X_M and Y_{M,p^*} a sequence of finite-dimensional operators (each of norm one) which tends pointwise to the identity operators of the spaces.

Combining Corollary 4.1 with Proposition 3.2 we obtain

PROPOSITION 4.2. *If* $\tilde{P}_M$ *is unbounded in* $L_{p^*}(T)$*-norm, then* $N_p(Y_{M,p^*}, X_M) \neq N_p^Q(Y_{M,p^*}, X)$ $(1 < p < +\infty)$.

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Multipliers and tensor products of L^p-spaces of locally compact groups*

by

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In an earlier paper concerned with induced representations [12] we introduced a definition of tensor product for Banach modules. Section 1 of the present paper contains general remarks on the relationship between multipliers of Banach modules and this definition of tensor product. In the following sections, motivated by theorems of Figà-Talamanca, Gaudry, Hörmander, and Eymard [5, 6, 10, 4] concerning multipliers of the L^p-spaces of locally compact groups, we give concrete representations as function spaces for the tensor products of these L^p-spaces, and we indicate how the theorems of the above-named authors can be reformulated in terms of these representations.

We would like to thank F. Greenleaf and L. Maté for several stimulating conversations about multipliers.

1. Multipliers and tensor products. Let A be a Banach algebra. By a *left (right) Banach A-module* we mean [12] a Banach space, V, which is a left (right) A-module in the algebraic sense, and for which

$$\|av\| \leqslant \|a\|\,\|v\| \quad \text{for all } a \in A \text{ and } v \in V.$$

If V and W are left (right) Banach A-modules, then $\mathrm{Hom}_A(V, W)$ will denote the Banach space of all continuous A-module homomorphisms from V to W with the operator norm. The elements of $\mathrm{Hom}_A(V, W)$ are traditionally called *multipliers* from V to W. If V is a left (right) Banach A-module, then V^*, the dual of V, is a right (left) Banach A-module under the adjoint action of A.

For completeness we include the definition of the tensor product of Banach modules which was introduced in [12]. Let V and W be respectively a left and right Banach A-module. Let $V \otimes_\gamma W$ denote the

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projective tensor product [8] of V and W as Banach spaces (so that γ is the greatest cross-norm [14]), and let K be the closed linear subspace of $V\otimes_\gamma W$ which is spanned by all the elements of the form

$$av\otimes w - v\otimes aw, \quad a\in A,\ v\in V,\ w\in W.$$

Then the *A-module tensor product*, $V\otimes_A W$, is defined to be the quotient Banach space $(V\otimes_\gamma W)/K$. Using the universal property of the projective tensor product with respect to bounded bilinear maps from $V\times W$, it is easily seen that $V\otimes_A W$ has the expected universal property with respect to A-balanced bounded bilinear maps from $V\times W$.

Now, with the situation as described in the previous paragraph, W^* is a left A-module, and it is not difficult to prove (2.12 and 2.13 of [12]), using the universal property just mentioned, that there is a natural isometric isomorphism

$$\text{(1.1)} \qquad \mathrm{Hom}_A(V, W^*) \cong (V\otimes_A W)^*,$$

under which the linear functional on $V\otimes_A W$ which corresponds to an operator $T\in\mathrm{Hom}_A(V, W^*)$ has value $\langle w, T(v)\rangle$ at the element $v\otimes w$ of $V\otimes_A W$. We thus obtain a representation of the space of multipliers from V to W^* as a dual Banach space, and this is the type of result sought by Figà-Talamanca [5] and Figà-Talamanca and Gaudry [6] for the special case of the L^p-spaces of locally compact groups. In fact, as we shall see in the next sections, some of their results can be interpreted as involving the construction of concrete representations as function spaces for the tensor products of these L^p-spaces.

But before we turn to this question, it will be useful to describe the topology on $\mathrm{Hom}_A(V, W^*)$ which corresponds to the weak*-topology on $(V\otimes_A W)^*$. Now it was shown by Grothendieck [8] (see also [13], p. 94) that every element, t, of $V\otimes_\gamma W$ has an absolutely convergent expansion of the form

$$\text{(1.2)} \qquad t = \sum_{i=1}^{\infty} v_i\otimes w_i, \quad v_i\in V,\ w_i\in W,$$

where

$$\sum_{i=1}^{\infty} \|v_i\|\,\|w_i\| < \infty.$$

It follows that every element of $V\otimes_A W$ also has such an expansion. Then the linear functional on $\mathrm{Hom}_A(V, W^*)$ which corresponds to $t\in V\otimes_A W$ has value

$$\sum_{i=1}^{\infty} \langle w_i, Tv_i\rangle$$

at $T\in\mathrm{Hom}_A(V, W^*)$, in terms of an expansion of type (1.2) for t. It is clear that the topology on $\mathrm{Hom}_A(V, W^*)$ defined by the linear functionals of this form is the topology which corresponds to the weak*-topology on $(V\otimes_A W)^*$. It is appropriate to call this topology the *ultraweak*-operator topology*, since it is easily verified that if W is reflexive, this topology is the same as the ultraweak operator topology as defined for example by Dixmier ([2], p. 35). In fact, from (1.1) one easily obtains the following generalization of Dixmier's result that a von Neumann algebra (defined in [2] as the commutant of a self-adjoint algebra of operators on a Hilbert space) is a dual Banach space.

1.3. Proposition. *If S is any set of operators on a reflexive Banach space, then the commutant of S is a dual Banach space.*

We summarize the results of this section as

1.4. Theorem. *Let V and W be respectively left and right A-modules. Then*

$$\mathrm{Hom}_A(V, W^*) \cong (V\otimes_A W)^*,$$

and the ultraweak-topology on $\mathrm{Hom}_A(V, W^*)$ corresponds to the weak*-topology on $(V\otimes_A W)^*$.*

2. The L^p-spaces of locally compact groups. Let G be a locally compact group. We choose a left Haar measure on G which will remain fixed throughout, and we let Δ denote the modular function for G. We let $L^p(G)$, $1\leqslant p\leqslant\infty$, denote the usual Lebesgue spaces with respect to left Haar measure on G. Then $L^1(G)$ is a Banach algebra under convolution, and $L^p(G)$ becomes a left $L^1(G)$-module when elements of $L^1(G)$ act on elements of $L^p(G)$ by convolution on the left.

We investigate first the adjoint action of $L^1(G)$ on the dual of $L^p(G)$. For any number p, $1\leqslant p\leqslant\infty$, we let p' denote the conjugate exponent defined by $1/p+1/p'=1$, and we define the dual pairing between $L^p(G)$ and $L^{p'}(G)$ by

$$\langle f, g\rangle = \int_G f(x)g(x)\,dx, \quad f\in L^p(G),\ g\in L^{p'}(G).$$

Then it is easily verified that

$$\langle \varphi * f, g\rangle = \langle f, \tilde\varphi * g\rangle$$

for $\varphi\in L^1(G)$, $f\in L^p(G)$, and $g\in L^{p'}(G)$, where $\tilde\varphi$ is defined by

$$\tilde\varphi(x) = \Delta(x^{-1})\varphi(x^{-1}).$$

Thus the adjoint action of an element φ of $L^1(G)$ on $L^{p'}(G)$, under which $L^{p'}(G)$ becomes a *right* $L^1(G)$-module, consists of convolution on the *left* by $\tilde\varphi$ (note that $\varphi\to\tilde\varphi$ is an isometric anti-automorphism of $L^1(G)$).

For any p we will let $\bar{L}^p(G)$ denote $L^p(G)$ viewed as a right $L^1(G)$-module with the adjoint action just described.

Then relation (1.1) applied to the $L^p(G)$ becomes

$$\mathrm{Hom}_G\big(L^p(G), L^{q'}(G)\big) \cong \big(L^p(G)\otimes_G \bar{L}^q(G)\big)^* \tag{2.1}$$

for $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$. (For simplicity we will always write $\otimes_G$ for $\otimes_{L^1(G)}$, and Hom_G for $\mathrm{Hom}_{L^1(G)}$.)

Thus to represent the elements of $\mathrm{Hom}_G\big(L^p(G), L^{q'}(G)\big)$ as elements of the dual of a concrete Banach function space, as is done in [5] and [6], it is sufficient to represent $L^p(G)\otimes_G \bar{L}^q(G)$ as such a function space. The following sections are devoted to the problem of obtaining such a representation [1].

We remark that relation (1.1) does not immediately apply to the case of $\mathrm{Hom}_G\big(L^p(G), L^1(G)\big)$ which is treated in [6], since $L^1(G)$ is not a dual space. Wendel [15] has shown that $\mathrm{Hom}_G\big(L^1(G), L^1(G)\big) \cong M(G) \cong \big(C_\infty(G)\big)^*$, where $C_\infty(G)$ is the space of continuous functions vanishing at infinity on G. If $p > 1$ and G is not compact, then Hörmander has shown ([10], Theorem 1.1) that $\mathrm{Hom}_G\big(L^p(G), L^1(G)\big)$ contains only the zero operator. Finally, if $1 < p < \infty$ and G is compact, then one can show that

$$\mathrm{Hom}_G\big(L^p(G), L^1(G)\big) \cong \mathrm{Hom}_G\Big(L^p(G), \big(L^1(G)\big)^{**}\Big),$$

and so (1.1) can be applied to obtain

$$\mathrm{Hom}_G\big(L^p(G), L^1(G)\big) \cong \big(L^p(G)\otimes_G \bar{L}^\infty(G)\big)^*,$$

thus again obtaining a representation for multipliers of the kind obtained in [6] (once we have given a concrete representation for $L^p(G)\otimes_G \bar{L}^\infty(G)$).

We turn now to the problem of representing $L^p(G)\otimes_G \bar{L}^q(G)$ as a function space. We can immediately dispatch with the case in which $p = 1$ or $q = 1$. For it follows from Theorem 4.4 of [12] that $L^1(G)\otimes_G \bar{L}^q(G) \cong L^q(G) \cong L^q(G)\otimes_G \bar{L}^1(G)$ if $1 \leqslant q < \infty$, and that $L^1(G)\otimes_G \bar{L}^\infty(G) \cong C_{lu}(G) \cong L^\infty(G)\otimes_G \bar{L}^1(G)$, where $C_{lu}(G)$ denotes the space of bounded left uniformly continuous functions on G. Thus we can assume hereafter that $p > 1$ and $q > 1$.

As suggested by the work of Hörmander, Figà-Talamanca and Gaudry, the situation for compact groups is somewhat different (and much simpler) than that for non-compact groups. For this reason we consider compact groups first.

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(1) Recently R. E. Harte sent us a preprint of a paper entitled *Tensor products of normed modules* in which he obtains similar representations for the tensor products of the L^p-spaces of measure spaces viewed as L^∞-modules under pointwise multiplication.

3. $L^p(G)\otimes_G \bar{L}^q(G)$ for compact G. Throught this section we assume that G is compact, and that Haar measure on G is normalized so that the measure of G is 1. The Banach space of continuous functions on G will be denoted by $C(G)$. We begin with a result which is essentially contained in [6].

3.1. PROPOSITION. *Let $f \in L^p(G)$ and $g \in L^q(G)$, where $1 \leqslant p, q \leqslant \infty$. Then $f*g$ is defined a.e., $f*g \in L^r(G)$, and*

$$\|f*g\|_r \leqslant \|f\|_p \|g\|_q,$$

where r is defined as follows:

(a) if $1/p+1/q > 1$, then $1/r = 1/p+1/q-1$;

(b) if $1/p+1/q \leqslant 1$, then $r = \infty$.

In fact, if $1/p+1/q \leqslant 1$, then $f*g \in C(G)$.

Proof. The case in which $1/p+1/q > 1$ is just a special case of Theorem 20.18 of [9]. If $1/p+1/q \leqslant 1$, then it is easily seen that $p' \leqslant q$. Since G is compact, $\|g\|_{p'} \leqslant \|g\|_q$, and so $g \in L^{p'}(G)$. Then, applying Theorem 20.16 of [9], we obtain $f*g \in C(G)$, and

$$\|f*g\|_\infty \leqslant \|f\|_p \|g\|_{p'} \leqslant \|f\|_p \|g\|_q.$$

We remark that for $1/p+1/q < 1$ this situation is in sharp contrast to that for noncompact groups. In fact, Rickert [11] has shown that if G is not compact and if $1/p+1/q < 1$, then $f \in L^p(G)$ and $g \in L^q(G)$ can be chosen so that $f*g$ is not even defined on a set of strictly positive measure.

In view of Proposition 3.1 we can define a bilinear map, b, from $L^p(G)\times \bar{L}^q(G)$ into $L^r(G)$ or $C(G)$ by

$$b(f, g) = \tilde{f}*g, \quad f \in L^p(G), g \in \bar{L}^q(G),$$

and $\|b\| \leqslant 1$. Then b lifts to a linear map, B, from $L^p(G)\otimes_\gamma \bar{L}^q(G)$ into $L^r(G)$ or $C(G)$, and $\|B\| \leqslant 1$. In analogy with definitions made in [5] and [6] we have

3.2. DEFINITION. The range of B, with the quotient norm, will be denoted by A_p^q.

Thus A_p^q is a Banach space of functions on G which can be viewed as a linear submanifold in $L^r(G)$ or $C(G)$. In view of the fact that every element of $L^p(G)\otimes_\gamma \bar{L}^q(G)$ has an expansion of the form (1.2), we see that A_p^q consists of exactly those functions, h, on G which have at least one expansion of the form

$$h = \sum_{i=1}^{\infty} \tilde{f}_i * g_i,$$

where $f_i \in L^p(G)$, $g_i \in L^q(G)$, and

$$\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty,$$

with the expansion converging in the norm of $L^r(G)$ or $C(G)$.

3.3. THEOREM. *If G is compact, $1 < p, q \leqslant \infty$, and either $p < \infty$ or $q < \infty$, then*

$$L^p(G) \otimes_G \bar{L}^q(G) \cong A_p^q,$$

the isomorphism being isometric.

Proof. It clearly suffices to show that the kernel of B is exactly K, the closed subspace of $L^p(G) \otimes_\gamma \bar{L}^q(G)$ spanned by elements of the form $(\varphi * f) \otimes g - f \otimes (\tilde{\varphi} * g)$, where $\varphi \in L^1(G), f \in L^p(G)$, and $g \in L^q(G)$. Now

$$B\big((\varphi * f) \otimes g\big) = (\varphi * f)^{\sim} * g = \tilde{f} * (\tilde{\varphi} * g) = B\big(f \otimes (\tilde{\varphi} * g)\big)$$

(b is $L^1(G)$-balanced), so that the kernel of B contains K.

Conversely, suppose that t is an element of the kernel of B. We will show that $t \in K$. Let

$$t = \sum_{i=1}^{\infty} f_i \otimes g_i$$

be an expansion of type (1.2) for t. Then the fact that t is in the kernel of B implies that

$$\sum_{i=1}^{\infty} \tilde{f}_i * g_i = 0,$$

where the sum converges absolutely in $L^r(G)$ or $C(G)$. Let $\{j_n\}$ be an approximate identity of norm 1 in $L^1(G)$ consisting of bounded functions, so that $j_n \in L^p(G)$ for each n (where n runs over a directed set). Suppose that $p < \infty$ (one has an analogous proof for the case in which $q < \infty$ instead). For each n define $t_n \in L^p(G) \otimes_\gamma \bar{L}^q(G)$ by

$$t_n = \sum_{i=1}^{\infty} (f_i * j_n) \otimes g_i.$$

Then $f_i * j_n$ converges to f_i for each i, since $p < \infty$, and from this it is easily seen that t_n converges to t. Since K is closed, it thus suffices to show that $t_n \in K$. But, given n and $\varepsilon > 0$, choose m_0 so that

$$\Big\|\sum_{i=1}^{m} \tilde{f}_i * g_i\Big\|_r < \varepsilon/2 \|j_n\|_p$$

whenever $m > m_0$. Choose $m_1 > m_0$ such that

$$\Big\|t_n - \sum_{i=1}^{m} (f_i * j_n) \otimes g_i\Big\| < \varepsilon/2$$

whenever $m > m_1$. Now

$$\sum_{i=1}^{m} (f_i * j_n) \otimes g_i$$

$$= \sum_{i=1}^{m} j_n \otimes (\tilde{f}_i * g_i) + \sum_{i=1}^{m} [(f_i * j_n) \otimes g_i - j_n \otimes (\tilde{f}_i * g_i)],$$

and it is clear that the second term on the right is in K. Furthermore, if $m > m_1$, then

$$\Big\|\sum_{i=1}^{m} j_n \otimes (\tilde{f}_i * g_i)\Big\| = \|j_n\|_p \Big\|\sum_{i=1}^{m} \tilde{f}_i * g_i\Big\|_q$$

$$\leqslant \|j_n\|_p \Big\|\sum_{i=1}^{m} \tilde{f}_i * g_i\Big\|_r < \varepsilon/2,$$

since it is easily seen that $r \geqslant q$. Thus the distance from t_n to K is less than ε for every $\varepsilon > 0$, and so $t_n \in K$.

We do not know whether a similar result is true for $L^\infty(G) \otimes_G \bar{L}^\infty(G)$.

4. G non-compact and $1/p + 1/q < 1$.

Throughout this section we assume that G is not compact. Suppose that $1/p + 1/q < 1$, $p < \infty$, $q < \infty$. Then $p > q'$, and Hörmander has shown ([10], Theorem 1.1) that in this case, since G is not compact, the only operator in $\mathrm{Hom}_G\big(L^p(G), L^{q'}(G)\big)$ is the zero operator. Then in view of (2.1), it follows that

4.1. THEOREM. *If G is not compact, and if $1/p + 1/q < 1$, $p < \infty$, $q < \infty$, then*

$$L^p(G) \otimes_G \bar{L}^q(G) = \{0\}.$$

Alternate proof. While this result immediately follows from the result of Hörmander mentioned above, we believe it may be of interest to include at least a sketch of a direct proof. What needs to be shown in this case is that K, the subspace of $L^p(G) \otimes_\gamma \bar{L}^q(G)$ spanned by elements of the form $(\varphi * f) \otimes g - f \otimes (\tilde{\varphi} * g)$, $\varphi \in L^1(G)$, $f \in L^p$, $g \in \bar{L}^q(G)$, is all of $L^p(G) \otimes_\gamma \bar{L}^q(G)$. To show this, it is easily seen that it suffices to show that for every $t \in L^p(G) \otimes_\gamma \bar{L}^q(G)$ the distance from t to K is less than $2^{(1/p)+(1/q)-1}\|t\|$. But, because $p < \infty$ and $q < \infty$, it is easily seen that to obtain this inequality it suffices to show

4.2. Lemma. *For any $f, g \in C_c(G)$ (the space of continuous functions of compact support) the distance from $f \otimes g$ to K is less than $2^{(1/p)+(1/q)-1}\|f \otimes g\|$.*

Proof. The proof uses the same type of device as is used in Hömander's theorem, namely that if f and xf have disjoint support, where xf denotes the left translate of f to the point $x \in G$, then

$$\|f+xf\|_p = 2^{1/p}\|f\|_p.$$

Let $\varepsilon > 0$ be given. Choose an integer m large enough so that $m^{(1/p)-1} < \varepsilon/(\|f\|_p\|g\|_q)$. Since G is not compact and f and g have compact support, we can find points $x_1, \ldots, x_m$ of G such that $f, x_1 f, \ldots, x_m f$ have disjoint support, and $g, x_1^{-1}g, \ldots, x_m^{-1}g$ have disjoint support. Then

$$\Big\| f\otimes g - (1/m)\sum_{i=1}^{m}(f-x_i f)\otimes g \Big\| = (1/m)\Big\|\sum_{i=1}^{m} x_i f\Big\|_p \|g\|_q$$
$$= m^{(1/p)-1}\|f\|_p\|g\|_q < \varepsilon.$$

But

$$(1/m)\sum_{i=1}^{m}(f-x_i f)\otimes g$$
$$= (1/2m)\sum_{i=1}^{m}(f-x_i f)\otimes(g-x_i^{-1}g)$$
$$+(1/2m)\sum_{i=1}^{m}[(f\otimes x_i^{-1}g - x_i f\otimes g)+(f\otimes g - x_i f\otimes x_i^{-1}g)].$$

By applying an approximate identity, we see that the second term on the right side of the equality is in K. Furthermore

$$\Big\|(1/2m)\sum_{i=1}^{m}(f-x_i f)\otimes(g-x_i^{-1}g)\Big\| \leqslant (1/2)2^{1/p}\|f\|_p 2^{1/q}\|g\|_q$$
$$= 2^{(1/p)+(1/q)-1}\|f\otimes g\|,$$

and the desired inequality follows.

Again we do not know whether a similar result is true if $p = \infty$ or $q = \infty$.

5. G non-compact and $1/p+1/q \geqslant 1$. Throughout this section we assume that $1/p+1/q \geqslant 1$, $1 < p < \infty$, and $1 < q < \infty$. We would like to prove a representation theorem for this case similar to Theorem 3.3. To do this it seems to be necessary, as suggested by the work of Figà-Talamanca and Gaudry, to know whether all the elements of $\mathrm{Hom}_G(L^p(G), L^{q'}(G))$ can be approximated by right convolution operators. To make this precise, we first note that $p \leqslant q'$. Then it is not difficult to prove

5.1. Proposition. *Let $1/p+1/q \geqslant 1$, and let $\varphi \in C_c(G)$. Define T_φ by $T_\varphi(f) = f*\varphi$, $f \in L^p(G)$. Then $T_\varphi \in \mathrm{Hom}_G(L^p(G), L^{q'}(G))$. In fact,*

$$\|T_\varphi\| \leqslant \|\tilde\varphi\|_q^{p/q'}\,\|(\Delta(\cdot))^{-1/p}\tilde\varphi\|_1^{1-(p/q')}.$$

The norm inequality follows from the Riesz convexity theorem ([3], VI. 10. 11).

5.2. Definition. A locally compact group G is said *to satisfy property P_p^q* (resp. Q_p^q) if every element of $\mathrm{Hom}_G(L^p(G), L^{q'}(G))$ can be approximated in the ultraweak operator topology (resp. boundedly in the strong operator topology) by operators of the form T_φ, $\varphi \in C_c(G)$.

It is easily seen that property Q_p^q implies property P_p^q. It does not seem to be known whether all groups satisfy property P_p^q. As far as we know, only the following results are known (2). Every Abelian group satisfies property P_p^q for all p and q. This follows from Theorem 1 of [6], where it is shown in fact that every Abelian group satisfies property Q_p^q (see also [1], Theorem 4.10). It is easily seen, using an appropriate approximate identity, that every compact group satisfies Q_p^q, and so P_p^q, for all p and q. Finally, every locally compact group G satisfies property P_2^2. This is shown by applying the Commutation Theorem for quasi-Hilbert algebras ([2], p. 69) to the quasi-Hilbert algebra for G as defined by Eymard ([4], p. 210). In fact, an application of the Kaplansky density theorem ([2], p. 46) then yields Q_2^2.

We will show that for any group satisfying property P_p^q there is a representation of $L^p(G)\otimes_G\bar L^q(G)$ similar to that of Theorem 3.3, and conversely. However, in defining the analogue of the bilinear map b used in Theorem 3.3, allowance must be made for the modular function. The appropriate analogue of Proposition 3.1 seems to be

5.3. Proposition. *Let $f \in L^p(G)$ and $g \in L^q(G)$, where $1/p+1/q \geqslant 1$, $1 \leqslant p, q < \infty$. Then $\tilde f * g$ is defined a.e., and $\|(\Delta(\cdot))^{1/p'}(f*g)\|_r \leqslant \|f\|_p\|g\|_q$, where $1/r = 1/p+1/q-1$. In fact, if $1/p+1/q = 1$, then*

$$(\Delta(\cdot))^{1/p'}(\tilde f * g) \in C_\infty(G).$$

Here $C_\infty(G)$ denotes the space of continuous functions vanishing at infinity on G. The proof is easily obtained by imitating the proofs of theorems 20.16 and 20.18 of [9].

Thus by Proposition 5.3 we can define a bilinear map, b, from $L^p(G)\times\bar L^q(G)$ into $L^r(G)$ or $C_\infty(G)$ by

$$b(f, g) = (\Delta(\cdot))^{1/p'}\tilde f * g,$$

(2) We have learned from A. Figà-Talamanca that C. S. Herz has recently shown that every amenable locally compact group satisfies property P_p^q.

and $\|b\| \leqslant 1$. Then b lifts to a linear map, B, from $L^p(G)\otimes_\gamma \bar{L}^q(G)$ into $L^r(G)$ or $C_\infty(G)$, and $\|B\| \leqslant 1$. In analogy with Definition 3.2 we have

5.4. Definition. The range of B, with the quotient norm, will be denoted by A_p^q [3].

Thus A_p^q is a Banach space consisting of those functions, h, on G which have at least one expansion of the form

$$h = \big(\Delta(\cdot)\big)^{1/p'} \sum_{i=1}^{\infty} \tilde{f}_i * g_i,$$

where $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$. As before, we let K denote the closed subspace of $L^p(G)\otimes_\gamma \bar{L}^q(G)$ spanned by elements of the form $(\varphi * f)\otimes g - f\otimes(\tilde{\varphi} * g)$, where $\varphi \in L^1(G), f\in L^p(G)$, and $g\in L^q(G)$.

5.5. Theorem. *Let $1/p+1/q \geqslant 1$. Then the following statements are equivalent*:

(a) *G satisfies property P_p^q.*

(b) *The kernel of B is K, so that $L_p(G)\otimes_G \bar{L}^q(G) \cong A_p^q$.*

Proof. As in Theorem 3.3, it is easily seen that K is always contained in the kernel of B (b is $L^1(G)$-balanced).

Suppose now that G satisfies property P_p^q. To show that the kernel of B is contained in K it suffices, by the Hahn-Banach theorem, to show that any bounded linear functional on $L^p(G)\otimes_\gamma \bar{L}^q(G)$ which annihilates K also annihilates the kernel of B. Let F be such a linear functional. Since F annihilates K, it is easily seen from (1.1) that there corresponds to F an operator $T\in \mathrm{Hom}_G\big(L^p(G), L^{q'}(G)\big)$ such that

$$\langle t, F\rangle = \sum_{i=1}^{\infty} \langle g_i, Tf_i\rangle$$

for any $t\in L^p(G)\otimes_\gamma \bar{L}^q(G)$ with expansion

$$t = \sum_{i=1}^{\infty} f_i \otimes g_i \tag{5.5}$$

of type (1.2).

Suppose now that t is in the kernel of B and has expansion (5.5). Then

$$\sum_{i=1}^{\infty} \big(\Delta(\cdot)\big)^{1/p'} (\tilde{f}_i * g_i) = 0,$$

the sum converging in the norm of $L^r(G)$. We wish to show that $\langle t, F\rangle = 0$, or equivalently, that

$$\sum_{i=1}^{\infty} \langle g_i, Tf_i\rangle = 0. \tag{5.6}$$

Now it is easily seen on examining the proof of theorem 6.4 of [13], p. 94, that in the expansion (5.5) for t the f_i and g_i can be chosen to be in $C_c(G)$. We assume that they are so chosen. The proof of (5.6) which now follows is just a reformulation of part of the proof of Theorem 2 of [6]. Since G is assumed to satisfy property P_p^q, there is a net $\{\varphi_j\}$, of elements of $C_c(G)$ such that the operators T_{φ_j} defined in Proposition 5.1 converge to T in the ultraweak operator topology. In particular,

$$\sum_{i=1}^{\infty} \langle g_i, f_i * \varphi_j\rangle \xrightarrow[j]{} \sum_{i=1}^{\infty} \langle g_i, Tf_i\rangle .$$

Thus to prove (5.6) it suffices to show that

$$\sum_{i=1}^{\infty} \langle g_i, f_i * \varphi_j\rangle = 0$$

for each j. But

$$\begin{aligned} \sum_{i=1}^{\infty} \langle g_i, \tilde{f}_i * \varphi_j\rangle &= \sum_{i=1}^{\infty} \langle \tilde{f}_i * g_i, \varphi_j\rangle \\ &= \Big\langle \sum_{i=1}^{\infty} \big(\Delta(\cdot)\big)^{1/p'} (\tilde{f}_i * g_i), \big(\Delta(\cdot)\big)^{-1/p'} \varphi_j\Big\rangle = 0, \end{aligned} \tag{5.7}$$

since $\big(\Delta(\cdot)\big)^{-1/p'} \varphi_j \in C_c(G)$ and so can be viewed as an element of $L^{r'}(G)$.

Suppose conversely that the kernel of B is K. To show that the operators of the form T_φ for $\varphi\in C_c(G)$ are dense in $\mathrm{Hom}_G\big(L^p(G), L^{q'}(G)\big)$ in the ultraweak operator topology, it is sufficient according to Theorem 1.4 to show that the corresponding functionals are dense in $\big(L^p(G)\otimes_G \bar{L}^q(G)\big)^*$ in the weak*-topology. But to show this it is sufficient to show that the annihilator of these functionals is $\{0\}$, or equivalently, if these functionals are viewed as functionals on $L^p(G)\otimes_\gamma \bar{L}^q(G)$, that their annihilator is K. But from (5.7) it is easily seen that the annihilator of these functionals is the kernel of B. Since our hypothesis is that the kernel of B is K, the proof is complete.

We remark that Eymard's theorem 3.10 [4] is essentially the analogue of our Theorem 1.3 and Theorem 5.5 for the case in which $L^2(G)$ is viewed as right and left $C_c(G)$-modules in which the action of $\varphi\in C_c(G)$ on $L^2(G)$ consists of convolution on the *right* by φ and $\tilde{\varphi}$, respectively. In particular, we have the identification

$$A(G) \cong L^2(G)\otimes_{C_c(G)} L^2(G),$$

(3) We have learned from A. Figà-Talamanca that C. S. Herz has shown that A_p^p is always a Banach *algebra* under pointwise multiplication. This is shown by making minor modifications to the prof for A_2^2 which he gave in *Remarque sur la note précédente de M. Varopoulos*, C. R. Acad. Sc. Paris 260 (1965), p. 6001-6004.

as seen from Theorem 3.4 E_{10} of [4], where $A(G)$ is Eymard's algebra of Fourier transforms.

We also remark that presumably the results of Gaudry in [7] can be interpreted as the identification

$$\mathrm{Hom}_{C_c(G)}\big(C_c(G), (C_c(G))^*\big) \cong \big(C_c(G)\otimes_{C_c(G)} C_c(G)\big)^*$$

(where tensor products must now be defined for modules which are locally convex topological vector spaces) together with a concrete representation of $C_c(G)\otimes_{C_c(G)} C_c(G)$ as a function space analogous to the representations given in Theorems 3.3 and 5.5.

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On invariant measures for expanding differentiable mappings

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This note concerns expanding differentiable mappings first studied by M. Shub, see [5] and [6]. These mappings are closely connected with Anosov diffeomorphisms. But while it is not known whether there always exists a finite Lebesgue measure invariant with respect to an Anosov diffeomorphism (see [1] and [6]), it turns out that such a measure always exists for any expanding differentiable mapping. The purpose of this note is to prove this fact. It seems that this may be of some interest and that is why we publish the proof although the arguments used in it have some points of similarity with the proof of Theorem 1 in [3], p. 483.

The authors are very much indebted to Professor J. G. Sinai for his valuable remarks concerning this paper.

In the sequel M will always denote a compact, connected differentiable manifold of class C^∞ unless stated otherwise. If φ is a map of class C^1 of M into itself, then $d\varphi$ will denote the derivative of φ which is the map of the tangent bundle $T(M)$ into itself. We shall say that φ *is expanding if* there exist a Riemannian metric $\|\cdot\|$ on M, a positive real number a and a real number c greater than 1 and such that

$$\|(d\varphi^n)(\mathfrak{a})\| \geqslant ac^n\|\mathfrak{a}\| \tag{1}$$

for each $\mathfrak{a}\in T(M)$ and $n = 1, 2, \ldots$

EXAMPLE. Let φ be a differentiable mapping of the 2-dimensional torus into itself given by the formula

$$\varphi(x, y) = \big(mx+ny+\varepsilon\cdot f(x, y),\ px+qy+\varepsilon g(x, y)\big)(\mathrm{mod}\,1),$$
$$0 \leqslant x < 1,\ 0 \leqslant y < 1,$$

where

(i) m, n, p, q are integers;

(ii) the eigenvalues of the matrix $\begin{pmatrix} m, & n\\ p, & q\end{pmatrix}$ are real and their moduli are greater than 1;

(iii) f and g are the real functions of class C^1 on R^2, periodic with period 1 with respect to each variable;

(iv) ε is a real positive number.