfrak{H}(\partial D)$* we mean a map

$$\mathscr{E}(\partial D) \supset \mathscr{V} \ni \varphi \to \sigma_\varphi \in \mathscr{E}(\boldsymbol{X}, \boldsymbol{X}),$$

where $\mathscr{V}$ is a neighbourhood of zero and σ satisfies the following conditions:

1° σ_φ is a C^∞-diffeomorphism;

2° the map $\sigma: \mathscr{V} \to \mathscr{E}(\boldsymbol{X}, \boldsymbol{X})$ is continuous if $\mathscr{E}(\boldsymbol{X}, \boldsymbol{X})$ is equipped with the topology of almost uniform convergence with all derivatives;

3° if $x \in \partial D$, then $\sigma_\varphi(x) = H(x, \varphi(x))$.

Lemma 10. *For every domain D and a transversal homotopy $H \in \mathfrak{H}(\partial D)$ there exists a dragging of D along H.*

Proof. The homotopy H defines in a neighbourhood of the set ∂D the coordinate system (p, t), where $p \in \partial D, t \in]-r, r[$. Suppose we are given a $\zeta \in C_0^\infty(\boldsymbol{R}^1)$ such that $\operatorname{supp}\zeta \subset]-r, r[$, $\zeta \geqslant 0$, $\zeta(0) = 1$.

Let

$$c := \sup_{t \in \boldsymbol{R}^1} \left| \frac{d\zeta}{dt}(t) \right|.$$

Now let us take

$$\mathscr{V} := \left\{ \varphi \in \mathscr{E}(\partial D) : |\varphi| \leqslant \frac{1}{2c} \right\}.$$

We set

$$\sigma_\varphi\big(H(p, t)\big) := H\big(p, t + \zeta(t)\varphi(p)\big) \quad \text{if } (p, t) \in \partial D \times]-r, r[,$$

$$\sigma_\varphi(x) := x \quad \text{if } x \notin H(\partial D \times]-r, r[).$$

The map σ_φ is bijective because

$$\frac{d}{dt}\big(t + \zeta(t)\varphi(p)\big) = 1 + \frac{d\zeta}{dt}(t)\varphi(p) \geqslant \frac{1}{2}.$$

Obviously, conditions 1°, 2° and 3° are satisfied, q. e. d.

Let $v \in \Gamma$; then its domain (resp. range) is denoted by D_v (resp. R_v). Since v is a C^∞-section on the set which is not open, we have (from the definition of differentiability on non-open sets) that there exists an extension $\tilde{v}$ of the section v onto a neighbourhood $\mathscr{O}$ of the domain D_v.

Let N_v (resp. $N_{\tilde{v}}$) be the vector bundle over R_v (resp. $R_{\tilde{v}}$) consisting of all vertical tangent vectors at points of the set R_v (resp. $R_{\tilde{v}}$). Obviously the bundle N_v is a restriction of the bundle $N_{\tilde{v}}$ to $R_v \subset R_{\tilde{v}}$.

It is easily seen that there exists a C^∞-diffeomorphism Φ from a neighbourhood of the range of zero section in the bundle $N_{\tilde{v}}$ onto a neighbourhood of the set $R_{\tilde{v}}$ in M (for example see [1]).

Let us choose $H \in \mathfrak{H}(\partial D_v)$, a dragging σ of D_v along H and a (linear) connection P in the bundle $N_{\tilde{v}}$. Let P_φ be the homomorphism of the bundle N_v into $N_{\tilde{v}}$ such that it is the parallel displacement of a fibre over $y \in R_v$ to the point $\sigma_\varphi(y)$ along the curve $[0, 1] \ni t \to \sigma_{t\varphi}(y) \in R_{\tilde{v}}$.

Let $\Gamma(R_v, N_v)$ be the vector space of all global sections of the bundle N_v. We equip it with the topology of uniform convergence of all derivatives.

Let us define the map

$$\mathscr{E} \times \Gamma(R_v, N_v) \supset \mathscr{V} \times V \ni (\varphi, \xi) \to \varkappa(\varphi, \xi) \in \Gamma,$$

where

$$\varkappa(\varphi, \xi) := \Phi \circ P_\varphi \circ \xi \circ v \circ \sigma_\varphi^{-1}$$

on $\sigma_\varphi(D) \subset \boldsymbol{X}$.

If for every $v \in \Gamma$ we take all systems $(H, \sigma, \tilde{v}, \Phi, P)$, then the maps $\varkappa$ defined by them equip Γ with the inductive topology. As in the case of the manifold $\mathscr{P}$ (cf. Preliminaries) we identify in some (not necessarily canonical) way the isomorphic spaces $\Gamma(R_v, N_v)$, where v's are homotopic sections of the bundle M.

Thus these spaces are isomorphic to a fixed topological vector space, which will be denoted by $\mathscr{C}$.

2. A differentiable structure in a manifold of sections. It is easily seen that the map $\varkappa$ is an injection. We shall show that every connected subspace of the topological space Γ which consists of homotopic sections of the bundle M is a C^∞-manifold modelled on the space $\mathscr{E} \times \mathscr{C}$, and that coordinate charts are given by the maps $\varkappa$. Since now we shall be interested in one of the above-mentioned connected subspaces of Γ, it will also be denoted by Γ.

We recall that a *coordinate chart* in a neighbourhood of a point $v \in \Gamma$ is given by the following elements:

H — transversal homotopy through the border ∂D_v;

σ — dragging of D_v along H;

$\tilde{v}$ — extension of the section v,

Φ — C^∞-diffeomorphism from a neighbourhood of the range of zero section in the bundle $N_{\tilde{v}}$ onto a neighbourhood of the set $R_{\tilde{v}}$ in M;

P — connection in $N_{\tilde{v}}$;

$I_1: \mathscr{E} \to \mathscr{E}(\partial D_v)$ — isomorphism;

$I_2: \mathscr{C} \to \Gamma(R_v, N_v)$ — isomorphism.

Suppose we are given two (homotopic) sections $v, w \in \Gamma$ and two systems $(H, \sigma, \tilde{v}, \Phi, P)$, $(G, \varrho, \tilde{w}, \Psi, Q)$.

Let $\varkappa, \lambda$ be the maps defined by these systems respectively,

$$\varkappa: \mathscr{V} \times V \to \Gamma, \qquad \lambda: \mathscr{W} \times W \to \Gamma,$$

where $\mathscr{V}, \mathscr{W} \subset \mathscr{E}$ and $V, W \subset \mathscr{C}$.

Let $z = \varkappa(\varphi, \xi) = \lambda(\psi, \eta) \in \Gamma$.

It follows from the proceeding chapter that $\varphi = S(\psi)$, where $S: \mathscr{V} \to \mathscr{E}$ is a C^∞-diffeomorphism and S depends on H and G only.

Let

$$\varkappa^{-1} \circ \lambda(\psi, \eta) = (\varphi, \xi) =: \big(S(\psi), T(\psi, \eta)\big).$$

LEMMA 11. *The map $T: \mathscr{W}\times W \to \mathscr{C}$ is of class C^∞.*

THEOREM 3. *The map $\varkappa^{-1}\circ\lambda$ is a C^∞-diffeomorphism. Thus Γ is a C^∞-manifold modelled on $\mathscr{E}\times\mathscr{C}$.*

Proof. We get from Lemma 11 that the map $\varkappa^{-1}\circ\lambda$ is of class C^∞. Since our problem is symmetric with respect to $\varkappa$ and λ, the map $\varkappa^{-1}\circ\lambda$ is a C^∞-diffeomorphism, q.e.d.

Proof of Lemma 11. Let $x \in D_z$; then

$$z(x) = \Phi\circ P_\varphi\circ\xi\circ v\circ\sigma_\varphi^{-1}(x) = \Psi\circ Q_\psi\circ\eta\circ w\circ\varrho_\psi^{-1}(x).$$

Let $v\circ\sigma_\varphi^{-1}(x) =: y \in R_v$; then $x = \sigma_\varphi\circ\pi(y)$, where π is the projection in the bundle M. Thus

$$\xi(y) = P_\varphi^{-1}\circ\Phi^{-1}\circ\Psi\circ Q_\psi\circ\eta\big(w\circ\varrho_\psi^{-1}\circ\sigma_\varphi\circ\pi(y)\big).$$

Let $\tau_\psi := w\circ\varrho_\psi^{-1}\circ\sigma_\varphi\circ\pi$, where $\varphi = S(\psi)$.

The map τ_ψ is a C^∞-diffeomorphism from R_v onto R_w. (The map $\tilde{w}\circ\varrho_\psi^{-1}\circ\sigma_v\circ\pi$ is a C^∞-differentiable extension of the map τ_ψ onto a neighbourhood of R_v in the set $R_{\tilde{v}}$.)

If we have in $C^\infty(R_v, R_w)$ the topology of uniform convergence with all derivatives, then the map

$$\mathscr{E}\supset\mathscr{W}\ni\psi\to\tau_\psi\in C^\infty(R_v, R_w)$$

is continuous.

Let

$$\Theta_\psi := P_\varphi^{-1}\circ\Phi^{-1}\circ\Psi\circ Q_\psi,$$

where $\varphi = S(\psi)$; then Θ_ψ is a diffeomorphism between some neighbourhoods of zero sections of the bundles $N_{\tilde{w}}$ and $N_{\tilde{v}}$, and a vector over a point $\tau_\psi(y)\in R_{\tilde{w}}$ is mapped into the fibre over $y\in R_{\tilde{v}}$.

It can be seen that the map

$$\mathscr{E}\supset\mathscr{W}\ni\psi\to\Theta_\psi\in C^\infty_{\mathrm{loc}}(N_{\tilde{w}}, N_{\tilde{v}})$$

is continuous if we have in C^∞_{loc} the topology of almost uniform convergence with all derivatives.

Thus

$$\xi = T(\psi,\eta) = \Theta_\psi\circ\eta\circ\tau_\psi. \tag{11}$$

Now let us assume that the bundles $N_{\tilde{w}}$ and $N_{\tilde{v}}$ are trivial, and let us take some global coordinate systems in $N_{\tilde{w}}$ and $N_{\tilde{v}}$. Their points will be denoted as $(y, a)\in R\times\boldsymbol{R}^k$, where R is one of $R_{\tilde{w}}$ and $R_{\tilde{v}}$, k is a dimension of the standard fibre of M.

Since now every map which can be transferred by the coordinate systems chosen above will be denoted by the same symbol before and after the transfer.

Let $\gamma: R_{\tilde{v}}\times\boldsymbol{R}^k\to\boldsymbol{R}^k$ be a C^∞-map such that

$$\Theta_\psi(y, a) = \big(\tau_\psi^{-1}(y), \gamma(y, a)\big),$$

where $\Theta_\psi: R_{\tilde{w}}\times\boldsymbol{R}^k\to R_{\tilde{v}}\times\boldsymbol{R}^k$.

In addition we define $\xi^\circ: R_{\tilde{v}}\to\boldsymbol{R}^k$, $\eta^\circ: R_{\tilde{w}}\to\boldsymbol{R}^k$ such that

$$\xi(y) = \big(y, \xi^\circ(y)\big)\in R_{\tilde{v}}\times\boldsymbol{R}^k,$$
$$\eta(y') = \big(y', \eta^\circ(y')\big)\in R_{\tilde{w}}\times\boldsymbol{R}^k$$

Then from (11) we have

$$\xi^\circ(y) = \gamma\big(\tau_\psi(y), \eta^\circ\circ\tau_\psi(y)\big). \tag{12}$$

Let $\bar{\gamma}: C^\infty(R_{\tilde{w}}, \boldsymbol{R}^k)\to C^\infty(R_{\tilde{v}}, \boldsymbol{R}^k)$ be defined as

$$\big(\bar{\gamma}(f)\big)(y) := \gamma\big(y, f(y)\big),$$

and the map

$$\mathscr{E}\supset\mathscr{W}\ni\psi\to K(\psi)\in\mathscr{L}\big(C^\infty(R_{\tilde{w}}, \boldsymbol{R}^k), C^\infty(R_{\tilde{v}}, \boldsymbol{R}^k)\big)$$

as

$$C^\infty(R_{\tilde{w}}, \boldsymbol{R}^k)\ni f\to K(\psi)f := f\circ\tau_\psi\in C^\infty(R_{\tilde{v}}, \boldsymbol{R}^k).$$

Now from (12) we get

$$\xi^\circ = \bar{\gamma}(\eta^\circ)\circ\tau_\psi = K(\psi)\bar{\gamma}(\eta^\circ).$$

The similarity of the notation in the two chapters is not accidental. It is justified because the proof of the C^∞-differentiability of the map

$$(\psi, \eta^\circ)\to K(\psi)\bar{\gamma}(\eta^\circ)\in C^\infty(R_{\tilde{v}}, \boldsymbol{R}^k)$$

proceeds like the analogical proof for S (see the proof of Theorem 2). Thus T is a C^∞-map.

If the bundles $N_{\tilde{w}}$ and $N_{\tilde{v}}$ are not trivial, then we proceed as at the beginning of Lemma 7, q.e.d.

References

[1] J. Eells, Jr., *On the geometry of function spaces*, Symp. Inter. de Topologie Alg., Mexico 1956 (1958), p. 303-308.

[2] — *A setting for global analysis*, Bull. Amer. Math. Soc. 72 (1966), p. 751-807.

[3] A. Grothendieck, *Sur les espaces $\mathscr{F}$ et $\mathscr{DF}$*, Summa Brasil. Math. 3 (1954), p. 57-123.

[4] J. Kijowski and W. Szczyrba, *On differentiability in an important class of locally convex spaces*, Studia Math. 30 (1968), p. 247-257.

[5] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology 2 (1963), p. 299-340.

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Reçu par la Rédaction le 9. 4. 1968