boundary problem with Cauchy's initial conditions. In fact, that problem can be considered as the limit case of ours, viz., when the thickness ε of I tends to 0. Note that, if we wish to admit, as solutions, functions f which are not of class C^m, then the Cauchy conditions become inapplicable. Still our formulation allows to extend the problem onto solutions f which are arbitrary distributions (Theorems 5 and 6).

The problem of extending solutions of (1) was also considered by Łojasiewicz in [1], but the purpose of his extension lemmas was quite different. It can be noted that neither Theorems 2, 3, 5 and 6 of the present paper nor the particular case, considered in this section can be deduced from Łojasiewicz lemmas.

References

[1] S. Łojasiewicz, *Sur la fixation des variables dans une distribution*, Studia Math. 17 (1958), p. 1-64.

[2] J. Mikusiński, *Irregular operations on distributions*, ibidem 20 (1961), p. 163-169.

[3] — and R. Sikorski, *The elementary theory of distributions, II*, Rozprawy Matem. 25 (1961).

[4] K. Skórnik, *Postać funkcji lokalnie całkowalnej, której m-ta pochodna okalna znika prawie wszędzie*, Zeszyty Naukowe W. S. P. w Katowicach (to be printed).

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On orbits of elements

by

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Let X be a linear metric space. Let A be a linear continuous operator mapping X into itself. Let $x \in X$. We shall write

$$\mathcal{O}(A:x) = \{A^n x : n = 0, 1, \ldots\}$$

and we shall call $\mathcal{O}(A:x)$ an *orbit* of x with respect to the operator A.

It is well known that if X is a space of finite dimension, then there are three possibilities:

1° $\lim A^n x = 0$,

2° $\lim \|A^n x\| = +\infty$,

3° the closure of the orbit $\mathcal{O}(A:x)$ is compact and 0 does not belong to this closure.

This follows for instance from [4], lemma 1, p. 270.

The purpose of the present paper is to show that in the infinite-dimensional case it is not so: it may happen that for some A and x, the orbit $\mathcal{O}(A:x)$ is dense in the whole space X. Examples of this situation in concrete spaces are given. It is not clear whether it may take place in an arbitrary infinite-dimensional separable Banach space (cf. Problem 1). Some related questions are also discussed.

The basic terminology and notation are the same as in Banach's book [1] and paper [3] of Mazur and Orlicz. In particular, by an *F-space* we mean any complete linear metric space and by a *B_0-space* we mean a locally-convex F-space. The norm in the sense of [1] (i.e. a subadditive, symmetric functional vanishing only at 0) is called in this paper an *F-norm*; norms and pseudonorms used here are always homogeneous and continuous.

THEOREM 1. *Let X be either l^p $(1 \leqslant p < +\infty)$ or c_0. For any arbitrary real $a > 1$, there are a linear continuos operator A and an element x_0 such that the orbit $\mathcal{O}(A:x_0)$ is dense in the whole of X.*

Proof. Let S be the left shift operator

$$S(\{x_1, x_2, x_3, \ldots\}) = \{x_2, x_3, \ldots\}$$

and S' be the right shift operator

$$S'(\{x_1, x_2, x_3, \ldots\}) = \{0, x_1, x_2, \ldots\}.$$

Let $A = aS$ and $B = S'/a$. Then $\|A\| = a$, $\|B\| = 1/a$, $AB = I$, where I, as usual, denotes the identity operator.

Now we shall start to construct the vector x_0. Let x^n be a sequence dense in X, with the property that all except finitely many coordinates of each element x^n are 0 (for instance we may take finite rational combinations of unit vectors). Let $k(n)$ denote the greatest index of the coordinate of x^n that does not vanish.

Let $r(n)$ be a sequence of positive integers such that

$$r(n) > \max_{1 \leqslant i \leqslant n} k(i) \tag{1}$$

and

$$\|B^{r(n)} x^n\| < \frac{1}{2^n}. \tag{2}$$

Let

$$p(n) = \sum_{i=1}^{n} r(i). \tag{3}$$

We write

$$x_0 = \sum_{n=1}^{\infty} B^{p(n)} x^n. \tag{4}$$

Formula (2) implies that series (4) is convergent.

Condition (1) implies that

$$A^{r(n)} x^i = 0 \quad \text{for } i < n. \tag{5}$$

Therefore

$$A^{p(n)} x_0 = x_n + \sum_{m=n+1}^{\infty} B^{p(m)-p(n)} x^m. \tag{6}$$

But

$$\Big\| \sum_{m=n+1}^{\infty} B^{p(m)-p(n)} x^m \Big\| \leqslant \sum_{m=n+1}^{\infty} \|B^{r(m)} x^m\| \leqslant \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n}. \tag{7}$$

Therefore

$$\|A^{p(n)} x_0 - x^n\| \leqslant \frac{1}{2^n}. \tag{8}$$

Let x be an arbitrary element. Since the sequence $\{x^n\}$ is dense in the space X, for each positive ε there is an x^n such that $\|x - x^n\| < \varepsilon$.

Without loss of generality we can assume that $1/2^n < \varepsilon$. Then

$$\|A^{p(n)}(x_0) - x\| \leqslant \|A^{p(n)} - x^n\| + \|x^n - x\| \leqslant \frac{1}{2^n} + \frac{1}{2^n} < 2\varepsilon.$$

Since ε is arbitrary, we infer, that the orbit $\mathcal{O}(A : x_0)$ is dense in X.

Let us remark that in the proof of theorem 1 the fact that we have considered the space of scalar sequences does not play any role. We can replace in this proof the field of scalars by an arbitrary separable Banach space, i.e. we can prove theorem 1 for spaces $(X)_{l^p}$ $(1 \leqslant p < +\infty)$ and $(X)_{c_0}$ of all sequences $x = \{x_n\}$ of elements of a separable Banach space X such that

$$x = \Big(\sum_{n=1}^{\infty} \|x_n\|_X^p\Big)^{1/p} < +\infty$$

$$\text{(respectively } \lim_{n\to\infty} \|x_n\|_X = 0)$$

with topology defined by the norm $|x|$ (respectively by the norm $|x| = \sup_n \|x_n\|_X$. (We denote by $\|x\|_X$ the norm in the space X.)

Therefore the following theorem holds:

THEOREM 1'. *Let X be either $(Y)_{l^p}$ or $(Y)_{c_0}$, Y being an arbitrary separable Banach space. Then for each real number a greater than 1 there are in X an element x_0 and a linear continuous operator A of norm a such that the orbit $\mathcal{O}(A : x_0)$ is dense in X.*

COROLLARY. *For an arbitrary a greater than 1 there are in each space $C[0,1]$ and $L^p[0,1]$, $1 \leqslant p < +\infty$, an operator A of the norm a and an element x_0 such that the orbit $\mathcal{O}^a_{x_0}$ of the element x_0 with respect to the operator A is dense in the whole space $C[0,1]$ (respectively $L^p[0,1]$).*

Proof. The space $C[0,1]$ (respectively $L^p[0,1]$) is isomorphic to the space $(C[0,1])_{c_0}$ (respectively $(L^p[0,1])_p$).

PROBLEM 1. Is it true that for every infinite-dimensional separable Banach space X there are such a linear continuous operator A and such an element x that $\mathcal{O}(A : x)$ is dense in X?

PROBLEM 2. Given a separable Banach space X. Characterize all continuous linear operators acting in X which admit a vector x such that $\mathcal{O}(A : x)$ is dense in X.

The operator A considered in theorem 1 is not invertible, but in a similar way we can construct an invertible operator with the same property. For this purpose let us remark that the space l^p $(1 \leqslant p < +\infty)$ and c_0 are isomorphic to the space l^p_a (c^a_0) of all sequences $x = \{x_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, such that

$$\|x\| = \Big(\sum_{n=-\infty}^{\infty} (a^{-|n|}|x_n|)^p\Big)^{1/p} < +\infty$$

(respectively $\lim_{|n|\to\infty} a^{-|n|}|x_n| = 0$) with the norm $\|x\|$ (respectively $\|x\| = \sup_n a^{-|n|}|x_n|$).

If a is greater than 1, then the shift operator S is of norm a and has the desired property; moreover, we can construct such an element x_0 that the orbits $\mathcal{O}^S_{x_0}$ and $\mathcal{O}^{S^{-1}}_{x_0}$ are both dense in the space l^p_a (respectively c^a_0). So far we have considered only powers of one operator. Similarly, we can construct a continuous semigroup [1] (or even a group) of continuous linear operators $T(s)$ such that there is an element x_0 such that the set

$$\mathcal{O}^{T(s)}_{x_0} = \{T(s)x_0 : s > 0\}$$

is dense in the whole space.

For example, let $L^p_a(-\infty, +\infty)$, $a > 1$, $1 \leqslant p < +\infty$, denote the space of all measurable functions $x = x(t)$ such that

$$\|x\| = \Big(\int_{-\infty}^{\infty} \big(a^{-|t|}|x(t)|\big)^p dt\Big)^{1/p} < +\infty$$

with the norm $\|x\|$. The space $L^p_a(-\infty, +\infty)$ is obviously isomorphic to the space $L^p[0,1]$.

The group of shift operators $T(s)$ defined as

$$T(s)[x(t)] = x(t-s)$$

is a continuous group of continuous operators. In the same way as in the proof of theorem 1, we can construct such an element x_0 that $\mathcal{O}^{T(s)}_{x_0}$ is dense in $L^p_a(-\infty, +\infty)$.

A similar construction can also be performed in the space $C^a_0(-\infty, \infty)$ $(a > 1)$ of all continuous functions such that

$$\lim_{|t|\to\infty} |a^{-|t|}x(t)| = 0$$

with the norm

$$\|x\| = \sup_t a^{-|t|}|x(t)|.$$

The space $C^a_0(-\infty, \infty)$ is isomorphic to the space $C[0,1]$.

If X is a Banach space and A is a continuous operator acting in X, then there is an $a > 0$ (namely any $0 < a < \|A\|^{-1}$) such that $\lim_{n\to\infty} (aA)^n x = 0$ for every $x \in X$, and therefore the orbit of no element with respect to aA is dense in X. If X is an F-space, there may exist operators S such that

[1] A semigroup $T(s)$ is called *continuous* if $\lim_{s\to s'} T(s)x = T(s')x$ for all x.

(*) for every $a > 0$ there is an x_a such that $\mathcal{O}(aS : x_a)$ is dense in X.

Proposition 1. *Let Y be an arbitrary F-space and let $X = (Y)_s$ be the space of all sequences $x = \{y_n\}$, with $y_n \in Y$ under the F-norm*

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2_n} \frac{|y_n|}{1+|y_n|},$$

where $|\ |$ denotes the F-norm in Y. Then the left shift operator S acting in X has the property (*).

Proof. The proof is on the same lines as the proof of theorem 1. It is only sufficient to remark that for all scalars a and $x \in X$

$$\|(aS')^n x\| < \frac{1}{2^n},$$

where, as before, S' denotes the right shift operator.

Remark. If X is an F-space and A is a continuous linear operator acting in X which admits a pseudonorm $\|\ \|$ such that A is continuous in $\|\ \|$, i.e. $\|Ax\| \leqslant b\|x\|$ for some $b > 0$ and all $x \in X$, then A does not satisfy (*). In fact, all elements $(b^{-1}A)^n x_0$ are in the set $\{x \in X : \|x\| < \|x_0\|\}$ and therefore the orbit of an element x_0 with respect to the operator $b^{-1}A$ cannot be dense in X.

Problem 3. Suppose that X is a separable B_0-space and A is a continuous linear opearator acting in X. Suppose that A does not satisfy (*). Does it imply that A is continuous with respect to some pseudonorm defined on X?

We will say that a B_0-space X is a *space with a norm* if there is such a pseudonorm $\|x\|$ in X that $\|x\| = 0$ implies $x = 0$. For example, the space $C^\infty[0,1]$ of infinite-differentiable functions $x(t)$ with topology given by a sequence of pseudonorms

$$\|x\|_m = \sup_{0\leqslant t\leqslant 1} \left|\frac{d^m x}{dt^m}\right|, \quad m = 0, 1, \ldots,$$

is a space with a norm. In fact, if $\|x\|_0 = 0$, then $x = 0$.

In paper [2] it is proved that a B_0-space X is a space with a norm if and only if X does not contain any subspace isomorphic to the space s of all sequences. Then obviously spaces $(Y)_s$ are not spaces with norms.

Problem 4. Does there exist a separable B_0-space X with a norm such that there are in X operators satisfying (*)?

Problem 5. Let X be a separable B_0-space with a norm. Let A be a continuous linear operator acting in X. Does there exist a pseudonorm $\|\ \|$ in which A is continuous?

Let X be a B_0-space. We say that a pseudonorm $\| \|$ defined in X is *infinite* (*finite*) *dimensional* if the quotient space $X/\{x : \|x\| = 0\}$ is infinite (finite) dimensional.

The following proposition, communicated to the author by Dr. C. Bessaga, is strictly connected with problem 5.

PROPOSITION 2. *There are infinite-dimensional B_0-space X and a continuous linear operator A acting in X which is not continuous in any infinite-dimensional pseudonorm.*

Proof. Let $X = M(n^m)$ be a space of all sequences $x = \{x_n\}$ such that

$$\|x\|_n = \sup n^m |x_n|.$$

$M(n^m)$ is a B_0-space with topology induced by pseudonorms $\|x\|_m$. Let A be defined by the formula

$$A(\{x_n\}) = \{nx_n\}.$$

The basis vectors $e_n = (0, 0, \ldots \underset{n\text{-th place}}{0, 1, 0}, \ldots)$ are eigenvectors of A respective to the eigenvalues $\lambda_n = n$. This implies that A is not continuous on any infinite-dimensional pseudonorm.

Remark. The example given in proposition 2 can be slightly extended. Namely, putting $T(s)\ (x_n) = n^s x_n$, we obtain in $M(n^m)$ a continuous group of continuous operators such that, for $s > 0$, $T(s)$ is not continuous in any infinite-dimensional pseudonorm.

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References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] C. Bessaga, A. Pełczyński and S. Rolewicz, *Some properties of the space* (*s*), Coll. Math. 7 (1957), p. 45-51.

[3] S. Mazur et W. Orlicz, *Sur les espaces linéaires métriques I*, Studia Math. 10 (1948), p. 184-208; *II*, ibidem 13 (1953), p. 137-179.

[4] A. C. Понтрягин, *Непрерывные группы*, Москва 1954.

[5] K. Singbal-Vedak, *A note on operators on a locally convex space*, Proc. Amer. Math. Soc. 16 (1965), p. 696-702.

[6] K. Yosida, *Functional analysis*, 1965.

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An analytic approach to semiclassical potential theory

by

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§ 0. Introduction. The aim of this paper is to give a new non-probabilistic approach to the semiclassical potential theory. The method used here is, may be, less interesting but much simpler. The semiclassical potential theory was started in 1950 by M. Kac who, using probabilistic methods, derived an analytic formula for the capacitory potential. Then it was systematically developed by Z. Ciesielski who indicated analogies between classical and semiclassical potential theories. Such notions as balayage, thinness, Dirichlet problem and barrier have their corresponding ones in the semiclassical theory. The sets of Lebesgue measure zero play essentially the same role as the polar sets. A brief, non-probabilistic account of this theory is given in § 2. For detailed treatment of this subject the reader is refered to [2] and [3]. Improving Kac's technique Stroock [7] has generalized the Kac formula on the strong balayage of an arbitrary superharmonic function. He has also obtained an analytic formula for the solution of the semiclassical Dirichlet problem. The method used in this article leads to the same formulas. We deal with this topic in § 3. § 4 is mainly devoted to non-probabilistic proofs of some Stroock's results (cf. [9]). In it a new method of solving the classical Dirichlet problem is estabilished. The solution is obtained as a limit of solutions of some integral equations (cf. Corollaries 4.5 and 4.6). We finish this paper by suggesting some possible generalizations.

The author wishes to thank Docent Z. Ciesielski for his guidance in the topic and much help and advice.

§ 1. Some basic lemmas. In the following U denotes a Greenian domain in the k-dimensional Euclidean space R^k and $G(x,y)$ the Green function for this domain. It will be convenient to employ the following notations:

$H_+^\uparrow(U)$ — the class of all positive and superharmonic functions on U;
$BH_+^\uparrow(U)$ — the class of all bounded $f \in H_+^\uparrow(U)$;
$CH_+^\uparrow(U)$ — the class of all continuous $f \in BH_+^\uparrow(U)$;