quence (f_n) weakly converges to zero. Hence, by [3], p. 156, there is a sequence of blocks (z_k) where

$$z_k = \sum_{i=m(k-1)+1}^{m(k)} c_i^{(k)} x_i, \quad 0 = m(0) < m(1) < m(2) < \ldots,$$

which is equivalent to a subsequence (f_{n_k}). Thus, by the well-known property of the unit vector basis in l_β, the sequence (z_k) is equivalent to the unit vector basis in l_β. Since for $1 < a < \beta < 2$ the space L_β does not have complemented subspaces isomorphic to l_β (cf. [12]), Lemmas 4 and 5 imply that there is no unconditional basis in L_a having a subbasis equivalent to (z_k).

This example answers in the negative a question of Ivan Singer.

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Additive functionals on Orlicz spaces

by

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This paper is concerned with obtaining integral representations of a class of non-linear functionals on Orlicz spaces. These functionals are known as additive functionals and their representation has been studied in Martin and Mizel [6], Mizel and Sundaresan [7]. For the importance of this class of functionals in generalized random processes we refer to Gel'fand and Vilenkin [2]. Further the representation theorems obtained here are of intrinsic interest and provide generalizations of results established in Halmos [3], Bartle and Joichi [1] and Krasnosel'skii [4].

We start with few definitions, remarks and establish a theorem useful in subsequent discussion.

Throughout this paper (T, Σ, μ) is a complete non-atomic totally σ-finite positive measure space. Φ (with or without a suffix) denotes a continuous non-zero Young function. L_Φ denotes the Banach space of real-valued measurable functions f on T such that for a positive number K (depending on f) $M(kf) = \int_T \Phi(k|f|)\,d\mu < \infty$ equipped with the norm

$$\|f\| = \inf\left\{\frac{1}{\xi} \mid \xi > 0,\ M(\xi f) \leqslant 1\right\}.$$

For a detailed discussion of this class of Banach spaces and for the undefined terms in this paper we refer to Luxemburg [5].

Next we proceed to define additive functionals. Throughout the rest of the paper $\int f d\mu$ denotes the definite integral $\int_T f d\mu$.

Definition. Let $\mathscr{F}$ be a linear space of measurable functions on a measure space (T, Σ, μ). A real-valued function F on $\mathscr{F}$ is said to be *additive* if (1) $F(x+y) = F(x)+F(y)$ for $x, y \in \mathscr{F}$ such that $\mu\{t \mid x(t)y(t) \neq 0\} = 0$ and (2) $F(x) = F(y)$ if x, y are equimeasurable functions in $\mathscr{F}$, i.e. $\mu(x^{-1}(B)) = \mu(y^{-1}(B))$ for all Borel sets B in R, the real line.

Remark 1. If x, y are integrable equimeasurable functions, it is verified that $\int x d\mu = \int y d\mu$ and further if f is a Borel measurable function on R, $f(x)$ and $f(y)$ are equimeasurable. It is also verified that for an additive functional F, $F(0) = 0$.

The problem of representing additive functionals on L_∞ has been studied in [6] under the additional assumption of continuity with respect to bounded a.e. convergence. The case of additive functionals on L_p $(p \geqslant 1)$ continuous with respect to various types of convergence in these spaces has been discussed in [7]. In this paper it is proposed to obtain integral representations of continuous additive functionals on Orlicz spaces L_Φ when Φ satisfies the growth conditions G_1 or G_2 (defined below) according as $\mu(T) < \infty$ or $\mu(T) = \infty$ respectively.

G_1: There exist positive numbers U_1 and k such that $\Phi(2U) \leqslant k\Phi(U)$ for $U \geqslant U_1$.

G_2: There exists a positive number k such that $\Phi(2U) \leqslant k\Phi(U)$ for $U \geqslant 0$.

The following theorem is a generalization of results established for the case of L_p-spaces in [1] and [3] to Orlicz spaces.

THEOREM 1. *If f is a continuous function on $R \to R$, then $x \in L_{\Phi_1}$ implies $f(x) \in L_{\Phi_2}$ if and only if for each positive number k there exist positive numbers λ_k and r_k such that*

(a) $\Phi_2(|\lambda_k f(r)|) \leqslant \Phi_1(|kr|)$ *for* $|r| \geqslant r_k$ *if* $\mu(T) < \infty$;

(b) $\Phi_2(|\lambda_k f(r)|) \leqslant \Phi_1(kr)$ *for* $r \geqslant 0$ *if* $\mu(T) = \infty$.

Proof. Let $\mu(T) < \infty$ and Φ_i $(i = 1, 2)$ satisfy condition (a). Let $x \in L_{\Phi_1}$. Thus there exists a positive number k such that $\int \Phi_1(Kx)\,d\mu < \infty$. Let $P = \{t \mid t \in T,\ |x(t)| > r_k\}$ and $P' = T \sim P$. Since Φ_2 and f are continuous, the function $\Phi_2(|\lambda_k f(x)|)$ is bounded on P' and hence is integrable on P'. This together with condition (a) yield

$$\int \Phi_2(|\lambda_k f(x)|)\,d\mu \leqslant \int_{P'} \Phi_2(|\lambda_k f(x)|)\,d\mu + \int_{P} \Phi_1(|kf(x)|)\,d\mu < \infty.$$

Thus $f(x) \in L_{\Phi_2}$.

If condition (a) is false, there exists a positive number k and a sequence of real numbers r_n such that $|r_n| \uparrow \infty$ and

$$\Phi_2\left(\frac{1}{2^{2n}}|f(r_n)|\right) > \Phi_1(k|r_n|).$$

Since Φ_2 is convex and $\Phi_2(0) = 0$, it follows that

$$\Phi_2\left(\frac{1}{2^n}|f(r_n)|\right) > 2^n\Phi_1(k|r_n|).$$

Further, Φ_1 is a non-zero increasing function and there exists an integer m such that $2^m\Phi_1(k|r_m|) > 2$. The non-atomicity of the measure space guarantees the existence of a sequence of mutually disjoint measurable sets $\{T_n\}_{n \geqslant m}$ such that

$$\mu(T_n) = \frac{\mu(T)}{2^n\Phi_1(k|r_n|)} \quad (n \geqslant m).$$

Let x be a function on T defined by $x(t) = r_n$ if $t \in T_n$, $n \geqslant m$ and $x(t) = 0$ if $t \notin \bigcup_{n \geqslant m} T_n$. Thus x is measurable and it is verified that

$$\int \Phi_1(k|x|)\,d\mu = \sum_{n \geqslant m} \frac{\mu(T)}{2^n} < \infty.$$

We verify that $f(x) \notin L_{\Phi_2}$. Let $c > 0$ and m_1, m_2 be two positive integers such that $1/2^{m_2} < c$ and $m_1 > \operatorname{Max}(m, m_2)$. With this choice of m_1 and the choice of $\{T_n\}_{n \geqslant m}$ it is verified that

$$\int \Phi_2(c|f(x)|)\,d\mu > \sum_{n \geqslant m} 2^n \int_{T_n} \Phi_1(k|r_n|)\,d\mu = \infty,.$$

Thus $f(x) \notin L\Phi_2$, completing the proof of the theorem for the case $\mu(T) < \infty$.

Next let $\mu(T) = \infty$. Clearly condition (b) on Φ_i $(i = 1, 2)$ implies $f(x) \in L_{\Phi_2}$.

Conversely, suppose that $x \in L_{\Phi_1}$ implies $f(x) \in L_{\Phi_2}$. We proceed to verify condition (b). Let $r_0 = \sup \Phi_1^{-1}(0)$. Claim that (b) holds if $|r| \leqslant r_0/k$.

Case 1. Let $\Phi_2(t) > 0$ if $t > 0$. If $|r| \leqslant r_0/k$ it is verified that $r\varkappa_T \in L_{\Phi_1}$. Thus $f(r\varkappa_T) \in L_{\Phi_2}$. Since $\Phi_2(t) > 0$ for $t > 0$ and $\mu(T) = \infty$, it follows that $f(r) = 0$. Thus with $\lambda_k = 1$, $\Phi_2(\lambda_k|f(r)|) = 0 \leqslant \Phi_1(k|r|)$.

Case 2. Suppose there exists a $t > 0$ such that $\Phi_2(t) = 0$. Let $r_1 = \sup \Phi_2^{-1}(0)$ and $r_2 = \sup |f(r)|$ on the closed interval $[-r_0/k, r_0/k]$. Thus if $|r| \leqslant r_0/k$ let $p > 0$ be such that $pr_2 < r_1$. For such a p

$$\Phi_2(p|f(r)|) \leqslant \Phi_2(|pr_2|) \leqslant \Phi_2(|r_1|) = 0 \leqslant \Phi_1(k|r|).$$

Thus inequality (b) holds for $|r| \leqslant r_0/k$. If (b) fails to hold for $|r| > r_0/k$, then there exists a sequence of real numbers r_n such that $|r_n| > r_0/k$ and

$$\Phi_2\left(\frac{1}{2^n}|f(r_n)|\right) > 2^n\Phi_1(k|r_n|).$$

Let $\{T_n\}_{n \geqslant 1}$ be a sequence of pairwise disjoint measurable sets such that $\mu(T_n) = 1/2^n\Phi_1(k_{r_n})$. Let x be the function on T such that $x(t) = r_n$ if $t \in T_n$ and $x(t) = 0$ if $t \notin \bigcup_{n \geqslant 1} T_n$. Thus $\int \Phi_1(k|x|)\,d\mu = 1$ and $x \in L_{\Phi_1}$.

However, for any real number $c > 0$ if m is an integer such that $1/2^m < c$, then

$$\int \Phi_2(c|f(x)|)\,d\mu \geqslant \sum_{n\geqslant m} \int_{T_n} \Phi_2(c|f(r_n)|)\,d\mu.$$

Since $\int_{T_n} \Phi_2(c|f(r_n)|)\,d\mu > 1$ for $n \geqslant m$ by our choice of r_n, it follows that $\Phi_2(c|f(x)|)$ is not μ-summable. Thus $f(x) \notin L_{\Phi_2}$. Hence f satisfies inequality (b).

Remark 2. If Φ_1 satisfies conditions G_1 or G_2, then conditions (a) and (b) of the theorem are respectively equivalent to (a′) and (b′) stated below:

(a′) $\qquad \Phi_2(\lambda|f(r)|) \leqslant \Phi_1(|r|) \quad$ for $r \geqslant r_1 \geqslant 0$;

(b′) $\qquad \Phi_2(\lambda|f(r)|) \leqslant \Phi_1(|r|) \quad$ for all real numbers r.

Indeed, with $k = 1$ it is verified that (a) ⇒ (a′). Next suppose Φ_1 satisfies the condition G_1, say, $\Phi_1(2r) \leqslant c\Phi_1(r)$ for $r \geqslant r_0$, where c could be assumed to be $\geqslant 1$. Let $k > 0$. Let m be an integer such that $1/2^m < k$. Then for $|r| \geqslant \operatorname{Max}(2^n r_1, r_0)$ it is verified that

$$\Phi_2(\lambda|f(r)|) \leqslant \Phi_1(|r|) \leqslant c^n \Phi_1\left(\left|\frac{r}{2^n}\right|\right) \leqslant c^n \Phi_1(kr).$$

Thus

$$\Phi_2\left(\frac{\lambda}{c^n}|f(r)|\right) \leqslant \frac{1}{c^n}\Phi_2(\lambda|f(r)|) \leqslant \Phi_1(k|r|)$$

if $|r| \geqslant \operatorname{Max}(2^n r_1, r_0)$ completing the proof (a′) ⇒ (a). The proof (b) ⇒ (b′) is similar and details are omitted.

Before proceeding to the representation theorems we state a lemma established in [5], p. 13, in a form suitable for our purpose.

Definition 2. An element $f \in L_\Phi$ is said to be of *absolutely continuous norm* if (1) given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $E \in \Sigma$, $\mu(E) < \delta$ implies $\|f\chi_E\| < \varepsilon$ and (2) if $\{E_m\}_{m\geqslant 1}$ is a sequence of measurable sets converging to a set of measure 0, then $\|f\chi_{E_m}\| \to 0$. A sequence $\{f_n\}_{n\geqslant 1}$ in L_Φ is said to be of *uniformly absolutely continuous norm* if for a given $\varepsilon > 0$ (1) above holds for all $f = f_n$ for the same $\delta > 0$ and if $\{E_m\}_{m\geqslant 1}$ is a sequence of measurable sets as in (2) above, then $\|f_n\chi_{E_m}\| \to 0$ uniformly in n as $m \to \infty$. If every element of L_Φ is of absolutely continuous norm, then L_Φ is said to be of *absolutely continuous norm*. It might be noted that L_Φ is of absolutely continuous norm if and only if for $x \in L_\Phi$, $\Phi(|x|)$ is μ-summable ([5], p. 58). It is also known ([5], Theorem 3, p. 58) that if (T, Σ, μ) is as in the introduction, then L_Φ is of absolutely continuous norm if and only if (1) $\mu(T) < \infty$ and Φ satisfies condition G_1 and (2) $\mu(T) = \infty$ and Φ satisfies condition G_2.

The analogue of Vitali's theorem providing a criterion for convergence in L_p-spaces ($p \geqslant 1$) is known for L_Φ-spaces of absolutely continuous norm ([5], Lemma 2, p. 13). The following is a corollary of this criterion and is stated for completeness:

LEMMA 1. *If L_Φ is of absolutely continuous norm, then if a sequence $\{f_n\}_{n\geqslant 1}$ in L_Φ converges to f_1, then (1) $f_n \to f$ in measure on sets of finite measure and (2) $\{\Phi(|f_n|)\}$ are of uniformly absolutely continuous L_1-norms.*

Proof. Suppose $f_n \to f$ in L_Φ-norm. Then by the criterion referred to above $f_n \to f$ in measure on measurable sets of finite measure and $\{f_n\}$ are of uniformly absolutely continuous norms. Thus if $\varepsilon > 0$, there exists a $\delta > 0$ such that $E \in \Sigma$, $\mu(E) < \delta$ implies $\int \Phi(|f_n|)\chi_E\,d\mu = \int \Phi(|f_n\chi_E|)\,d\mu \leqslant \|f_n\chi_E\| < \varepsilon$, since ε might be assumed to be less than 1. Similarly, it is verified that if $\{E_m\}_{m\geqslant 1}$ is a sequence of measurable sets converging to a set of measure 0, then $\int \Phi(|f_n|)\chi_{E_m}\,d\mu \to 0$ uniformly as $m \to \infty$.

LEMMA 2. *A sequence x_n converges to x in L_Φ if and only if $\int \Phi(k|x_n - x|)\,d\mu \to 0$ as $n \to \infty$ for every $k > 0$.*

For a proof we refer to [5], Theorem 1, p. 45.

THEOREM 2. *Let Φ be a Young function satisfying the growth condition G_1 and (T, Σ, μ) be as in the introduction, $\mu(T) < \infty$. A functional F on L_Φ is continuous and additive if and only if there exists a continuous function $f: R \to R$ such that (1) $f(0) = 0$, (2) there exist positive numbers a and r_0 such that $|f(r)| \leqslant a\Phi(|r|)$ if $|r| \leqslant r_0$, (3) $F(x) = \int f(x)\,d\mu$ for all $x \in L_\Phi$. Such a representing function f is unique.*

Proof. Let F be a continuous additive functional on L_Φ. Since $\mu(T) < \infty$, it is verified that $L_\infty \subset L_\Phi$. Hence by [5], Theorem 4, p. 51, it follows that $\|x\|_\Phi \leqslant A\|x\|_\infty$ for all $x \in L_\infty$ and for some constant $A > 0$. Thus if $\{x_n\}_{n\geqslant 1}$ is a sequence in L_∞ converging to $x \in L_\infty$ boundedly a.e., since $\Phi(0) = 0$ and Φ is continuous, it follows that $\Phi(k|x_n - x|) \to 0$ boundedly a.e. for every $k > 0$. Thus by lemma 2, $\|x_n - x\| \to 0$. Since F is continuous, $F(x_n) \to F(x)$. Hence by Theorem 1 in [6] it follows that there exists a unique continuous function $f: R \to R$ with $f(0) = 0$ such that for all $x \in L_\infty$, $F(x) = \int f(x)\,d\mu$.

Next we establish that the function f satisfies the growth condition (2) in the theorem. Since Φ satisfies the condition G_1, there exist two positive numbers U_1, c such that $\Phi(2U) \leqslant c\Phi(U)$ for $U \geqslant U_1$. If f does not satisfy condition (2), there exists a sequence r_n such that $|r_n| \uparrow \infty$ and $|f(r_n)| > 2^n\Phi(|r_n|)$, where $|r_n| \geqslant U_1$. Further, since Φ is a non-zero increasing function, there exists an integer m such that $2^m\Phi(|r_m|) \geqslant 2$. Since the measure space is non-atomic, there exists a sequence of measurable sets $\{E_n\}_{n\geqslant m}$ such that $\mu(E_n) = \mu(T)/|f(r_n)|$. We verify that $\|r_n\chi_{E_n}\| \to 0$ as $n \to \infty$. Let k be a real number $k > 0$ and p be a positive integer such that $k \leqslant 2^p$. Thus, since $|r_n| \geqslant U_1$,

$$\int \Phi(k|r_n\chi_{E_n}|)\,d\mu \leqslant \int \Phi(2^p|r_n\chi_{E_n}|)\,d\mu \leqslant c^p \int \Phi(|r_n\chi_{E_n}|)\,d\mu \leqslant \frac{c^p\mu(T)}{2^n}$$

for $n \geqslant m$ as a consequence of condition G_1 and our choice of $\{E_n\}_{n\geqslant m}$.

Thus $\int \Phi(k|r_n\chi_{E_n}|)\,d\mu \to 0$ for $k > 0$. Hence $\|r_n\chi_{E_n}\| \to 0$. But $F(r_n\chi_{E_n}) = \int f(r_n\chi_{E_n})\,d\mu = \pm 1\mu(T)$, a contradiction on the continuity of F since $F(0) = 0$. Hence f satisfies condition (2).

Now to complete the proof it is enough to prove that for all $x \in L_\Phi$, $F(x) = \int f(x)\,d\mu$. Let $x \in L_\Phi$. Since $\mu(T) < \infty$, from the remark on p. 55 of [7] we conclude that L_∞ is a dense subspace. Thus there exists a sequence $x_n \in L_\infty$ such that $\|x_n - x\|_\Phi \to 0$. We claim that $\int f(x_n)\,d\mu \to \int f(x)\,d\mu$ as $n \to \infty$. Since Φ satisfies the condition G_1, L_Φ is of absolutely continuous norm. Thus since $\|x_n - x\|_\Phi \to 0$ by lemma 1, it follows that (1) $x_n \to x$ in measure on sets of finite measure and (2) $\{\Phi(|x_m|)\}$ are of uniformly absolutely continuous L_1-norms. Since f is continuous, (1) above implies that (*) $f(x_n) \to f(x)$ in measure on sets of finite measure. Further (2) implies that if $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that if $E \in \Sigma$ and $\mu(E) < \delta_1$, then $\int \Phi(|x_m\chi_E|)\,d\mu < \varepsilon/2a$ (a as in condition (2) of the theorem). Now with r_0 as in the theorem let $k = \sup_{|r|\leqslant r_0} |f(r)|$. Let $0 < \delta < \operatorname{Min}(\varepsilon/2k\mu(T), \delta_1)$ if $k \neq 0$ and $0 < \delta < \delta_1$ if $k = 0$. If $F \in \Sigma$ is such that $\mu(F) < \delta$, then from condition (2) of the theorem it follows that

$$\int_F |f(x_n)|\,d\mu \leqslant k\mu(F) + a\int_{F_1} \Phi(|x_n|)\,d\mu,$$

where $F_1 = \{t \mid t \in F,\ |x_n(t)| \geqslant r_0\}$. Thus (**) $\int_F |f(x)_n|\,d\mu < \varepsilon$ if $\mu(F) < \delta$.

Next if E_m is a sequence of measurable sets converging to a set of measure 0 with k as before from condition (2) of the theorem it is verified that

$$\int f(x_n\chi_{Em})\,d\mu \leqslant k\int_{E_m} d\mu + \int \Phi(|x_n\chi_{E_m}|)\,d\mu.$$

Since $\Phi(|x_n|)$ are of uniformly absolutely continuous norms, the second integral $\to 0$ uniformly (in n) as $m \to \infty$. Thus (***) $\int |f(x_n\chi_{E_m})|\,d\mu \to 0$ uniformly as $m \to \infty$. The statements (*) (**) and (***) imply that $\int f(x_n)\,d\mu \to \int f(x)\,d\mu$. Hence from the continuity of F it follows that

$$F(x) = \lim F(x_n) = \lim \int f(x_n)\,d\mu = \int f(x)\,d\mu.$$

It is verified that such a representing function f is unique by evaluating $F(r\chi_T)$ for real numbers r.

Conversely, if f is a continuous real-valued function on R satisfying conditions (1) and (2), then by remark 2 it is verified that $x \in L_\Phi \Rightarrow f(x) \in L_1$.

Thus if F is defined on L_Φ by setting $F(x) = \int f(x)\,d\mu$, then F is verified to be additive on L_Φ. Further, by arguing as in the preceding paragraph it is found that F is continuous.

THEOREM 3. *If (T, Σ, μ) is as in the introduction and $\mu(T) = \infty$ and Φ satisfies condition G_2, then F is a continuous additive functional on L_Φ if and only if there exists a continuous real-valued function f on R such that (1) $f(0) = 0$, (2) there exists a constant $a > 0$ such that $|f(r)| \leqslant a\Phi(|r|)$ for all $r \in R$ and (3) $F(x) = \int f(x)\,d\mu$ for all $x \in L_\Phi$. Such a representing function f is unique.*

Proof. Let F be a continuous additive functional on L_Φ. For $B \in \Sigma$, $0 < \mu(B) < \infty$, let us define a functional F_B on $L(T, \Sigma, \mu_B)$, where μ_B is the contraction of μ to B by setting $F_B(y) = F(y\chi_B)$. It is verified that F_B is a continuous additive functional on $L_\Phi(\mu_B)$. Hence by theorem 2 there exists a unique continuous function f on $R \to R$ satisfying conditions (1), (2) and (3) of the preceeding theorem representing F_B. We claim that the function f representing F_B is independent of B. For if $C \in \Sigma$, $0 < \mu(C) < \infty$ and $\mu(C) \leqslant \mu(B)$, by the nonatomicity of the measure space there exists a set $B_1 \in \Sigma$, $B_1 \subset B$ such that $\mu(C) = \mu(B_1)$. If g represents F_C, then since for each real number, r, $r\chi_C$ and $r\chi_{B_1}$ are equimeasurable $F_C(r\chi_C) = F(r\chi_C) = F(r\chi_{B_1}) = F_B(r\chi_{B_1})$. Thus $\int f(r\chi_{B_1})\,d\mu = \int g(r\chi_C)\,d\mu_C$. Hence $f(r) = g(r)$.

With f chosen as above we note that if $x \in L_\Phi$ and $\mu(S(x)) < \infty$, where $S(x)$ is the support of x, then $F(x) = F_{S(x)}(x) = \int_{S(x)} f(x)\,d\mu = \int f(x)\,d\mu$ since $f(0) = 0$. Next we verify that f satisfies condition (2) in the theorem. Since Φ is a non-zero Young function satisfying condition G_2, it is verified that $\Phi(r) \neq 0$ for $r > 0$. Hence to show that f fulfills condition (2) it is enough to show that

$$\text{(i)} \qquad \overline{\lim_{r\to\infty}} \frac{f(r)}{\Phi(|r|)} < \infty,$$

since f is already known to verify condition (2) of the preceding theorem. Suppose (i) is false. Then there exists a sequence $\{r_n\}$ of real numbers such that $r_n \to 0$ and $|f(r_n)| > 2^n\Phi(|r_n|)$. Let $\{B_n\}_{n\geqslant 1}$ be a sequence of measurable sets such that $\mu(B_n) = 1/|f(r_n)|$. Proceeding as in the paragraph 2 of the proof of the preceding theorem it is verified that $\|r_n\chi_{B_n}\| \to 0$ as $n \to \infty$. However, $F(r_n\chi_{B_n}) = \int f(r_n\chi_{B_n})\,d\mu = \pm 1$, contradicting the continuity of F since $F(0) = 0$. Hence f satisfies condition (2) in the theorem.

The proof of this part is complete if it is shown that for all $x \in L_\Phi$, $F(x) = \int f(x)\,d\mu$. If $x \in L_\Phi$, let $E_n = \{t \mid t \in T,\ |x(t)| \geqslant 1/n\}$. Thus if $x_n = x\chi_{E_n}$, then $|x_n| \leqslant |x|$ and $x_n \to x$ pointwise. Further, since $x \in L_\Phi$ and Φ satisfies G_2, $\Phi(|x|)$ is summable. Hence $F(x_n) = F_{E_n}(x_n) = \int f(x_n)\,d\mu$.

Also since f verifies condition (2) of the theorem,

$$|f(x_n)| \leqslant a\Phi(|x_n|) \leqslant a\Phi(|x|),$$

and since $f(x_n) \to f(x)$ a.e., it follows by Lebesgue theorem on dominated convergence that

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int f(x_n)\,d\mu = \int f(x)\,d\mu,$$

thus completing the representation of F. The uniqueness of f is verified as in theorem 2.

Conversely, if f is a real-valued continuous function on R satisfying conditions (1) and (2) of the theorem, then from Remark 2 it follows that the functional $F(x) = \int f(x)\,d\mu$ is well defined on L_Φ and is additive. Next we verify that F is continuous. Let x_n be a sequence in L_Φ converging to x. Thus by lemma 1 since f is continuous, $f(x_n) \to f(x)$ converges in measure on sets of finite measure and further the inequality $\int f(x_n \chi_E)\,d\mu \leqslant a \int \Phi(|x_n \chi_E|)\,d\mu$ implies that $\{f(x_n)\}_{n\geqslant 1}$ are of uniformly absolutely continuous L_1-norms. Hence $\int f(x_n)\,d\mu \to \int f(x)\,d\mu$. Thus $F(x_n) \to F(x)$.

In conclusion it might be mentioned that the problem of representing additive functionals on Orlicz spaces L_Φ, when the space is not of absolutely continuous norm, is not considered here and it is conjectured that non-trivial continuous additive functionals do not exist in such spaces.

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Banach spaces of functions satisfying a modulus of continuity condition*

by

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1. Introduction and terminology. A function $\beta\colon [0,\infty) \to [0,\infty]$ will be called a *modulus of continuity* if it is monotone increasing, continuous at zero, and zero at zero only. Note that it need not be subadditive. For pseudometric spaces (X, d) and (Y, e), a function $f\colon (X, d) \to (Y, e)$ will be said *to satisfy a modulus of continuity condition* β (locally) if there is some positive real M (and some positive real ε) such that $e(f(x), f(y)) \leqslant Md(x, y)$ (whenever $d(x, y) < \varepsilon$) for all x and y in X. Obviously, such a function is uniformly continuous.

Let F denote the real or complex numbers with the usual metric. For a pseudometric space (X, d), let $\mathrm{Lip}(X, \beta\circ d)$ be the set of bounded F-valued functions on X which satisfy a modulus of continuity condition β locally. When $\beta(t) = t$, we will denote the set by $\mathrm{Lip}(X, d)$. If only one metric is being considered on X, we will denote $\mathrm{Lip}(X, \beta\circ d)$ by $\mathrm{Lip}(X, \beta)$. It is known that if β is subadditive (so that $\beta\circ d$ is a pseudometric) and the functions satisfy the modulus of continuity condition β globally, then $\mathrm{Lip}(X, \beta\circ d)$ is a Banach space with a natural norm [4].

Let (X, d), (X, d') and (Y, e) be pseudometric spaces. If there exist $M, \varepsilon > 0$ such that $d(x, y) \leqslant Md'(x, y)$ whenever $d'(x, y) < \varepsilon$, we indicate it by writing $d \ll d'$. Then to say that $f\colon (X, d) \to (Y, e)$ satisfies a local Lipschitz condition can be denoted $e\circ f_2 \ll d$, where $f_2(x, y) = (f(x), f(y))$. If $d \ll d'$ and $d' \ll d$, we say that d and d' are *strongly equivalent* (in contrast to topologically or uniformly equivalent) and denote it by $d \approx d'$.

We attempt to describe how the various spaces $\mathrm{Lip}(X, \beta\circ d)$ are related, if one considers different pseudometrics on X or different moduli of continuity. In the first section, we give a natural norm for $\mathrm{Lip}(X, \beta\circ d)$, under which it is a Banach space. Then we show that $\mathrm{Lip}(X, d)$ is con-

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