As we saw in the proof of theorem 4, f is an extreme point in the unit sphere of A since the range of f contains infinitely many scalars of modulus 1. Now we need to show that f is not an extreme point in the unit sphere of the closure of A. The set E does not separate the plane and has no interior, consequently by Mergelyan's theorem any continuous function on E is the uniform limit of polynomials [1]. Given the function $h(x,y) = x(1-x)$ for $y=0$ and $h(x,y)=0$ for $y\neq 0$, there is a sequence $g_n(z)=\sum a_{mn}z^m$ converging uniformly to h on E and since $h(0,0)=0$, we may take $g_n(0)=0$ for all n. Then $g_n(f)$ is a Cauchy sequence in A and so converges to a function g in the closure of A which is not zero. Since

$$|f(s)\pm g(s)| = |f(s)\pm h(\mathrm{Re} f(s), \mathrm{Im} f(s))| \leqslant 1,$$

$\|f\pm g\|\leqslant 1$ and f is not extreme in the unit sphere of the closure of A.

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Lebesgue and Lipschitz spaces and integrals of the Marcinkiewicz type

by

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§ 1. Introduction. A theorem of Zygmund [16] states that for $1<p<\infty$ the L^p-norm of

$$(Mf)(x) = \left(\int_0^{2\pi}\left|\frac{F(x+t)+F(x-t)-2F(x)}{t}\right|^2\frac{dt}{t}\right)^{1/2}$$

satisfies

$$\|Mf\|_p \leqslant A_p\|f\|_p$$

and, if $\int_0^{2\pi} f(x)\,dx = 0$,

$$\|f\|_p \leqslant A_p\|Mf\|_p,$$

where

$$F(x) = \int_0^x f(u)\,du.$$

The integral Mf is called the (*first*) *Marcinkiewicz integral* of F and is related in a rather natural way to the Hilbert transform of f. In fact, proceeding formally,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x-t)\frac{dt}{t} &= -\int_0^{\infty}[f(x+t)-f(x-t)]\frac{dt}{t}\\ &= -\int_0^{\infty}\frac{d}{dt}[F(x+t)+F(x-t)-2F(x)]\frac{dt}{t}\\ &= -\int_0^{\infty}\left(\frac{F(x+t)+F(x-t)-2F(x)}{t}\right)\frac{dt}{t}.\end{aligned}$$

It was exactly this relation which led Stein in [9] to define an n-dimensional version of the Marcinkiewicz integral (1). Let $\Omega(z)$, $z\in E_n$,

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(1) For another generalization of Mf to E_n, see [11].

be homogeneous of degree zero, sufficiently smooth and satisfy

$$\int_{\Sigma} \Omega(z')\,dz' = 0, \tag{1.1}$$

where $z' = z/|z|$ for $|z| \neq 0$ and Σ is the unit sphere in E_n. Again proceeding formally, the singular integral

$$\int_{E_n} f(x-z)\frac{\Omega(z')}{|z|^n}\,dz = \omega_n \int_0^\infty \left[\int_{\Sigma} f(x-tz')\Omega(z')\,dz'\right]\frac{dt}{t}$$

$$= \int_0^\infty \frac{d}{dt}\big(F_x(t)\big)\frac{dt}{t} = \int_0^\infty \left(\frac{F_x(t)}{t}\right)\frac{dt}{t},$$

where

$$F_x(t) = \int_{|z|<t} f(x-z)\frac{\Omega(z')}{|z|^{n-1}}\,dz.$$

In analogy with the 1-dimensional situation, Stein set

$$\mu(f)(x) = \left(\int_0^\infty \left|\frac{F_x(t)}{t}\right|^2\frac{dt}{t}\right)^{1/2}$$

and proved that for sufficiently smooth Ω satisfying (1.1) and certain values of p, $1<p<\infty$,

$$\|\mu(f)\|_p \leqslant A_p\|f\|_p.$$

He proved, moreover, that for $1<p<\infty$

$$\|f\|_p \leqslant A_p\sum_{j=1}^{n}\|\mu_j(f)\|_p,$$

where $\mu_j(f)$ is formed from

$$F_x^{(j)}(t) = \int_{|z|<t} f(x-z)\frac{z_j'}{|z|^{n-1}}\,dz.$$

Stein's results were later improved by Hörmander in his paper [8] where the following results are proved.

THEOREM 1. *Let $f\in L^p$, $1\leqslant p<\infty$, and let Ω be a real-valued function homogeneous of degree zero satisfying* (1.1). *In addition, let the modulus of continuity $\omega(\delta)$ of Ω on Σ satisfy the Dini condition*

$$\int_0^1 \frac{\omega(\delta)}{\delta}\,d\delta < +\infty,$$

and set

$$\mathscr{F}(x,t) = t^{-\beta}\int_{|z|<t} f(x-z)\frac{\Omega(z')}{|z|^{n-\beta}}\,dz, \qquad \beta>0,$$

$$\mu(f)(x) = \left(\int_0^\infty |\mathscr{F}(x,t)|^2\frac{dt}{t}\right)^{1/2}.$$

Then the transformation $f\to\mu(f)$ is bounded from L^1 into weak L^1 and from L^p into L^p, $1<p<\infty$.

Conversely,

THEOREM 2. *If $f\in L^p$, $1<p<\infty$, then*

$$\|f\|_p \leqslant A_{p,\beta,m}\sum\|\mu_j(f)\|_p,$$

where $\mu_j(f)$ is formed from

$$\mathscr{F}_j(x,t) = t^{-\beta}\int_{|z|<t} f(x-z)\frac{\Omega_j(z')}{|z|^{n-\beta}}\,dz,$$

$\{\Omega_j\}$ being a normalized basis for the spherical harmonics of a fixed degree m, $m\neq 0$.

The restriction in Theorem 2 to the spherical harmonics is stronger than is absolutely necessary and for a more general statement we refer the reader to [8].

The purpose of this paper is to study Marcinkiewicz type integrals formed from hypersingular integrals rather than singular integrals. The hypersingular integral

$$\int_{E_n} [f(x-z)-f(x)]\frac{\Omega(z')}{|z|^{n+a}}\,dz,$$

$0<a<2$, may be written (by changing to polar coordinates $z = tz'$)

$$\int_0^\infty \frac{d}{dt}\big(F_x(t)\big)\frac{dt}{t^{a+1}} = (a+1)\int_0^\infty \left(\frac{F_x(t)}{t^{a+1}}\right)\frac{dt}{t},$$

where

$$F_x(t) = \int_{|z|<t} [f(x-z)-f(x)]\frac{\Omega(z')}{|z|^{n-1}}\,dz.$$

It then seems natural to consider L^p-norms of

$$\left(\int_0^\infty \left|\frac{F_x(t)}{t^{a+1}}\right|^2\frac{dt}{t}\right)^{1/2}$$

or, more generally, L^p-norms of

$$\left(\int_0^\infty \left|\frac{\mathcal{F}(x,t)}{t^\alpha}\right|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$\mathcal{F}(x,t) = t^{-\beta}\int_{|z|<t} [f(x-z)-f(x)]\,\frac{\Omega(z')}{|z|^{n-\beta}}\,dz.$$

Before proceeding, we introduce some standard notation. Given a measurable function $g(x,t)$, $x\in E_n$, $0<t<\infty$, write (see [8])

$$X^pT^2 g = \left(\int_{E_n}\left(\int_0^\infty |g(x,t)|^2\frac{dt}{t}\right)^{p/2} dx\right)^{1/p}$$

for $1\leqslant p<\infty$. If $f\in L^p$, $1\leqslant p<\infty$, define $J^\alpha f$ by

$$(J^\alpha f)^\wedge(x) = (1+|x|^2)^{-\alpha/2}\hat{f}(x),$$

where $^\wedge$ denotes the Fourier transform in the sense of tempered distributions, the Fourier transform of an integrable function being defined

$$\hat{f}(x) = (2\pi)^{-n/2}\int_{E_n} f(z)e^{-i(x\cdot z)}\,dz.$$

$J^\alpha f$ is called the *Bessel potential of order* α of f and it is known that for $0<\alpha<2$, $J^\alpha f = f*G_\alpha$, where $G_\alpha\geqslant 0$ and $G_\alpha\in L^1(E_n)$. We denote by L^p_α the class of all L^p-functions $f=J^\alpha\Phi$, where $\Phi\in L^p$, and write $\Phi = J^{-\alpha}f$, $\|f\|_{p,\alpha}=\|\Phi\|_p$. For a discussion of the L^p_α-spaces, see [1] and [3]. We will prove the following companion results to Theorems 1 and 2:

THEOREM 3. *Let $f\in L^p_\alpha$, $0<\alpha<2$, $1\leqslant p<\infty$, and*

$$\mathcal{F}(x,t) = t^{-\beta}\int_{|z|<t} [f(x-z)-f(x)]\,\frac{\Omega(z')}{|z|^{n-\beta}}\,dz$$

for $\beta>0$, where $\Omega\in L^1(\Sigma)$ and for $1\leqslant\alpha<2$

(1.2) $$\int_\Sigma z'_j\,\Omega(z')\,dz' = 0,\quad j=1,\ldots,n.$$

Then

$$X^pT^2(t^{-\alpha}\mathcal{F}(x,t))\leqslant c\|f\|_{p,\alpha},\quad 1<p<\infty,$$

and

$$\left|\left\{x: T^2\left(t^{-\alpha}\mathcal{F}(x,t)\right)>s\right\}\right| \leqslant \frac{c}{s}\|f\|_{1,\alpha}\ (^2)$$

with c independent of f.

(2) $|\{\ldots\}|$ denotes the measure of the set $\{\ldots\}$.

Conversely,

THEOREM 4. *Let $f\in L^p$, $1<p<\infty$, and suppose each $X^pT^2\left(t^{-\alpha}\mathcal{F}_j(x,t)\right)<+\infty$, where*

$$\mathcal{F}_j(x,t) = t^{-\beta}\int_{|z|<t} [f(x-z)-f(x)]\,\frac{\Omega_j(z')}{|z|^{n-\beta}}\,dz,$$

$0<\alpha<2$, $\beta>0$, and $\{\Omega_j\}$ is a normalized basis for the spherical harmonics of a fixed degree m, $m\neq 1$ when $1\leqslant\alpha<2$. Then $f\in L^p_\alpha$ and

$$\|f\|_{p,\alpha}\leqslant c\Big[\sum_j X^pT^2\left(t^{-\alpha}\mathcal{F}_j(x,t)\right)+\|f\|_p\Big]$$

with c independent of f.

In view of the statements of Theorems 1 and 2 it is natural to ask if these theorems have versions when $X^pT^2\left(t^{-\alpha}\mathcal{F}(x,t)\right)$ is replaced by $T^qX^p\left(t^{-\alpha}\mathcal{F}(x,t)\right)$, where for $1\leqslant p\leqslant\infty$

$$T^qX^p g(x,t) = \left(\int_0^\infty \|g(x,t)\|_p^q\,\frac{dt}{t}\right)^{1/q},\quad 1\leqslant q<\infty,$$

$$T^\infty X^p g(x,t) = \sup_t \|g(x,t)\|_p.$$

The answer is yes if we replace the Lebesgue spaces L^p_α by the Lipschitz spaces $\Lambda(\alpha,p,q)$ studied in [13]. For the sake of simplicity, we shall define $\Lambda(\alpha,p,q)$ only when $0<\alpha<2$, $1\leqslant p,q\leqslant\infty$. There are many equivalent definitions for these spaces (see [13], p. 421) but for our purposes the most natural one is in terms of the Poisson integral

$$f(x,t) = \int_{E_n} f(x-z)\,\frac{t}{(t^2+|z|^2)^{(n+1)/2}}\,dz$$

of f. We say that *a function $f\in L^p(E_n)$ belongs to $\Lambda(\alpha,p,q)$*, $0<\alpha<2$, $1\leqslant p,q\leqslant\infty$, if

$$T^qX^p\left(t^{2-\alpha}f_{tt}(x,t)\right)<+\infty,$$

and write $\|f\|_{\alpha,p,q} = \|f\|_p+T^qX^p\left(t^{2-\alpha}f_{tt}(x,t)\right)$. For the properties of the $\Lambda(\alpha,p,q)$-spaces we refer the reader to [13] and [14]. We will prove the following two theorems:

THEOREM 5. *Let $f\in\Lambda(\alpha,p,q)$ for $0<\alpha<2$, $1\leqslant p,q\leqslant\infty$. If Ω satisfies the hypothesis of Theorem 3, then*

$$T^qX^p\left(t^{-\alpha}\mathcal{F}(x,t)\right)\leqslant c\|f\|_{\alpha,p,q}$$

with c independent of f.

THEOREM 6. *Let* $f\in L^p$, $1<p<\infty$, *and suppose each* $T^q X^p\big(t^{-\alpha}\mathcal{F}_j(x,t)\big)<+\infty$, *where* $1<q<\infty$, $0<\alpha<2$ *and the* $\mathcal{F}_j(x,t)$ *are as in Theorem* 5. *Then* $f\in\Lambda(\alpha,p,q)$ *and*

$$\|f\|_{\alpha,p,q}\leqslant c\Big[\sum_j T^q X^p\big(t^{-\alpha}\mathcal{F}_j(x,t)\big)+\|f\|_p\Big]$$

with c *independent of* f. *The theorem is valid for* $1\leqslant p,q\leqslant\infty$ *if* $m=0$.

Although we will always restrict α to the range $0<\alpha<2$, Theorems 3-6 have analogues for larger α and we will indicate what they are. Given an integer $k\geqslant 1$, let $f\in L^p_{k-1}$. If $\nu=(\nu_1,\ldots,\nu_n)$, where the ν_j are non-negative integers, $|\nu|=\nu_1+\ldots+\nu_n$, $\nu!=\nu_1!\ldots\nu_n!$, $z^\nu=z_1^{\nu_1}\ldots z_n^{\nu_n}$, then for $|\nu|\leqslant k-1$, let $f_\nu(x)$ be the L^p-function which is the derivative of f of order ν. For $k-1<\alpha<k+1$ and $\beta>0$, let

$$\mathcal{F}(x,t)=t^{-\beta}\int_{|z|<t}\Big[f(x+z)-\sum_{|\nu|\leqslant k-1}\frac{f_\nu(x)}{\nu!}z^\nu\Big]\frac{\Omega(-z')}{|z|^{n-\beta}}\,dz,$$

where Ω is a real-valued function which is homogeneous of degree zero and integrable over Σ and, in addition, satisfies for $k\leqslant\alpha<k+1$

$$\int_\Sigma z'^\nu\,\Omega(z')\,dz'=0$$

for all ν with $|\nu|=k$. Theorems 3-6 are concerned with the case $k=1$ of this set-up and it is clear what their analogues are for other k, e.g., if $f\in L^p_\alpha$, then $X^pT^2\big(t^{-\alpha}\mathcal{F}(x,t)\big)\leqslant c\|f\|_{p,\alpha}$, etc.

We would like to emphasize that the novelty of Theorems 3-6 is that we have assumed Ω is merely integrable. For $0<\alpha<1$ and bounded Ω or for $1\leqslant\alpha<2$ and bounded even Ω, Theorems 3 and 5 are simple corollaries of stronger results of Strichartz [12] and Taibleson [13] respectively. (See the remarks at the end of sections 2 and 3.)

In proving the theorems we depend very heavily on the methods developed in [9], [8], [2] and [13], and in many cases our proofs are just slight modifications of those given there. Moreover, we require familiarity with the basic facts in the theory of the L^p_α and $\Lambda(\alpha,p,q)$-spaces.

§ 2. Proof of Theorem 5.

We begin with Hardy's classical inequality.

LEMMA 1. *Let* $h(t)$ *be a non-negative function defined on* $0<t<\infty$. *Given* $\gamma\neq 0$, *let*

$$H(s)=\int_0^s h(t)\,dt\quad\text{for } \gamma<0$$

and

$$H(s)=\int_s^\infty h(t)\,dt\quad\text{for } \gamma>0.$$

Then

$$\Big[\int_0^\infty \big(s^\gamma H(s)\big)^q\frac{ds}{s}\Big]^{1/q}\leqslant\frac{1}{|\gamma|}\Big[\int_0^\infty\big(s^{\gamma+1}H(s)\big)^q\frac{ds}{s}\Big]^{1/q}$$

for $1\leqslant q\leqslant\infty$.

For a proof see [7], p. 239-246.

Let $f\in L^p$, $1\leqslant p\leqslant\infty$, and let $f(x,y)$, $x\in E_n$, $y>0$, be its Poisson integral. Given $\varepsilon>0$, repeated integration by parts yields

$$f(u,\varepsilon)=\int_\varepsilon^t yf_{yy}(u,y)\,dy-tf_y(u,t)+\varepsilon f_y(u,\varepsilon)+f(u,t)$$

so that

$$\int_{|z|<t}[f(x-z,\varepsilon)-f(x,\varepsilon)]\frac{\Omega(z')}{|z|^{n-\beta}}\,dz$$

$$=\int_\varepsilon^t y\,dy\int_{|z|<t}[f_{yy}(x-z,y)-f_{yy}(x,y)]\frac{\Omega(z')}{|z|^{n-\beta}}\,dz-$$

$$-t\int_{|z|<t}[f_y(x-z,t)-f_y(x,t)]\frac{\Omega(z')}{|z|^{n-\beta}}\,dz+\varepsilon\int_{|z|<t}[f_y(x-z,\varepsilon)-f_y(x,\varepsilon)]\times$$

$$\times\frac{\Omega(z')}{|z|^{n-\beta}}\,dz+\int_{|z|<t}[f(x-z,t)-f(x,t)]\frac{\Omega(z')}{|z|^{n-\beta}}\,dz=I_1+I_2+I_3+I_4.$$

By Minkowski's inequality, the L^p-norm of I_1 with respect to x does not exceed

$$2\int_0^t y\|f_{yy}(x,y)\|_p\,dy\int_{|z|<t}\frac{|\Omega(z')|}{|z|^{n-\beta}}\,dz=ct^\beta\int_0^t y\|f_{yy}(x,y)\|_p\,dy$$

since $\Omega\in L^1$.

If $z=\varrho z'$, then

$$f_y(x-z,\varepsilon)-f_y(x,\varepsilon)=\int_0^\varrho\frac{d}{dr}[f_y(x-rz',\varepsilon)]\,dr$$

$$=-\sum_{j=1}^n z'_j\int_0^\varrho f_{yx_j}(x-rz',t)\,dr$$

and by Minkowski's inequality, the L^p-norm of I_2 with respect to x does not exceed

$$t\sum_{j=1}^n\|f_{yx_j}(x,t)\|_p\int_{|z|<t}\frac{|\Omega(z')|}{|z|^{n-\beta-1}}\,dz\leqslant ct^{\beta+2}\sum_{j=1}^n\|f_{yx_j}(x,t)\|_p.$$

The L^p-norm of I_3 is majorized by

$$2\varepsilon\|f_y(x,\varepsilon)\|_p\int\limits_{|z|<t}\frac{|\Omega(z')|}{|z|^{n-\beta}}\,dz\leqslant c\varepsilon t^{\beta}\|f_y(x,\varepsilon)\|_p$$

since $\Omega\in L^1(\Sigma)$. To estimate the L^p-norm of I_4, suppose first $0<a<1$ and write

$$f(x-z,t)-f(x,t)=\int\limits_0^{\varrho}\frac{d}{dr}[f(x-rz',t)]\,dr,$$

$z=\varrho z'$. Arguing as for I_2,

$$\|I_4\|_p\leqslant ct^{\beta+1}\sum_{j=1}^{n}\|f_{x_j}(x,t)\|_p.$$

If, on the other hand, $1\leqslant a<2$, then Ω is orthogonal to polynomials of degree 1 by assumption, and I_4 is not changed if we replace $f(x-z,t)-f(x,t)$ in the integrand by

$$f(x-z,t)-f(x,t)+\sum_{j=1}^{n}z_jf_{x_j}(x,t)$$

$$=-\sum_{j=1}^{n}z_j'\int\limits_0^{\varrho}[f_{x_j}(x-rz',t)-f_{x_j}(x,t)]\,dr$$

$$=-\sum_{j=1}^{n}z_j'\int\limits_0^{\varrho}dr\int\limits_0^{r}\frac{d}{ds}[f_{x_j}(x-sz',t)]\,ds=\sum z_i'z_j'\int\limits_0^{\varrho}dr\int\limits_0^{r}f_{x_ix_j}(x-sz',t)\,ds.$$

Hence,

$$\|I_4\|_p\leqslant ct^{\beta+2}\sum_{i,j}\|f_{x_ix_j}(x,t)\|_p.$$

Collecting these estimates in the case $1\leqslant a<2$, we see the L^p-norm of

$$t^{-\beta}\int\limits_{|z|<t}[f(x-z,\varepsilon)-f(x,\varepsilon)]\frac{\Omega(z')}{|z|^{n-\beta}}\,dz$$

is majorized by a constant times

$$\int\limits_0^t y\|f_{yy}(x,y)\|_p\,dy+t^2\sum\|f_{yx_j}(x,t)\|_p+\varepsilon\|f_y(x,\varepsilon)\|_p+t^2\sum\|f_{x_ix_j}(x,t)\|_p.$$

Since $f(x,\varepsilon)$ is the Poisson integral of f and $\varepsilon\|f_y(x,\varepsilon)\|_p$ tends to zero with ε (see [13], p. 426) it follows that

$$\|\mathscr{F}(x,t)\|_p\leqslant c\Big[\int\limits_0^t y\|f_{yy}(x,y)\|_p\,dy+t^2\sum\|f_{yx_j}(x,t)\|_p+t^2\sum\|f_{x_ix_j}(x,t)\|_p\Big].$$

For $q<\infty$,

$$T^qX^p\big(t^{-a}\mathscr{F}(x,t)\big)=\Big(\int\limits_0^{\infty}(t^{-a}\|\mathscr{F}(x,t)\|_p)^q\frac{dt}{t}\Big)^{1/q}$$

and by Hardy's inequality

$$\Big(\int\limits_0^{\infty}\Big[t^{-a}\int\limits_0^{t}y\|f_{yy}(x,y)\|_p\,dy\Big]^q\frac{dt}{t}\Big)^{1/q}\leqslant\frac{1}{a}\Big(\int\limits_0^{\infty}[t^{2-a}\|f_{tt}(x,t)\|_p]^q\frac{dt}{t}\Big)^{1/q}$$
$$=\frac{1}{a}T^qX^p\big(t^{2-a}f_{tt}(x,t)\big)\leqslant\frac{1}{a}\|f\|_{a,p,q}.$$

Since $a<2$ (see [13], p. 420),

$$\Big(\int\limits_0^{\infty}[t^{2-a}\|f_{tx_j}(x,t)\|_p]^q\frac{dt}{t}\Big)^{1/q}\leqslant c\|f\|_{a,p,q}$$

and

$$\Big(\int\limits_0^{\infty}[t^{2-a}\|f_{x_ix_j}(x,t)\|_p]^q\frac{dt}{t}\Big)^{1/q}\leqslant c\|f\|_{a,p,q}$$

which proves Theorem 5 for $1\leqslant a<2$, $q<+\infty$. For $q=\infty$

$$T^{\infty}X^p\big(t^{-a}\mathscr{F}(x,t)\big)=\sup_t\big(t^{-a}\|\mathscr{F}(x,t)\|_p\big)$$

and the argument is the same using the $q=\infty$ version of Hardy's inequality. Finally, for $0<a<1$ only the part of the argument concerning I_4 requires comment. The contribution of $\|I_4\|_p$ to $\|\mathscr{F}(x,t)\|_p$ is at most

$$t\sum_j\|f_{x_j}(x,t)\|_p$$

and the theorem follows from [13], p. 420, and the fact that

$$T^qX^p\big(t^{1-a}f_t(x,t)\big)\leqslant\|f\|_{a,p,q}$$

when $0<a<1$ ([13], p. 421).

Remarks. (1) The argument we have given above differs only slightly from that used in proving part of Theorem 4 of [13]. In fact, if we were willing to allow Ω to be bounded when $0<a<1$ or bounded and even when $1\leqslant a<2$, then Theorem 1 has stronger versions which follow easily from [13]. For example, we can replace $\mathscr{F}(x,t)$ by

$$t^{-\beta}\int\limits_{|z|<t}|f(x+z)-f(x)|\frac{dz}{|z|^{n-\beta}}\qquad\text{for } 0<a<1$$

and by

$$t^{-\beta}\int\limits_{|z|<t}|f(x+z)+f(x-z)-2f(x)|\,\frac{dz}{|z|^{n-\beta}}\qquad \text{for } 1\leqslant a<2$$

and the conclusion of Theorem 1 remains valid.

(2) In case $p=q=2$,

$$A(a,2,2)=L_a^2 \quad\text{and}\quad T^2X^2\big(t^{-a}\mathcal{F}(x,t)\big)=X^2T^2\big(t^{-a}\mathcal{F}(x,t)\big).$$

Hence Theorem 5 includes the $p=2$ version of Theorem 3.

§ 3. Proof of theorem 3. The proof of theorem 3 follows the well-known method in [2] (see also [9], [8] and [13]). We begin with two lemmas. L_0^∞ denotes the space of bounded functions with compact support.

LEMMA 1. *For $\Phi\in L_0^\infty$, let*

$$(A\Phi)(x,t)=\int\limits_{E_n}\Phi(y)Q(x-y,t)\,dy.$$

If $T^2Q(x,t)$ is finite for almost all x and both

(a) $X^2T^2A\Phi\leqslant c\|\Phi\|_2$

and

(b) $\int\limits_{|x|\geqslant\lambda d}T^2[Q(x-y,t)-Q(x,t)]\,dx\leqslant c$

independently of d for $|y|<d$ and sufficiently large fixed λ, then

$$X^pT^2A\Phi\leqslant c\|\Phi\|_p$$

for $\Phi\in L_0^\infty$ and $1<p<\infty$.

Lemma 1 is a special case of theorem 2 of [2].

LEMMA 2. *If $\Phi\in L^p$, $1\leqslant p<\infty$, then for $0<a<n$, $J^a\Phi=\Phi*G_a$, where*

(a) $G_a\geqslant 0$, $\int G_a(x)\,dx<\infty$

and

(b) *G_a is infinitely differentiable for $x\neq 0$, and*

$$\left|\frac{\partial^\nu}{\partial x^\nu}G_a(x)\right|\leqslant c_{a,\nu}|x|^{a-n-|\nu|}\qquad \text{for}\qquad |\nu|\geqslant 0.$$

For a proof, see for example [5], p. 191-192.

If $\Phi\in L_0^\infty$, $f=J^a\Phi$ and

$$t^{-a}\mathcal{F}(x,t;f)=\int\limits_{E_n}[f(x-z)-f(x)]K(z,t)\,dz,$$

where $K(z,t)=t^{-a-\beta}|z|^{\beta-n}\Omega(z')$ for $|z|<t$ and $K(z,t)=0$ otherwise, we may interchange the order of integration to obtain

$$t^{-a}\mathcal{F}(x,t;f)=\int\Phi(y)Q(x-y,t)\,dy=(A\Phi)(x,t),$$

where

$$Q(x,t)=K_a(x,t)=\int\{G_a(x-z)-G_a(x)\}K(z,t)\,dz.$$

It is a simple matter to show that for any $c>0$

$$\int\limits_{|x|>c}T^2K_a(x,t)\,dx<\infty,$$

since, if for example $1\leqslant a<2$, then

$$K_a(x,t)=\int\limits_{|z|<c/2}\Big\{G_a(x-z)-G_a(x)+\sum_{j=1}^{n}z_jG_a^{(j)}(x)\Big\}K(z,t)\,dz+ \\ +\int\limits_{|z|>c/2}\{G_a(x-z)-G_a(x)\}K(z,t)\,dz,$$

where $G_a^{(j)}=\partial G_a/\partial x_j$.

But

$$T^2K(z,t)=\frac{|\Omega(z')|}{|z|^{n-\beta}}\left(\int\limits_{|z|}^{\infty}\frac{dt}{t^{2a+2\beta+1}}\right)^{1/2}=c\,|\Omega(z')|\,|z|^{-n-a}. \tag{3.1}$$

Hence,

$$T^2K_a(x,t)\leqslant c\int\limits_{|z|<c/2}|G_a(x-z)-G_a(x)+\sum z_jG_a^{(j)}(z)|\,\frac{|\Omega(z')|}{|z|^{n+a}}\,dz+ \\ +c\int\limits_{|z|>c/2}\{G_a(x-z)+G_a(x)\}\,\frac{|\Omega(z')|}{|z|^{n+a}}\,dz.$$

The last integral on the right is clearly integrable over $|x|>c>0$. By lemma 2 and the mean value theorem, the first is majorized in $|x|>c$ by a constant times

$$|x|^{-n-2+a}\int\limits_{|z|<c/2}|z|^2\,\frac{|\Omega(z')|}{|z|^{n+a}}\,dz.$$

Since $a<2$, it is integrable over $|x|>c>0$.

Hence if we apply lemma 1, we will obtain

$$X^pT^2\big(t^{-a}\mathcal{F}(x,t;f)\big)\leqslant c_p\|f\|_{p,a} \tag{3.2}$$

for $1<p<\infty$ if we show that for large fixed λ,

$$\int\limits_{|x|>\lambda d}T^2[K_a(x-y,t)-K_a(x,t)]\,dx\leqslant c \tag{3.3}$$

for $|y|<d$, with c independent of d (3). Once (3.2) and (3.3) are proved, it will be easy to remove the restriction $f=J^a\Phi$ for $\Phi\in L_0^\infty$.

(3) To verify (a) of lemma 1, we use theorem 5 for $p=2$. However, theorem 3 for $p=2$ can be proved directly by Parseval's formula. See section 4.

In proving (3.3) we will consider separately the cases $0 < \alpha < 1$ and $1 \leqslant \alpha < 2$. Let us take $1 \leqslant \alpha < 2$ and indicate later the changes necessary when $0 < \alpha < 1$.

We have

$$K_\alpha(x-y,t)-K_\alpha(x,t) = \int \{G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+ + G_\alpha(x)\}K(z,t)\,dz = \int\limits_{|z|<d} + \int\limits_{|z|>d} = A+B.$$

Using the orthogonality of $\Omega(z')$ and z_j',

$$T^2 A \leqslant T^2 \int\limits_{|z|<d} \Big\{G_\alpha(x-y-z)-G_\alpha(x-y)+\sum_{j=1}^{n} z_j G_\alpha^{(j)}(x-y)\Big\} K(z,t)\,dz + $$
$$+T^2 \int\limits_{|z|<d} \Big\{G_\alpha(x-z)-G_\alpha(x)+\sum_{j=1}^{n} z_j G_\alpha^{(j)}(x)\Big\} K(z,t)\,dz,$$

where $G_\alpha^{(j)} = \partial G_\alpha/\partial x_j$. Since $|y| < d$, these two terms are essentially of the same form and we consider the second. By the mean-value theorem and Lemma 2 above,

$$\Big|G_\alpha(x-z)-G_\alpha(x)-\sum z_j G_\alpha^{(j)}(x)\Big| \leqslant c\frac{|z|^2}{|x|^{n+2-\alpha}}$$

for $|z| < d$, $|x| > \lambda d$. Hence

$$\int\limits_{|x|>\lambda d} T^2 A\,dx \leqslant c \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}}\, T^2\Big(\int\limits_{|z|<d} |z|^2 |K(z,t)|\,dz\Big).$$

From (3.1) we obtain

$$\int\limits_{|x|>\lambda d} T^2 A\,dx \leqslant c \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}} \int\limits_{|z|<d} \frac{|\Omega(z')|}{|z|^{n+\alpha-2}}\,dz = O(d^{\alpha-2})\,O(d^{2-\alpha}) = O(1)$$

since $\Omega \in L^1(\Sigma)$ and $\alpha < 2$.

We now write

$$B = \int\limits_{|z|>d} \{G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)\}K(z,t)\,dz$$
$$= \int\limits_{d<|z|<|x|/2} + \int\limits_{|z|\geqslant |x|/2} = B_1+B_2.$$

Write the integrand of B_1 as $K(z,t)$ times

$$\Big\{G_\alpha(x-y-z)-G_\alpha(x-z)+\sum_{j=1}^{n} y_j G_\alpha^{(j)}(x-z)\Big\} -$$
$$-\Big\{G_\alpha(x-y)-G_\alpha(x)+\sum_{j=1}^{n} y_j G_\alpha^{(j)}(x)\Big\} - \sum_{j=1}^{n} y_j\{G_\alpha^{(j)}(x-z)-G_\alpha^{(j)}(x)\}.$$

Since $|z| < |x|/2$ in B_1 and $|y| < d \leqslant |x|/\lambda$, the first two terms above are $O(|y|^2/|x|^{n+2-\alpha})$ and the third is $O(|y|\,|z|/|x|^{n+2-\alpha})$. Hence for $|y| < d$,

$$\int\limits_{|x|>\lambda d} T^2 B_1\,dx \leqslant c \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}}\, T^2 \int\limits_{d<|z|<\frac{|x|}{2}} (d^2+d|z|)\,|K(z,t)|\,dz$$
$$\leqslant cd^2 \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}} \int\limits_{|z|>d} \frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz +$$
$$+cd \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}} \int\limits_{d<|z|<\frac{|x|}{2}} \frac{|\Omega(z')|}{|z|^{n+\alpha-1}}\,dz$$

by (3.1). Here

$$d^2 \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}} \int\limits_{|z|>d} \frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz = O(1)$$

and

$$d \int\limits_{|x|>\lambda d} \frac{dx}{|x|^{n+2-\alpha}} \int\limits_{d<|z|<|x|/2} \frac{|\Omega(z')|}{|z|^{n+\alpha-1}}\,dz \leqslant d \int\limits_{|z|>d} \frac{|\Omega(z')|}{|z|^{n+\alpha-1}}\,dz \int\limits_{|x|>2|z|} \frac{dx}{|x|^{n+2-\alpha}}$$
$$\leqslant cd \int\limits_{|z|>d} \frac{|\Omega(z')|}{|z|^{n+1}}\,dz = O(1).$$

Next

$$B_2 = \int\limits_{|z|>|x|/2} \{G_\alpha(x-y-z)-G_\alpha(x-y)-G_\alpha(x-z)+G_\alpha(x)\}K(z,t)\,dz$$

and the part

$$\{G_\alpha(x-y)-G_\alpha(x)\} \int\limits_{|z|>|x|/2} K(z,t)\,dz$$

has T^2-norm majorized by a constant times

$$\frac{d}{|x|^{n+1-\alpha}} \int\limits_{|z|>|x|/2} \frac{|\Omega(z')|}{|z|^{n+\alpha}}\,dz = c\frac{d}{|x|^{n+1}},$$

whose integral over $|x| > \lambda d$ is bounded. Finally, the remaining part of B_2 is

$$\int\limits_{|z|>|x|/2} \{G_\alpha(x-y-z)-G_\alpha(x-z)\}K(z,t)\,dz = \int\limits_{\substack{|z|>|x|/2\\ |x-z|>2d}} + \int\limits_{\substack{|z|>|x|/2\\ |x-z|<2d}} = B_2'+B_2''.$$

Since $|x-z| > 2d$ in B_2' and $|y| < d$, it follows from the mean-value theorem and (3.1) that

$$T^2 B_2' \leqslant cd \int\limits_{\substack{|z|>|x|/2\\ |x-z|>2d}} \frac{|\Omega(z')|}{|x-z|^{n+1-\alpha}}\,\frac{dz}{|z|^{n+\alpha}}.$$

In this domain of integration $|x-z| \leqslant |x|+|z| < 3|z|$. Hence choosing $0<\delta<1$ (then $\delta<\alpha$),

$$T^2 B_2' \leqslant cd \int_{\substack{|z|>|x|/2\\ |x-z|>2d}} \frac{|\Omega(z')|}{|x-z|^{n+1-\delta}} \frac{dz}{|z|^{n+\delta}}$$

so that

$$\int_{|x|>\lambda d} T^2 B_2' dx \leqslant cd \int_{|z|>\lambda d/2} \frac{|\Omega(z')|}{|z|^{n+\delta}} dz \int_{|x-z|>2d} \frac{dx}{|x-z|^{n+1-\delta}} = O(1).$$

In B_2'', we estimate

$$\int_{\substack{|z|>|x|/2\\ |x-z|<2d}} G_\alpha(x-y-z)K(z,t)\,dz, \qquad \int_{\substack{|z|>|x|/2\\ |x-z|<2d}} G_\alpha(x-z)K(z,t)\,dz$$

separately. Consider for example the second. Its T^2-norm is less than a constant times

$$\int_{\substack{|z|>|x|/2\\ |x-z|<2d}} \frac{|\Omega(z')|}{|x-z|^{n-\alpha}} \frac{dz}{|z|^{n+\alpha}},$$

whose integral over $|x|>\lambda d$ is less than

$$\int_{|z|>\lambda d/2} \frac{|\Omega(z')|}{|z|^{n+\alpha}} dz \int_{|x-z|<2d} \frac{dx}{|x-z|^{n-\alpha}} = O(1).$$

This completes the proof for $1\leqslant \alpha<2$. For $0<\alpha<1$, the only changes necessary are those in the arguments for A, B_1 and B_2'. In A and B_1 we would not introduce the auxiliary terms $\sum z_j G_\alpha^{(j)}(x)$, etc., but majorize $G_\alpha(x-y-z)-G_\alpha(x-y)$ and $G_\alpha(x-z)-G_\alpha(x)$ in A by $O(|z|/|x|^{n+1-\alpha})$ and $G_\alpha(x-y-z)-G_\alpha(x-z)$ and $G_\alpha(x-y)-G_\alpha(x)$ in B_1 by $O(|y|/|x|^{n+1-\alpha})$ for $|z|<|x|/2$. In B_2', we would pick $0<\delta<\alpha$ (then $0<\delta<1$). In particular, the argument would not require that Ω be orthogonal to polynomials of degree 1 for $0<\alpha<1$. This completes the proof of (3.3).

To remove the restriction in (3.2) that $f=J^\alpha\Phi$, $\Phi\in L_0^\infty$, we argue as follows. Given $f\in L_\alpha^p$, $f=J^\alpha\Phi$, choose $\Phi_m\in L_0^\infty$ with $\|\Phi-\Phi_m\|_p\to 0$. Write $f_m=J^\alpha\Phi_m$, $A\Phi_m = t^{-\alpha}\mathscr{F}(x,t;f_m)$ and $A\Phi = t^{-\alpha}\mathscr{F}(x,t;f)$. Then $A\Phi_m$ is a Cauchy sequence in X^pT^2 norm, and there exists $g\in X^pT^2$ with $X^pT^2(A\Phi_m-g)\to 0$. In particular, $X^pT^2g\leqslant c\|f\|_{p,\alpha}$ and it is enough to show that $g=A\Phi$ for almost all (x,t). However, for fixed t,

$$\|A\Phi_m - A\Phi\|_p \leqslant 2\|f_m-f\|_p t^{-\alpha-\beta} \int_{|z|<t} \frac{|\Omega(z')|}{|z|^{n-\beta}} dz \leqslant ct^{-\alpha}\|f_m-f\|_p.$$

Hence $\|A\Phi_m-A\Phi\|_p\to 0$ uniformly in $t\geqslant\delta>0$ and $A\Phi_m$ converges in x-measure to $A\Phi$ uniformly in $t\geqslant\delta>0$. Thus for all $t\geqslant\delta$ there is a single $\{m_k\}$ such that $A\Phi_{m_k}\to A\Phi$ for almost all x. Hence for almost all x, $A\Phi_{m_k}\to A\Phi$ for almost all $t\geqslant\delta$.

On the other hand, since $X^pT^2(A\Phi_{m_k}-g)\to 0$ there is a subsequence m_k' of m_k with $T^2(A\Phi_{m_k'}-g)\to 0$ almost everywhere. For such x, there is a subsequence m_k'' of m_k' with $A\Phi_{m_k''}\to g$ for almost all t. It follows that $g=A\Phi$ for almost all (x,t), $t\geqslant\delta$. Since δ is arbitrary, the proof is complete.

It remains only to prove the weak-type conclusion of theorem 3 for $p=1$. Although $p=1$ is not considered in [2], it is easy to check that theorem 3 for $p=1$ is a corollary of (3.3) and the case $p=2$ of theorem 3. We omit the proof.

Remark. In case Ω is bounded and $0<\alpha<1$ or Ω is bounded and even and $1\leqslant\alpha<2$, theorem 3 has a stronger conclusion due to Strichartz [12]. In fact, the conclusion remains true if we replace $\mathscr{F}(x,t)$ by

$$t^{-\beta}\int_{|z|<t} |f(x-z)-f(x)|\frac{dz}{|z|^{n-\beta}}, \qquad 0<\alpha<1,$$

and by

$$t^{-\beta}\int_{|z|<t} |f(x+z)+f(x-z)-2f(x)|\frac{dz}{|z|^{n-\beta}}, \qquad 1\leqslant\alpha<2.$$

The method we followed in proving theorem 3 has the same general outline as that used by Strichartz.

§ 4. Proof of theorems 4 and 6. One can obtain theorem 4 as a corollary of theorem 4 of [2]. However, since theorem 4 of [2] does not apply directly to theorem 6, we will follow a method which can be applied to either theorem 4 or theorem 6.

For $f\in L^2$,

$$(Bf)(x,t) = \int [f(x-z)-f(x)]K(z,t)\,dz,$$

where $K(z,t)=t^{-\alpha-\beta}|z|^{\beta-n}\Omega(z')$ for $|z|<t$ and $K(z,t)=0$ otherwise, belongs to L^2 as a function of x for each fixed $t>0$. Its Fourier transform with respect to x is

$$(4.1)\quad (Bf)(\hat{x},t) = \hat{f}(x)k(x,t), \quad \text{where} \quad k(x,t)=\int K(z,t)[e^{-i(x\cdot z)}-1]\,dz.$$

$$(X^2T^2Bf)^2 = \int_{E_n}\int_0^\infty |(Bf)(x,t)|^2\frac{dt}{t}\,dx = \int_0^\infty \frac{dt}{t}\int_{E_n} |(Bf)(\hat{x},t)|^2\,dx$$
$$= \int_{E_n} |\hat{f}(x)|^2 (T^2k(x,t))^2\,dx.$$

Since $T^2k(x,t)$ positively homogeneous of degree a in $x = |x|x'$,

$$(X^2T^2Bf)^2 = \int |\hat{f}(x)|^2 |x|^{2a} \big(T^2k(x',t)\big)^2 dx. \tag{4.2}$$

We now observe that $\big(T^2k(x',t)\big)^2$ is bounded if $\Omega \in L(\Sigma)$ and is orthogonal to polynomials of degree 1 when $1 \leqslant a < 2$. For

$$|k(x',t)| \leqslant 2t^{-a-\beta} \int_{|z|<t} \frac{|\Omega(z')|}{|z|^{n-\beta}} dz = O(t^{-a})$$

and therefore

$$\int_1^\infty |k(x',t)|^2 \frac{dt}{t} \leqslant c \int_1^\infty \frac{dt}{t^{2a+1}} < \infty.$$

For $0 < a < 1$, $e^{-i(x'\cdot z)} - 1 = O(|z|)$ and

$$|k(x',t)| \leqslant ct^{-a-\beta} \int_{|z|<t} \frac{|\Omega(z')|}{|z|^{n-\beta-1}} dz = O(t^{1-a})$$

and

$$\int_0^1 |k(x',t)|^2 \frac{dt}{t} \leqslant c \int_0^1 \frac{dt}{t^{2a-1}} < \infty.$$

If $1 \leqslant a < 2$, we may replace $e^{-i(x'\cdot z)} - 1$ in the integrand of k by $e^{-i(x'\cdot z)} - 1 + i(x'\cdot z) = O(|z|^2)$ and argue in the same way.

If Ω is a spherical harmonic of degree m, $\Omega = Y_m$, $m \neq 1$ when $1 \leqslant a < 2$, then

$$k(x',t) = \begin{cases} t^{-a-\beta} \displaystyle\int_{|z|<t} \frac{Y_m(z')}{|z|^{n-\beta}} e^{-i(x'\cdot z)} dz, & m \neq 0, \\ t^{-a-\beta} \displaystyle\int_{|z|<t} \frac{Y_0(z')}{|z|^{n-\beta}} [e^{-i(x'\cdot z)} - 1] dz, & m = 0. \end{cases}$$

Changing to polar coordinates $z = \varrho z'$, $\varrho = |z|$, and applying the formula ([6], p. 247 and p. 178)

$$\int_\Sigma Y_m(z') e^{-is(x'\cdot z')} dz' = i^m (2\pi)^\gamma \frac{J_{m+\gamma}(s)}{s^\gamma} Y_m(-x'),$$

where $\gamma = (n-2)/2$ and J_ν is the Bessel function of order ν, we obtain

$$k(x',t) = c_m \mu_m(t) Y_m(x')$$

where, for $m \geqslant 1$,

$$\mu_m(t) = t^{-a-\beta} \int_0^t \varrho^{\beta-\gamma-1} J_{m+\gamma}(\varrho) d\varrho$$

and

$$\mu_0(t) = t^{-a-\beta} \int_0^t \varrho^{\beta-1} \left[\frac{J_\gamma(\varrho)}{\varrho^\gamma} - \frac{1}{2^\gamma \Gamma(\gamma+1)} \right] d\varrho.$$

We observe that the coefficients c_m and the multipliers $\mu_m(t)$ depend only on the degree of Ω and not on Ω itself. Hence since $T^2k(x',t)$ is finite ($m \neq 1$ when $1 \leqslant a < 2$),

$$\big(T^2k(x',t)\big)^2 = C_m Y_m^2(x'). \tag{4.3}$$

For $f \in L_a^2$,

$$(X^2T^2Bf)^2 = (Bf, Bf) = \langle B^*Bf, f \rangle,$$

where $\langle\,,\,\rangle$ is the inner product between L_{-a}^2 and L_a^2. Taking successively for Y_m each element $Y_{m,j}$ of an orthonormal basis for the spherical harmonics of a fixed degree m, $m \neq 1$ if $1 \leqslant a < 2$, denoting by $B_j = B_{m,j}$ the corresponding operator and adding over j, we obtain

$$\Big\langle \sum B_j^* B_j f, f \Big\rangle = C_m \int |\hat{f}(x)|^2 |x|^{2a} dx$$

from (4.2), (4.3) and the fact that $\sum_j Y_{m,j}^2$ is a constant depending on m ([6], p. 243). Since $B_j^* B_j$ is self-adjoint, we may polarize this identity to get

$$\Big\langle \sum B_j^* B_j f, g \Big\rangle = C_m \int \hat{f}(x) \bar{\hat{g}}(x) |x|^{2a} dx$$

for $f, g \in L_a^2$.

Lemma 1. *For $\delta > 0$,*

$$(1+|x|^2)^{\delta/2} = |x|^\delta d\hat{\sigma} + d\hat{\tau},$$

where $d\hat{\sigma}$ and $d\hat{\tau}$ denote Fourier transforms of finite measures $d\sigma$ and $d\tau$.

See [10], p. 103.

If $[\,,\,]$ denotes the inner product in L^2, Parseval's formula and Lemma 1 give

$$\begin{aligned}[J^{-a}f, J^{-a}g] &= \int \hat{f}(x)\bar{\hat{g}}(x)(1+|x|^2)^{2a/2} dx \\ &= \int \hat{f}(x)\hat{g}(x)|x|^{2a} d\hat{\sigma} dx + \int \hat{f}(x)\bar{\hat{g}}(x) d\hat{\tau} dx\end{aligned}$$

for $f, g \in L_a^2$. Incorporating the constant C_m in $d\sigma$, we obtain for such f and g

$$[J^{-a}f, J^{-a}g] = \Big[J^a \sum B_j^* B_j (f * d\sigma), J^{-a}g\Big] + [J^a(f * d\tau), J^{-a}g]$$

and since $g \in L_a^2$ is arbitrary,

$$J^{-a}f = J^a \sum B_j^* B_j (f * d\sigma) + J^a (f * d\tau) \tag{4.4}$$

for $f \in L_a^2$.

If we consider $B = B_p$ as an operator from L^p_a to X^pT^2, its adjoint B^*_p is bounded from $X^{p'}T^2$ to $L^{p'}_{-a}$, $1/p+1/p' = 1$. Thus in (4.4), $B^*_jB_j = B^*_{j,2}B_{j,2}$ and we note that for $f \in L^p_a \cap L^2_a$, $B^*_2B_2f = B^*_{p'}B_pf$. For if $U(x,t)$ is infinitely differentiable and has compact support in $E_n \times (0, \infty)$, it is easy to see that

$$B^*_2U = B^*_{p'}U = \iint [U(x-z,t) - U(x,t)]K(-z,t)\,dz\frac{dt}{t}.$$

By approximating, $B^*_2U = B^*_{p'}U$ for $U \in X^2T^2 \cap X^pT^2$. But if $f \in L^p_a \cap L^2_a$, then $B_2f = B_pf \in X^2T^2 \cap X^pT^2$.

In particular, if $f \in L^p_a \cap L^2_a$, $1 < p < \infty$, then $B^*_{j,2}B_{j,2}(f*d\sigma) = B^*_{j,p'}B_{j,p}(f*d\sigma)$ in (4.4) and taking L^p-norms,

$$\|f\|_{p,a} = \|J^{-a}f\|_p \leqslant c\sum X^pT^2B_j(f*d\sigma) + \|f*d\tau\|_{p,-a}.$$

For fixed t, $B_j(f*d\sigma) = (B_jf)*d\sigma$ by (4.1) so that

$$X^pT^2B_j(f*d\sigma) \leqslant c_\sigma X^pT^2B_jf$$

by Young's theorem. Therefore, for $f \in L^p_a \cap L^2_a$,

$$\|f\|_{p,a} \leqslant c\Big[\sum_j X^pT^2B_jf + \|f\|_p\Big]. \tag{4.5}$$

It follows from Theorem 3 and an approximation argument that (4.5) holds for any $f \in L^p_a$.

Finally suppose $f \in L^p$, $1 < p < \infty$, and each $X^pT^2B_jf < +\infty$, where $\{Y_{m,j}\}$ is a normalized basis for the spherical harmonics of degree m, $m \neq 1$ when $1 \leqslant a < 2$. To prove $f \in L^p_a$, let $f_\delta = f*\Phi_\delta$, where $\Phi_\delta(x) = \delta^{-n}\Phi(x/\delta)$, $\delta > 0$, $\Phi \geqslant 0$, is a rapidly decreasing function (i.e., a member of the Schwartz space of testing functions) and $\int \Phi dx = 1$. Then $f_\delta \in L^p_a$ for each $\delta > 0$ and

$$(B_jf_\delta)(x,t) = \int (B_jf)(x-y,t)\Phi_\delta(y)\,dy$$

for almost all (x,t), so that

$$X^pT^2B_jf_\delta \leqslant (X^pT^2B_jf)\int \Phi_\delta(y)\,dy = X^pT^2B_jf.$$

By (4.5), $\|f_\delta\|_{p,a}$ is uniformly bounded in $\delta > 0$ by

$$c\Big[\sum X^pT^2B_jf + \|f\|_p\Big].$$

The fact that $f \in L^p_a$ and $\|f\|_{p,a}$ is bounded by the expression above follows exactly as in [15], section 2.

The proof of Theorem 6 is practically the same as that of Theorem 4 and we shall be brief. Since $\Lambda(a,2,2) = L^2_a$ and $T^2X^2 = X^2T^2$, (4.4) is true for $f \in \Lambda(a,2,2)$. If we consider $B = B_{pq}$ as an operator from $\Lambda(a,p,q)$ to T^qX^p, its adjoint B^*_{pq} is bounded from $T^{q'}X^{p'}$ to $\Lambda(-a,p',q')$, $1/p+1/p' = 1$, $1/q+1/q' = 1$, $1 < p, q < \infty$, these being the dual spaces of T^qX^p and $\Lambda(a,p,q)$ resp. (see [14]). Moreover, for $f \in \Lambda(a,p,q) \cap \Lambda(a,2,2)$, $B^*_{22}B_{22}f = B^*_{p'q'}B_{pq}f$ by the same kind of argument earlier. In particular, taking $\Lambda(0,p,q)$-norms in (4.4),

$$\begin{aligned}\|f\|_{a,p,q} &\leqslant c\sum T^qX^pB_j(f*d\sigma) + \|f*d\tau\|_{-a,p,q} \\ &\leqslant c\Big[\sum T^qX^pB_jf + \|f\|_p\Big]\end{aligned} \tag{4.6}$$

by [13], p. 437. A simple approximation argument and Theorem 5 show that (4.6) holds for any $f \in \Lambda(a,p,q)$.

Suppose now that $f \in L^p$, $1 < p < \infty$, and each $B_jf \in T^qX^p$, $1 < q < \infty$, where $\{Y_{m,j}\}$ is a normalized basis for the spherical harmonics of degree m, $m \neq 1$ when $1 \leqslant a < 2$. Let $f_\delta(x) = f(x,\delta)$ be the Poisson integral of f. Then $f_\delta \in \Lambda(a,p,q)$ for each $\delta > 0$ and since

$$(B_jf_\delta)(x,t) = \int (B_jf)(x-z,t)\frac{\delta}{(\delta^2+|z|^2)^{(n+1)/2}}\,dz,$$

we have

$$T^qX^p(B_jf_\delta) \leqslant T^qX^p(B_jf), \qquad \delta > 0.$$

Theorem 6 now follows from (4.6) and [13], p. 426.

That Theorem 6 is true for $1 \leqslant p, q \leqslant \infty$ when $m = 0$ (that is, when $\Omega = 1$) was known by Stein and Taibleson, at least in the case $\beta = n$. For arbitrary $\beta > 0$, we use a method similar to that used in proving Theorem 4, p. 421, of [13]. If $f \in L^p$, $1 \leqslant p \leqslant \infty$,

$$f_{yy}(x,y) = \int f(x-z)P_{yy}(y,z)\,dz = \int [f(x-z)-f(x)]P_{yy}(y,z)\,dz,$$

where $P(y,z) = y(y^2+|z|^2)^{-(n+1)/2}$ is the Poisson kernel. Letting

$$G_x(t) = \int_{|z|<t} [f(x-z)-f(x)]\frac{dz}{|z|^{n-\beta}}$$

and changing to polar coordinates,

$$\begin{aligned}f_{yy}(x,y) &= \int_0^\infty t^{n-\beta}P_{yy}(y,t)\,dG_x(t) \\ &= t^{n-\beta}P_{yy}(y,t)G_x(t)\Big|_{t=0}^\infty - \int_0^\infty G_x(t)\frac{d}{dt}\big(t^{n-\beta}P_{yy}(y,t)\big)\,dt.\end{aligned}$$

Assuming for the time being that the integrated term is zero,

$$\|f_{yy}(x,y)\|_p \leqslant c\int_0^\infty \|G_x(t)\|_p\frac{yt^{n-\beta-1}}{(y^2+t^2)^{(n+3)/2}}\,dt.$$

If $q = \infty$,

$$\|f_{yy}(x, y)\|_p \leqslant cT^\infty X^p\left(t^{-\alpha-\beta}G_x(t)\right)\int_0^\infty \frac{yt^{n+\alpha+1}}{(y^2+t^2)^{(n+3)/2}}\,dt \leqslant cy^{\alpha-2}(T^\infty X^p Bf),$$

$$Bf = t^{-\alpha-\beta}G_x(t).$$

If $q < \infty$, then

$$\left[\int_0^\infty \left(y^{2-\alpha}\|f_{yy}(x, y)\|_p\right)^q \frac{dy}{y}\right]^{1/q} \leqslant c\left[\int_0^\infty \left(y^{-n-\alpha}\int_0^y \|G_x(t)\|_p t^{n-\beta-1}\,dt\right)^q \frac{dy}{y}\right]^{1/q} +$$

$$+ c\left[\int_0^\infty \left(y^{3-\alpha}\int_y^\infty \|G_x(t)\|_p t^{-\beta-4}\,dt\right)^q \frac{dy}{y}\right]^{1/q} \leqslant cT^q\left(t^{-\alpha-\beta}\|G_x(t)\|_p\right) = cT^qX^pBf,$$

by Lemma 1 of section 2.

To show the integrated term is zero for almost all x, it is enough to show that

$$\frac{t^{n-\beta}}{(y^2+t^2)^{(n+3)/2}}\int_{|z|<t}|f(x-z)|\frac{dz}{|z|^{n-\beta}}\Bigg|_{t=0}^{\infty} = 0$$

almost everywhere, since

$$\int_{|z|<t}|f(x)|\frac{dz}{|z|^{n-\beta}} \leqslant c|f(x)|t^\beta \quad \text{and} \quad |P_{yy}(t, y)| \leqslant y/(y^2+t^2)^{(n+3)/2}.$$

If $p = \infty$, then

$$\int_{|z|<t}|f(x-z)|\frac{dz}{|z|^{n-\beta}} = O(t^\beta)$$

and we are done. For large t and $f\epsilon L^p$, $1 \leqslant p < \infty$,

$$\int_{|z|<t}|f(x-z)|\frac{dz}{|z|^{n-\beta}} = \int_{|z|<1} + \int_{1<|z|<t} \leqslant M_x + t^\beta \int_{1<|z|<t}|f(x-z)|\frac{dz}{|z|^n},$$

where M_x is finite almost everywhere by Young's theorem and

$$\int_{|z|>1}|f(x-z)|\frac{dz}{|z|^n} < \infty$$

by Hölder's inequality. Hence the integrated term is zero at $t = \infty$ almost everywhere. For small t and $1 \leqslant p < \infty$,

$$\int_{|z|<t}|f(x-z)|\frac{dz}{|z|^{n-\beta}} \leqslant \int_{|z|<1}|f(x-z)|\frac{dz}{|z|^{n-\beta}} < \infty$$

almost everywhere. Hence the integrated term is zero at $t = 0$ if $n-\beta > 0$. If $n = \beta$, then

$$\int_{|z|<t}|f(x-z)|\,dz = o(1).$$

If $n-\beta < 0$, then

$$\int_{|z|<t}|f(x-z)|\frac{dz}{|z|^{n-\beta}} \leqslant t^{\beta-n}\int_{|z|<t}|f(x-z)|\,dz = o(t^{\beta-n}).$$

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