To see that I^{-1} is continuous, let B be open X/E. Then $I(B) = u(p^{-1}(B))$. Now $p^{-1}(B) = [B]$ is a collection of cosets in X and is open; we want to show $I(B) = u([B])$ is open. Suppose $x_n \in F$, $x_n \xrightarrow{I} x \in u([B])$. Because $x_n \to x$ type I, there is an $f \neq 0$ in $\mathscr{L}$ such that $\varrho_f[(x_n, f), (x, f)] \to 0$ and since $x \in u([B])$, $(x, f) \in [B]$, thus (x_n, f) is eventually in $[B]$ and x_n is eventually in $u([B])$. Thus $u([B]) = I(B)$ is open in F.

Conclusion. Some of the unresolved questions with respect to the sequential topology for type I convergence are as follows. First what is the connection between convergence in the topology and type II convergence. If $x_n \xrightarrow{II} x$ and the regularizing sequence f_n, such that $f_n \xrightarrow{\mathscr{C}} f \neq 0$ and $f_n x_n \xrightarrow{\mathscr{C}} fx$, can be chosen so that $f \in \mathscr{C}_0$, then Theorem 4 shows that in fact $x_n \to x$ in the topology. If a regularizing sequence with $f \in \mathscr{C}_0$ can be chosen, then, in particular, $\overline{\lim}\, \alpha(x_n) \leqslant \alpha(x)$. A reasonable conjecture is that if $x_n \xrightarrow{II} x \neq 0$, then $x_n \to x$ if and only if $\overline{\lim}\, \alpha(x_n) \leqslant \alpha(x)$.

If O is such that $O \cap B_f$ is open in B_f for each $f \in \mathscr{L} - \{0\}$, then O is open in F. An unresolved question is: if V is such that $V \cap B_f$ contains an open neighborhood of the origin in B_f for each $f \in \mathscr{L} - \{0\}$ does V necessarily contain an open neighborhood of the origin in F?

Is F Hausdorff?

References

[1] T. Boehme, *On sequences of continuous functions and convolution*, Studia Math. 25 (1965), p. 333-335.

[2] — *Concerning convolution on the half-line*, Archive for Rat. Mech. Anal. 10 (1962), p. 220-228.

[3] — *Linear S-spaces*, Proceedings on the Symposium on Convergence Structures, Univ. of Oklahoma, 1965.

[4] R. Dudley, *On sequential convergence*, Trans. Amer. Math. Soc. 112 (1964), p. 483-507.

[5] S. P. Franklin, *Spaces in which sequences suffice*, Coll. Math. 57 (1965), p. 107-116.

[6] J. Kisyński, *Convergence du type L*, ibidem 7 (1959), p. 205-211.

[7] D. Norris, *A topology for Mikusiński operators*, Studia Math. 24 (1964), p. 245-255.

[8] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*, Paris 1966.

[9] K. Urbanik, *Sur la structure non-topologique du corps des opérateurs*, Studia Math. 14 (1954), p. 243-246.

UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIF.

Reçu par la Rédaction le 23. 2. 1968

STUDIA MATHEMATICA T. XXXIII. (1969)

Reflexivity and summability: the Nakano $l(p_i)$ spaces

by

D. WATERMAN*, T. ITO, F. BARBER, and J. RATTI (Detroit)

1. A classical theorem of Banach and Saks asserts that every bounded sequence in L_p, $p > 1$, has a subsequence whose $(C, 1)$ means converge. Nishiura and Waterman [4] showed that a Banach space is reflexive if and only if, for every bounded sequence, there is a summability method T of a particular kind and a subsequence whose T-means converge (either weakly or strongly). This has been discussed further by Singer [7], Pełczyński [5], and Waterman [8].

In his review [6] of the paper of Nishiura and Waterman, Sakai raised the following question: Is there a reflexive space for which $(C, 1)$ is not the suitable method? Klee [1] attempted to answer this and showed that certain $l(p_i)$-spaces of Nakano contained bounded sequences with no $(C, 1)$ summable subsequences. In section 2 we will show that these spaces exhibit a more striking property, namely that, for any regular method T or any regular* method T^* of Zygmund [10], p. 202-205, there exists a bounded sequence without T (T^*)-summable subsequences. However, as we will show in section 3, it is precisely these $l(p_i)$-spaces which are *not* reflexive. Thus the question of Sakai remains unanswered. The result in section 3 was stated in our review [9] of [1].

2. Let $\{p_i\}$ be a sequence of real numbers, $1 \leqslant p_i \leqslant \infty$. Then $l(p_i)$ denotes the set of all real sequences $x = \{t_i\}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{p_i} |at_i|^{p_i} < \infty$$

for some $a > 0$ depending on x. We adopt the convention that, for a function f of a finite real variable, the value at ∞ is given by

$$f(\infty) = \lim_{u\to\infty} f(u).$$

* Supported in part by National Science Foundation Grant GP 7358.

Then $l(p_i)$ is a linear space and if we define the modular

$$m(x) = \sum_{1}^{\infty} \frac{1}{p_i} |t_i|^{p_i}$$

and

$$\|x\| = \inf\{1/a : a > 0 \text{ and } m(ax) \leqslant 1\},$$

then $l(p_i)$ is a Banach space with $\|\cdot\|$ as norm ([2] and [3], § 89).

A useful observation, which is easily verified, is that $\|x\| \leqslant 1$ if and only if $m(x) \leqslant 1$.

We will now suppose, as did Klee [1], that $\limsup p_i = \infty$. Let (c_{mn}) be a regular* summability method, that is,

(i) $\lim\limits_{m\to\infty} \sum\limits_{n} c_{mn} = 1$,

(ii) $\lim\limits_{m\to\infty} c_{mn} = 0$ for every n.

Let $\{N_m\}$ be an increasing sequence of integers such that

$$\Big|\sum_{n=N}^{\infty} c_{mn}\Big| < \tfrac{1}{2}$$

if $N \geqslant N_m$ and set

$$K_{mn} = \begin{cases} 1 & \text{for } n \geqslant N_m, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{p_{i_k}\}$ be a subsequence of $\{p_i\}$ such that $p_{i_k} \geqslant k$ for every k. We now define

$$a_m = \begin{cases} p_m^{1/p_m} & \text{for } m \in \{i_k\}, \\ 0 & \text{otherwise,} \end{cases}$$

and let $y = \{a_m\}$, $y_n = \{K_{mn} a_{mn}\}$. Then $y_n \to y$ coordinatewise. Also

$$m(y/2) = \sum_{1}^{\infty} 2^{-p_{i_k}} \leqslant 1,$$

implying $y \in l(p_i)$ with $\|y\| \leqslant 2$ and $y_n \in l(p_i)$ with $\|y_n\| \leqslant 2$. Let $\{y_{n_a}\}$ be a subsequence of $\{y_n\}$. We will show that the means of $\{y_{n_a}\}$ do not converge to the coordinatewise limit y.

If $\{t_m\}$ is the sequence of (c_{mn})-means of $\{y_{n_a}\}$, then

$$y - t_m = \Big\{a_i \Big(1 - \sum_{a=1}^{\infty} K_{in_a} c_{ma}\Big)\Big\}$$

and, for $\lambda > 0$,

$$m\big((y-t_m)/\lambda\big) = \sum_{k=1}^{\infty} \Big(|1 - \sum_{a=1}^{\infty} K_{i_k n_a} c_{ma}|/\lambda\Big)^{p_{i_k}}.$$

Consider the term in this sum with $i_k = M$. We have

$$\sum_{a=1}^{\infty} K_{Mn_a} c_{ma} = \sum_{a\in A} c_{ma},$$

where $A = \{a : n_a \geqslant N_M\}$. If M is chosen so large that $n_a \geqslant N_M$ implies $a > N_m$, then

$$\Big|\sum_{a=1}^{\infty} K_{Mn_a} c_{ma}\Big| < \tfrac{1}{2}.$$

Thus

$$\Big(\Big|1 - \sum_{a=1}^{\infty} K_{Mn_a} c_{ma}\Big|/\lambda\Big)^{p_M} > (1/2\lambda)^{p_M},$$

and

$$1 \geqslant m\big((y-t_m)/\lambda\big) > (1/2\lambda)^{p_M}$$

requires $\lambda > \frac{1}{2}$, implying

$$\|y - t_m\| > \tfrac{1}{2}.$$

3. We shall now establish the following result:

THEOREM. *The space $l(p_i)$ is reflexive if and only if*

$$1 < \liminf p_i \leqslant \limsup p_i < \infty.$$

Our proof of this result requires two lemmas. In the following, the symbol $\cong$ will denote isomorphism, *not* isometry.

LEMMA 1. *If $\limsup p_i = \infty$, then $l(p_i)$ contains a subspace isomorphic to l^∞.*

Proof. Without loss of generality, we may assume that $p_i \geqslant i$ for all i. Then if $x \in l(p_i)$ and $\|x\| \leqslant 1$, we have

$$\|\tfrac{1}{2}x\|_\infty = \sup_i |\tfrac{1}{2}t_i| \leqslant 1,$$

for if, for some N, $|\frac{1}{2}t_N| > 1$, then

$$\sum \frac{1}{p_i} |t_i|^{p_i} > \frac{1}{p_N} 2^{p_N} > 1,$$

implying $m(x) > 1$, a contradiction.

Conversely, if $x \in l^\infty$ and $\|x\|_\infty < \frac{1}{2}$, then

$$m(x) \leqslant \sum \frac{1}{i} \|x\|_\infty^i \leqslant \sum \|x\|_\infty^i = \|x\|_\infty/(1 - \|x\|_\infty) < 1.$$

It is clear then that, as vector spaces, $l(p_i)$ and l^∞ are identical and, further,

$$\tfrac{1}{2}\|x\| \leqslant \|x\|_\infty \leqslant 2\|x\|$$

for all x since, for $x \neq (0, 0, \ldots)$,

$$m\left(\frac{x}{2\|x\|_\infty}\right) = \sum \frac{1}{p_i}\left|\frac{t_i}{2\sup t_n}\right|^{p_i} \leqslant 1,$$

implying $2\|x\|_\infty \geqslant \|x\|$ and similarly,

$$m\left(\frac{2x}{\|x\|_\infty}\right) > 1,$$

implying $2\|x\| \geqslant \|x\|_\infty$. Thus $l(p_i) \cong l^\infty$.

LEMMA 2. *If* $\limsup p_i < \infty$, *then* $l(p_i)^* \cong l(q_i)$, *where* $1/p_i + 1/q_i = 1$.

Proof. Let $1/p + 1/q = 1$, $1 \leqslant p < \infty$. Then

$$|ts| \leqslant \frac{1}{p}|t|^p + \frac{1}{q}|s|^q$$

with equality if and only if $t = |s|^{q-1}$. Thus, for every s,

$$\frac{1}{q}|s|^q = \sup\left\{ts - \frac{1}{p}|t|^p : -\infty < t < \infty\right\}.$$

Let $y = \{s_i\} \in l(q_i)$, $x = \{t_i\} \in l(p_i)$. Then, letting $\overline{m}$ denote the modular corresponding to $l(q_i)$,

$$\sum |t_i s_i| \leqslant \sum \frac{1}{p_i}|t_i|^{p_i} + \sum \frac{1}{q_i}|s_i|^{q_i} = m(x) + \overline{m}(y).$$

There are $\alpha, \beta > 0$ such that $m(\alpha x)$ and $\overline{m}(\beta y)$ are finite, implying

$$\sum \alpha\beta |t_i s_i| \leqslant m(\alpha x) + \overline{m}(\beta y) < \infty$$

or

$$\sum |t_i s_i| < \infty.$$

Thus, for every $y \in l(q_i)$, we can define a linear functional f_y on $l(p_i)$ by

$$f_y(x) = \sum t_i s_i.$$

If $h = f_{y/\|y\|} = f_y/\|y\|$, then

$$|h(x)| \leqslant m(x) + \overline{m}(y/\|y\|)$$

implying

$$\|h\| = \sup\{|h(x)| : \|x\| \leqslant 1\} \leqslant 2$$

and so

$$\|f_y\| \leqslant 2\|y\|.$$

Thus $y \to f_y$ is a continuous 1-1 mapping of $l(q_i)$ into $l(p_i)^*$.

Next we show that this mapping is onto. Suppose $f \in l(p_i)^*$ and $\|f\| \leqslant 1$. Let e^i denote the sequence which has 1 in the i-th place and 0 elsewhere and let $s_i = f(e^i)$. Then

$$\sum_1^N \frac{1}{q_i}|s_i|^{q_i} = \sup\left\{\sum_1^N t_i s_i - \sum_1^N \frac{1}{p_i}|t_i|^{p_i} : -\infty < t_i < \infty\right\}$$

$$\leqslant \sup\{f(x) - m(x) : x \in l(p_i)\} \leqslant 1.$$

To justify this last inequality, we consider two cases, first supposing $\|x\| \leqslant 1$. Then

$$f(x) - m(x) \leqslant \|f\| \cdot \|x\| - m(x) \leqslant \|x\| \leqslant 1.$$

It is clear that for $0 \leqslant a \leqslant 1$, $m(ax) \leqslant am(x)$. We also have $m(x/\|x\|) \geqslant 1$. Thus for $\|x\| > 1$,

$$\|x\| \leqslant \|x\| m(x/\|x\|) \leqslant m(x),$$

from which it follows that

$$f(x) - m(x) \leqslant 0.$$

We have then $y = \{s_i\} \in l(q_i)$ and $\overline{m}(y) \leqslant 1$.

We now show that $\{e^i\}$ generates a dense subset of $l(p_i)$, implying $f = f_y$ on $l(p_i)$. To that end we let $\|x\| \leqslant 1$; then $m(x) \leqslant 1$ and

$$\left\|x - \sum_1^M t_i e^i\right\| = \inf\left\{1/a : a > 0 \text{ and } \sum_{M+1}^\infty \frac{1}{p_i}|at_i|^{p_i} \leqslant 1\right\}.$$

There exist p_∞ and N_0 such that $p_i < p_\infty < \infty$ for $i > N_0$. For any $\varepsilon \in (0, 1)$, there is an $M_\varepsilon > N_0$ such that, for $M > M_\varepsilon$,

$$\sum_{M+1}^\infty \frac{1}{p_i}|t_i/\varepsilon|^{p_i} \leqslant (1/\varepsilon^{p_\infty}) \sum_{M+1}^\infty \frac{1}{p_i}|t_i|^{p_i} \leqslant 1.$$

Thus

$$\left\|x - \sum_1^M t_i e^i\right\| \leqslant \varepsilon$$

for $M > M_\varepsilon$.

Now we note that we have shown that $\|f_y\| = 1$ implies $\overline{m}(y) \leqslant 1$ and, therefore, $\|y\| \leqslant 1$. Thus, for any $f = f_y \in l(p_i)^*$, $\|f_y\| \geqslant \|y\|$, implying that the correspondence $y \leftrightarrow f_y$ is an isomorphism.

Proof of the theorem. If $l(p_i)$ is reflexive, then $\limsup p_i < \infty$ by lemma 1. Lemma 2 tells us that $l(p_i)^* \cong l(q_i)$, implying that $l(q_i)$ is reflexive. Then $\limsup q_i < \infty$ by lemma 1 and, therefore, $\liminf p_i > 1$.

Conversely, if $1 < \liminf p_i$ and $\limsup p_i < \infty$, then lemma 2 shows us that $l(p_i) \cong l(p_i)^{**}$ and, since this isomorphism is the natural imbedding, $l(p_i)$ is reflexive.

References

[1] V. Klee, *Summability in $l(p_1, p_2, \ldots)$-spaces*, Studia Math. 25 (1965), p. 277-280.

[2] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad. 27 (1951), p. 508-512.

[3] — *Topology and linear topological spaces*, Tokyo 1951.

[4] T. Nishiura and D. Waterman, *Reflexivity and summability*, Studia Math. 23 (1963), p. 53-57.

[5] A. Pełczyński, *A remark on the preceding paper of I. Singer*, ibidem 26 (1965), p. 115-116.

[6] S. Sakai, *Review of* [4], Math. Reviews 27 (1964), p. 974.

[7] I. Singer, *A remark on reflexivity and summability*, Studia Math. 26 (1965), p. 113-114.

[8] D. Waterman, *Reflexivity and summability II*, ibidem 32 (1969), p. 61-63.

[9] — *Review of* [1], Math. Reviews 33 (1967), p. 1359.

[10] A. Zygmund, *Trigonometric series*, Vol. I, Cambridge 1959.

WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN

Reçu par la Rédaction le 15. 4. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

Boundedness in certain topological linear spaces

by

B. A. BARNES* and A. K. ROY (Bombay)

1. Introduction. Throughout this paper we assume that $\{p_k\}$ is a sequence of real numbers such that $0 < p_k \leqslant 1$ for all $k \geqslant 1$. We also write this sequence as $\{p(k)\}$ when this is convenient. Several authors have considered the topological linear space $l(p_k)$ of complex sequences $\{b_k\}$ with the property that

$$\varrho(\{b_k\}) = \sum_{k=1}^{\infty} |b_k|^{p(k)} < +\infty,$$

where the function ϱ defines an invariant metric on $l(p_k)$ by $d(\{b_k\}, \{a_k\}) = \varrho(\{b_k - a_k\})$ (see [4] and the references of [4]). $l(p_k)$ is a complete metric linear space with this metric by [4], Lemma 1, p. 423. Most of the interest in the spaces $l(p_k)$ has been confined to the cases where $\inf p_k > 0$. Then $l(p_k)$ is a locally bounded topological linear space in its metric topology by [4], Theorem 6, p. 430. Also in this case a set is bounded if and only if it is bounded in metric by the same theorem. The space $l(p_k)$ has quite different topological properties when $\inf p_k = 0$. In this paper we investigate the bounded sets of $l(p_k)$ in the case $\lim p_k = 0$ and the weakly bounded sets in $l(p_k)$ with a slightly stronger assumption on $\{p_k\}$. Our results contrast sharply with those concerning boundedness and weak boundedness in the case $\inf p_k > 0$. We prove in Section 2 that if $\lim p_k = 0$, then a bounded set in $l(p_k)$ is always totally bounded. In Section 3, with a slightly stronger hypothesis on $\{p_k\}$, we prove that a weakly bounded set in $l(p_k)$ is always totally weakly bounded. The last section is devoted to the consideration of questions concerning boundedness with respect to k-pseudometrics.

After this paper was sent for publication, we learnt that S. Rolewicz had considered some of the matter presented here in an earlier paper [2].

* The research for this paper was done while this author was a visiting fellow at Tata Institute of Fundamental Research, Bombay.