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Piecewise flatness and surface area

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1. In 1868 Serret [8] proposed that the area of a surface S be defined as "the limit of the elementary areas of the polyhedral surfaces inscribed on S as these approach S." In 1882, Schwarz [7] showed that the set of the areas of the polyhedra inscribed on such a simple surface as a circular cylinder is unbounded. In 1902, Lebesgue [3] proposed that surface area be defined as the G. L. B. of the set of limit inferiors of the sequences of areas of polyhedral surfaces which converge uniformly to the given surface. This paper stimulated a great number of investigations. Most of these used Lebesgue's definition as a point of departure. However, many mathematicians [1], [2], [5] and [10] have been interested in a more geometric definition. The present work is one in this direction.

2. The notion of piecewise flatness was introduced in [9]. However, the discussion there was confined to continuously partially differentiable surfaces. In the present work we consider the general case of a continuous non-parametric surface.

Let $S = f(E)$ be a continuous non-parametric surface, E a closed region on the xy plane bounded by a simple closed polygon. Let $0 < a < \pi/2$. We shall consider triangular polyhedra Π inscribed on S such that:

1) The projection $P(\Pi)$ of Π on the xy plane is identical with $P(S)$, the projection of S on the xy plane, i.e., $P(\Pi) = P(S) = E$.

2) Every face of Π has an angle which lies between a and $\pi - a$.

3) If θ denotes the acute angle between the z-axis and the normal line to a face of Π, then, considering the set of all such polyhedra, $\sec\theta$ is bounded. We will denote the L. U. B. of $\sec\theta$ by m.

4) The projection on the xy plane of each face of each polyhedron (a closed triangle) is a subset of $P(S)$.

We shall refer to such polyhedra as admissible polyhedra.

A triangle T inscribed on S is said to be admissible if:

1) One angle of T lies between a and $\pi - a$.

2) The secant of the acute angle between the z-axis and the normal line to T is less than m.

3) $P(T) \subset P(S)$.

Every face of an admissible polyhedron is an admissible triangle.

DEFINITIONS. Let T be a face of an admissible inscribed polyhedron. By a *subface* Q of T we mean an admissible triangle inscribed on S such that $P(Q) \subset P(T)$.

By the *directional deviation* $D_a(T)$ of T we mean the L. U. B. of the set of acute angles φ between the normal line to T and the normal line to every subface of T.

In the sequel, the term inscribed polyhedra shall mean inscribed admissible polyhedra. We shall generally refer to such polyhedra as (a, m) polyhedra and to its faces as (a, m) faces. We shall, however, often omit the "(a, m)."

Given admissible polyhedra Π_1 and Π_2 inscribed on S, we say that Π_2 is a refinement of Π_1 if every vertex of Π_1 is a vertex of Π_2. Similarly, if T is an admissible triangle inscribed on S and K is a polyhedron inscribed on S such that $P(K) = P(T)$, every vertex of T is a vertex of K and every face of K is an admissible triangle, then we shall refer to K as a refinement of T.

DEFINITIONS. Given an admissible polyhedron Π inscribed on S, by the *norm* of Π we mean the greatest of the diameters of its faces.

By the *deviation norm* of Π we mean the greatest of the directional deviations of its faces.

DEFINITION. Given $S = f(E)$, by an (a, m) *regular sequence* of inscribed admissible polyhedra $(\Pi_1, \Pi_2, \Pi_3, \ldots)$ we mean an infinite sequence of inscribed (a, m) admissible polyhedra having the following properties:

1) The corresponding sequence $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ of deviation norms converges to zero.

2) The corresponding sequence $(N_1, N_2, N_3, \ldots)$ of norms converges to zero.

DEFINITION. $S = f(E)$ is said to be an (a, m) *regular surface* if it admits a regular sequence of (a, m) inscribed admissible polyhedra.

THEOREM 1. *Let $S = f(E)$ be (a, m) regular. Then for each (a, m) regular sequence of inscribed polyhedra, the corresponding sequence $(A_1, A_2, A_3, \ldots)$ of polyhedral areas converges.*

Proof. Let $(\Pi_1, \Pi_2, \Pi_3, \ldots)$ be any (a, m) regular sequence of admissible polyhedra inscribed on S. Let the corresponding sequence of norms, deviation norms, and polyhedral areas be, respectively,

$$(N_1, N_2, N_3, \ldots), \quad (\varphi_1, \varphi_2, \varphi_3, \ldots), \quad (A_1, A_2, A_3, \ldots).$$

We wish to show that $(A_1, A_2, A_3, \dots)$ converges.

LEMMA 1. *There exists a positive constant M, depending only on S, a, and m such that if θ_1 is the acute angle between the z-axis and the normal to a face of Π_i and θ_2 is the acute angle between the z-axis and the normal to a face of Π_j, then $|\sec\theta_1 - \sec\theta_2| < M|\theta_1 - \theta_2|$.*

Proof. $\operatorname{Sec}\theta$ is continuously differentiable on a closed and bounded set. Hence $\sec\theta$ is uniformly Lipschitzian on this set.

LEMMA 2. *Let T be a face of Π_n. Let B denote the area of T. Let K be any refinement of T. Let its faces be $Q_1, Q_2, \dots, Q_r$ with respective areas, $B_1, B_2, \dots, B_r$. Let a = area of $P(K)$ and a_i = area of $P(Q_i)$, $i = 1, 2, \dots, r$. Let θ denote the acute angle between the z-axis and the normal to T; θ_i, the acute angle between the z-axis and the normal to Q_i, $i = 1, 2, \dots, r$. Finally let $B^* = B_1 + B_2 + \dots + B_r$. Then $|B - B^*| < aM\varphi_n$.*

Proof.

$$B = a_1\sec\theta + a_2\sec\theta + \dots + a_r\sec\theta ,$$

$$B^* = a_1\sec\theta_1 + a_2\sec\theta_2 + \dots + a_r\sec\theta_r ,$$

$$|B - B^*| \leqslant a_1|\sec\theta - \sec\theta_1| + a_2|\sec\theta - \sec\theta_2| + \dots + a_r|\sec\theta - \sec\theta_r|$$
$$< a_1M\varphi_n + a_2M\varphi_n + \dots + a_rM\varphi_n < aM\varphi_n .$$

We now proceed to the proof of Theorem 1.

Let $\varepsilon > 0$ be given. There exists a positive integer N such that, if $n > N$, then

$$\varphi_n < \frac{\varepsilon}{2AM}, \qquad \text{where } A = \text{area of } E.$$

Let $n_1 > N$ and $n_2 > N$. Consider Π_{n_1} and Π_{n_2}. Let Π be the common refinement of Π_{n_1} and Π_{n_2}. Let A_{n_1,n_2} denote the area of Π. We will compare A_{n_1,n_2} with both A_{n_1} and A_{n_2}

$$|A_{n_1} - A_{n_1,n_2}| < AM\varphi_{n_1} < AM\frac{\varepsilon}{2AM} = \frac{\varepsilon}{2},$$

$$|A_{n_2} - A_{n_1,n_2}| < AM\varphi_{n_2} < AM\frac{\varepsilon}{2AM} = \frac{\varepsilon}{2}.$$

Hence $|A_{n_1} - A_{n_2}| < \varepsilon$. Hence the sequence $(A_1, A_2, A_3, \dots)$ converges.

THEOREM 2. *Let $S = f(E)$ be (a, m) regular. Then for all regular sequences of (a, m) admissible polyhedra inscribed on S the corresponding sequences of polyhedral areas converge to the same real number as a limit.*

Proof. Let $(\Pi_1, \Pi_2, \Pi_3, \dots)$ and $(\Pi_1', \Pi_2', \Pi_3', \dots)$ be two (a, m) regular sequences of admissible polyhedra inscribed on S. Let the corresponding sequences of polyhedral areas be $(A_1, A_2, A_3, \dots)$ and $(A_1', A_2', A_3', \dots)$, respectively.

The sequence $(\Pi_1, \Pi_1', \Pi_2, \Pi_2', \ldots)$ is an (a, m) regular sequence. Hence the corresponding sequence of polyhedral areas $(A_1, A_1', A_2, A_2', \ldots)$ converges. Since $(A_1, A_2, A_3, \ldots)$ and $(A_1', A_2', A_3', \ldots)$ are subsequences of $(A_1, A_1', A_2, A_2', \ldots)$ these three sequences all converge to the same real number as a limit.

It is easy to see that if $S = f(E)$ is both (a_1, m_1) regular and (a_2, m_2) regular, the unique limit of the sequences of polyhedral areas corresponding to regular (a_1, m_1) sequences of inscribed polyhedra is equal to the unique limit of the sequences of polyhedral areas corresponding to (a_2, m_2) sequences of inscribed polyhedra.

DEFINITION. $S = f(E)$ is said to be *piecewise flat* if there exists a pair (a, m) such that S is (a, m) regular.

DEFINITION. Let $S = f(E)$ be piecewise flat. Then by the ***area*** of S we mean the unique limit of the sequences of polyhedral areas corresponding to regular sequences of polyhedra inscribed on S.

In [9] we showed that a sufficient condition for $S = f(E)$ to be piecewise flat is that f be continuously partially differentiable, i.e., f is continuously partially differentiable on E.

We now wish to obtain a necessary condition that $S = f(E)$ be piecewise flat.

Consider a regular sequence $(\Pi_1, \Pi_2, \Pi_3, \ldots)$ of admissible polyhedra inscribed on a piecewise flat surface S. Let $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ be the corresponding sequence of deviation norms.

Let Π_n be any element of the above sequence of polyhedra and let K_{nm} be any of the faces of Π_n. Each point p of the interior of K_{nm} has the property that if L_1 and L_2 are any two subfaces of K_{nm} containing p (not necessarily an interior point), then the acute angle between the normals to L_1 and L_2 is less than $2\varphi_n$. Let E_{φ_n} denote the union of the projections of the interiors of the faces of Π_n. The set $F_{\varphi_n} = E - E_{\varphi_n}$ is of 2-dimensional measure zero. The union $U = \bigcup_{n=1}^{\infty} F_{\varphi_n}$ is then also of measure zero. Let $F = E - U$. By a regular point q of S we shall mean a point S such that $P(q) \epsilon F$.

THEOREM 3. *Let q be a regular point of a piecewise flat surface S. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if T_1 and T_2 are any two admissible triangles inscribed on S of diameters less than δ, such that $P(q) \epsilon P(T_1) \frown P(T_2)$, then the acute angle between the normals to T_1 and T_2 is less than ε.*

Proof. There exists a positive integer N such that, if $n > N$, then $\varphi_n < \frac{1}{2}\varepsilon$. Let $n > N$ and let T be the face of Π_n such that $P(q)$ is an interior point of $P(T)$. Take $\delta =$ the distance from $P(q)$ to the boundary of $P(T)$.

COROLLARY. *Let q be a regular point of S. Let $(T_1, T_2, T_3, \ldots)$ be an infinite sequence of admissible triangles inscribed on S such that, for each i, $P(q) \in P(T_i)$ and such that the corresponding sequence $(D_1, D_2, D_3, \ldots)$ of diameters converges to zero. Let $(V_1, V_2, V_3, \ldots)$ be the corresponding sequence of unit vectors such that each V_i is normal to T_i. Then the sequence $(V_1, V_2, V_3, \ldots)$ converges to a unique unit vector $V(q)$.*

Proof. Let $\varepsilon > 0$ be given. Take the δ referred to in Theorem 3 corresponding to $\theta = (1-\varepsilon^2/2)$.

There exists a positive integer N such that if $n > N$, then $D_n < \delta$.

Let $n_1 > N$ and $n_2 > N$. Then the acute angle between V_{n_1} and V_{n_2} is less than θ. Thus

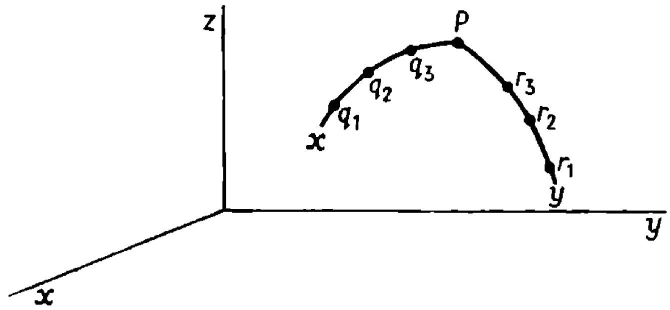
$$|V_{n_1} - V_{n_2}| < \sqrt{2}\sqrt{1-\cos\theta} < \varepsilon .$$

Hence the sequence $(V_1, V_2, V_3, \ldots)$ converges.

Given two sequences $(T_1, T_2, T_3, \ldots)$ and $(T'_1, T'_2, T'_3, \ldots)$ having the above mentioned property, the sequence $(T_1, T'_1, T_2, T'_2, \ldots)$ also has that property. It follows that all sequences $(V_1, V_2, V_3, \ldots)$ converge to the same limit vector.

THEOREM 4. *Let p be a regular point of a piecewise flat surface S. Then f is partially differentiable at $P(p)$.*

Proof. Let C_x and C_y be the curves of intersection of S with the planes through p parallel to the xz plane and the yz plane, respectively.



Let $(q_1, q_2, q_3, \ldots)$ be an infinite sequence of points on C_x converging to p. Let $(V_{q_1}, V_{q_2}, V_{q_3}, \ldots)$ be the corresponding sequence of unit vectors from p through $q_1, q_2, q_3, \ldots$, respectively.

If the set $\{V_{q_1}, V_{q_2}, V_{q_3}, \ldots\}$ is finite, then there exists a convergent subsequence of $(V_{q_1}, V_{q_2}, V_{q_3}, \ldots)$. If the set $\{V_{q_1}, V_{q_2}, V_{q_3}, \ldots\}$ is infinite, then there exists a vector limit point of the set. There exists then a subsequence of $(V_{q_1}, V_{q_2}, V_{q_3}, \ldots)$ which converges to this vector limit point. Thus, in either case there exists a convergent subsequence.

Similarly, if $(r_1, r_2, r_3, \ldots)$ is any sequence of points on C_y converging to p, and $(W_{r_1}, W_{r_2}, W_{r_3}, \ldots)$ is the corresponding sequence of unit vectors from p through $(r_1, r_2, r_3, \ldots)$, respectively, then there exists a convergent subsequence of $(W_{r_1}, W_{r_2}, W_{r_3}, \ldots)$.

Suppose now that there exist two subsequences of $(V_{q_1}, V_{q_2}, V_{q_3}, \ldots)$ which converge to two distinct limit vectors. Let these be V_x^* and V_x^{**}. Let a subsequence of $(W_{r_1}, W_{r_2}, W_{r_3}, \ldots)$ converge to V_y. Then $V_x^* \times V_y \neq V_x^{**} \times V_y$. This contradicts the corollary to Theorem 3.

Since the sequences $(V_{q_1}, V_{q_2}, V_{q_3}, \ldots)$ converge to a unique limit vector, it follows that $\partial f/\partial x$ exists at $P(p)$. Similarly, $\partial f/\partial y$ exists at $P(p)$.

COROLLARY. *Let p be a regular point of $S = f(E)$. Then the unique limit vector $V(p)$ is normal to S at p.*

Proof. If C is the intersection of S with any plane through p and parallel to the z-axis, then the directional tangent vector to C at p exists and is normal to $V(p)$.

THEOREM 5. *Let p be a regular point of a piecewise flat surface $S = f(E)$. Then f is continuously partially differentiable at p.*

Proof. Suppose that there exists $\varepsilon > 0$, such that for every $\delta > 0$, there exists $q \in E$ such that $|q - P(p)| < \delta$, f differentiable at q, and the acute angle between $V(p)$ and $V(f(q))$ is not less than ε.

Now there exists $\delta > 0$ such that:

1) If T_1 and T_2 are any two admissible triangles inscribed on S and such that $P(T_1) \cup P(T_2)$ is a subset of the open ball $S(f(p), \delta)$, then the acute angle between the normals to T_1 and T_2 is less than $\varepsilon/4$.

2) The acute angle between $V(p_1)$ and the normal to T_1 is less than $\varepsilon/4$.

3) If $q \in S(P(p), \delta)$ and $f(q)$ is a regular point of S, then the acute angle between $V(f(q))$ and the normal to T_2 is less than $\varepsilon/4$.

It follows from these that the acute angle between $V(p)$ and $V(f(q))$ is less than ε. This exhibits a contradiction. Hence f is continuously partially differentiable at p.

We have just shown that a necessary condition that $S = f(E)$ be piecewise flat is that f be continuously differentiable over E except on a set of measure zero. In [9] we showed that a sufficient condition that $S = f(E)$ be piecewise flat is that f be continuously partially differentiable on E.

3. We will now give an expression for the area of a piecewise flat surface $S = f(E)$.

THEOREM 6. *Let $S = f(E)$ be piecewise flat. Let E^* be the subset of E on which f is continuously partially differentiable.*

Let

$$F(x, y) = \begin{cases} \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}, & \text{if } (x, y) \in E^*, \\ 0, & \text{if } (x, y) \in E - E^*. \end{cases}$$

Then the area of S is given by the Lebesgue integral $\int_E F(x, y)$.

Proof. $F(x, y)$ is Lebesgue integrable on E.

Consider a regular sequence of admissible polyhedra inscribed on S

$$(\Pi_1, \Pi_2, \Pi_3, \ldots)$$

with corresponding sequences

$$(N_1, N_2, N_3, \ldots), \quad (\varphi_1, \varphi_2, \varphi_3, \ldots), \quad (A_1, A_2, A_3, \ldots).$$

Let the faces of Π_n be $T_{n1}, T_{n2}, \ldots, T_{nm(n)}$. Let their projections be $R_{n1}, R_{n2}, \ldots, R_{nm(n)}$, respectively. $A_n = R_{n1}\sec\theta_{n1} + R_{n2}\sec\theta_{n2} + \ldots + R_{nm(n)}\sec\theta_{nm(n)}$, where θ_{n1} is the acute angle between the z-axis and the normal to T_{n1}, etc.

On R_{n1} there exists a point q_{n1} such that $f(q_{n1})$ is a regular point of S. Similarly, on R_{n2}, there exists a point q_{n2} such that $f(q_{n2})$ is a regular point of S, etc.

Let

$$A_n^* = R_{n1}\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}\big|_{q_{n1}} + R_{n2}\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}\big|_{q_{n2}} + \ldots + R_{nm(n)}\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}\big|_{q_{nm(n)}},$$

$$|A_n - A_n^*| \leqslant R_{n1}d_{n1} + R_{n2}d_{n2} + \ldots + R_{nm(n)}d_{nm(n)},$$

where $d_{n1} < M\varphi_n$, $d_{n2} < M\varphi_n$, ..., $d_{nm(n)} < M\varphi_n$, M being the positive real number introduced earlier. $|A_n - A_n^*| < EM\varphi_n$.

Since $(\varphi_1, \varphi_2, \varphi_3, \ldots)$ converges to zero and $(A_1, A_2, A_3, \ldots)$ converges to the area of S, it follows that $(A_1^*, A_2^*, A_3^*, \ldots)$ also converges to the area of S. Thus the area of S is given by the above Lebesgue double integral.

4. Let $S = f(E)$ be a piecewise flat surface. Let $(\Pi_1, \Pi_2, \Pi_3, \ldots)$ be an (a, m) regular sequence of polyhedra inscribed on S. Each polyhedron Π_n may be represented by a quasi-linear function $F_n(p)$, in the sense of McShane [4].

For each n, $F_n(p)$ is uniformly continuous on E. The function $f(p)$ is also uniformly continuous on E. Thus for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\varrho(p_1, p_2)$, the distance between p_1 and p_2, elements of E, is less than δ, then $|f(p_1) - f(p_2)| < \varepsilon/2$ and $|F_n(p_1) - F_n(p_2)| < \varepsilon/2$.

Now, let $\varepsilon > 0$ be given. Since the corresponding sequence $(N_1, N_2, N_3, \ldots)$ of norms converges to zero and since $\sec\theta$, the secant of the acute angle between the z-axis and the normal to any face of any of the polyhedra $\Pi_1, \Pi_2, \Pi_3, \ldots$ is less than m, there exists a positive integer N such that if $n > N$, then if p is any element of E, there exists p^*, an element of E such that $p^* \epsilon S \cap \Pi_n$ and $\varrho(p, p^*) < \delta$

$$|f(p) - f(p^*)| < \varepsilon/2 \quad \text{and} \quad |F_n(p) - F_n(p^*)| < \varepsilon/2.$$

Since $f(p^*) = F_n(p^*)$, it follows that $|f(p)-F_n(p)| < \varepsilon$.

Thus, for every $\varepsilon > 0$, there exists a positive integer N such that if $n = N$, then $|f(p)-F_n(p)| < \varepsilon$, for every $p \in E$, i.e., the sequence $(F_1(p), F_2(p), F_3(p), \ldots)$ converges uniformly to $f(p)$ on E.

Since the sequence $(F_1(p), F_2(p), F_3(p), \ldots)$ converges uniformly to $f(p)$ on E and in view of the fact, shown earlier, that the area of the surface S, as we defined this area, is given by the above Lebesgue integral, it follows from a result of Rado [6] that, for a piecewise flat surface S, the area of S, as we defined this term, is precisely equal to the Lebesgue area of S.

References

[1] Z. de Geocze, *Quadrature de surfaces courbes*, Math. Naturwiss. Berichte Ung., 1910.

[2] S. Kempisty, *Sur la méthode triangulaire du calcul de l'aire d'une surface courbe*, Bull. Soc. Math. France, 1936.

[3] H. Lebesgue, *Integrale, longueur, aire*, Ann. Mat. Pura Appl. (1902).

[4] E. J. McShane, *On the semi-continuity of double integrals in the calculus of variations*, Ann. of Math. 33 (1932).

[5] M. Rademacher, *Ueber Partielle und Totale Differenzierbarkeit II*, Math. Ann. 81 (1920).

[6] T. Rado, *On the semi-continuity of double integrals in parametric form*, Trans. Amer. Math. Soc. 51 (1942).

[7] H. A. Schwarz, *Sur une definition erronée de l'aire d'une surface courbe*, Ges. Math. Abhandl. II (1882).

[8] J. A. Serret, *Cours de calcul differential et integral*, 1868.

[9] L. V. Toralballa, *Directional deviation norms and surface area*, L'Enseignement Mathématique II^e Serie, Tome XIII, Fascicule 2, 1967.

[10] M. W. H. Young, *On the triangulation method of defining the area of a surface*, Proc. London Math. Soc., 2nd Series, 19 (1919).

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