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Estimates for the coefficients of polynomials and trigonometric polynomials

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1. Let $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n such that $|P_n(z)| \leqslant M$ for $|z| = 1$. By a very simple method Visser [13] proved that

$$|a_0| + |a_n| \leqslant M . \tag{1.1}$$

By the method of Visser [13], van der Corput and Visser [4] later proved the following more general

THEOREM A. *Let* $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *such that* $|P_n(z)| \leqslant M$ *for* $|z| \leqslant 1$. *If* a_u, a_v $(u < v)$ *are two coefficients such that for no other coefficient* $a_w \neq 0$ *we have* $w = u \bmod (v-u)$, *then*

$$|a_u| + |a_v| \leqslant M . \tag{1.2}$$

Also

$$|a_u| + |a_v| \leqslant \tfrac{1}{4} \int_0^{2\pi} |P_n(e^{i\theta})|\, d\theta . \tag{1.3}$$

Besides, they proved that if $F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a real trigonometric polynomial, i.e. $c_{-k} = c_k$, then for $k > n/2$

$$|c_0| + 2\,|c_k| \leqslant \max |F(\theta)| , \tag{1.4}$$

$$|c_0| + \tfrac{2}{3}\,|c_k| \leqslant \tfrac{1}{4} \int_0^{2\pi} |F(\theta)|\, d\theta . \tag{1.5}$$

The constant $\frac{1}{4}$ in (1.5) was improved by Boas [2] to $\frac{1}{2}(1+\frac{1}{3}\sqrt{2})/\pi = 0.234\ldots$ He later obtained [3] the best possible result and proved that for $k > n/2$ and any positive γ

$$|c_0| + 2\gamma\,|c_k| \leqslant A_\gamma \int_0^{2\pi} |F(\theta)|\, d\theta , \tag{1.6}$$

where A_γ is given by

$$A_\gamma = \frac{1}{2\pi - 4\varphi}, \tag{1.7}$$

and φ is the smallest positive root of $\sin\varphi = \frac{1}{2}\gamma(\pi - 2\varphi)$.

The more general problem of determining the best possible estimate for

$$\max \frac{\lambda_0|c_0| + \lambda_k|c_k|}{\int_0^{2\pi} |F(\theta)|\,d\theta}, \quad 0 < k \leqslant n,$$

where λ_0, λ_k are given non-negative numbers was solved by Geronimus [5]. He proved in particular the following:

THEOREM B. *If* $F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ *is a real trigonometric polynomial, then*

$$|c_k| \leqslant \frac{1}{8h} \int_0^{2\pi} |F(\theta)|\,d\theta,$$

where h is the smallest positive root of the algebraic equation

$$\begin{vmatrix} m_0 & m_1 & \dots m_\beta \\ m_1 & m_0 & \dots m_{\beta-1} \\ \vdots & \vdots & \vdots \\ m_{-\beta} & m_{-\beta-1} & \dots m_0 \end{vmatrix} = 0 \tag{1.8}$$

with $m_0 = 2$, $m_s = m_{-s} = (2h)^s/s!$ $(s = 1, 2, \dots, \beta = [n/k])$. *The result is the best possible.*

In the spacial case, when $k > n/3$ this result gives the estimate

$$|c_k| \leqslant \tfrac{1}{8} \int_0^{2\pi} |F(\theta)|\,d\theta, \tag{1.9}$$

whereas, for $n/3 \geqslant k > n/5$ we get

$$|c_k| \leqslant 0.142 \int_0^{2\pi} |F(\theta)|\,d\theta. \tag{1.10}$$

Generalizing (1.3) and (1.6) respectively Rahman ([8], Theorem 2 and 3) has recently proved the following theorems:

THEOREM C. *If $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n and a_u, a_v $(u < v)$ are two coefficients such that for no other coefficient $a_w \neq 0$ we have $w = u \bmod. (v-u)$, then for every $\delta \geqslant 1$*

$$|a_u| + |a_v| \leqslant 2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^\delta d\theta\right)^{1/\delta},$$

where

$$C_\delta = \frac{2\pi}{\int_0^{2\pi} |1+e^{i\varphi}|^\delta d\varphi} = \frac{2^{-\delta}\sqrt{\pi}\,\Gamma(\frac{1}{2}\delta+1)}{\Gamma(\frac{1}{2}\delta+\frac{1}{2})}. \tag{1.11}$$

The result is best possible.

THEOREM D. *If $F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a real trigonometric polynomia and $\delta \geqslant 1$, then for $k > n/2$ and any positive γ*

$$|c_0| + 2\gamma|c_k| \leqslant (C_{\gamma,\delta})^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi} |F(\theta)|^\delta d\theta\right)^{1/\delta},$$

where

$$C_{\gamma,\delta} = \max_{0<r<\infty} \frac{(1+2\gamma r)^\delta}{\frac{1}{2\pi}\int_0^{2\pi} |1+2r\cos\psi|^\delta d\psi}.$$

Real-valued trigonometric polynomials have been studied in detail by Rogosinski [9] and [10], and Mulholland [7]. The following theorem, equivalent to a result of Mulholland ([7], Theorem 1) stands in analogy with Theorem B.

THEOREM E. *If $F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a real trigonometric polynomial such that $|F(\theta)| \leqslant M$ for $0 \leqslant \theta < 2\pi$, then for $k \geqslant 1$*

$$|c_k| \leqslant \frac{M}{l}\cot\frac{\pi}{2l}, \quad \text{where } l = \left[\frac{n}{2k}+\frac{3}{2}\right]. \tag{1.12}$$

The maximum is attained, in particular, for the polynomial

$$\frac{1}{l^2}\sum_{r=1}^{l-1} (-1)^{r-1}\{e^{ik(2r-1)\theta}+e^{-ik(2r-1)\theta}\} \times \left\{(2l-2r+1)\cot\frac{(2r-1)\pi}{2l}+\cot\frac{\pi}{2l}\right\}.$$

THEOREM F. *If* $P_n(x) = \sum_{\nu=0}^{n} A_\nu x^\nu$ *is a polynomial of degree* n *such that* $|P_n(x)| \leqslant 1$ *for* $-1 \leqslant x \leqslant 1$, *then*

$$|A_n| \leqslant 2^{n-1} \quad \text{(Tchebycheff)},$$

$$|A_{n-1}| \leqslant 2^{n-2} \quad \text{(W. Markoff)}.$$

Generalizing Theorems B and E we obtain an estimate for $|c_k|$ in terms of

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |F(\theta)|^\delta d\theta\right)^{1/\delta}$$

for every $\delta \geqslant 1$.

THEOREM 1. *If* $F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ *is a real trigonometric polynomial and* $\delta \geqslant 1$, *then for* $k \geqslant 1$

$$|c_k| \leqslant (C_\delta)^{1/\delta}\left(\frac{2}{l}\cot\frac{\pi}{2l}\right)\left(\frac{1}{2\pi}\int_0^{2\pi} |F(\theta)|^\delta d\theta\right)^{1/\delta}, \tag{1.13}$$

where

$$l = \left[\frac{n}{2k} + \frac{3}{2}\right] \quad \text{and} \quad C_\delta = \frac{2\pi}{\int_0^{2\pi} |1+e^{i\varphi}|^\delta d\varphi}.$$

On letting $\delta \to \infty$ we get Theorem E. However, we do not assert that the result is best possible. Its interest lies in the fact that for δ other than 1 and 2 *it is the first and only result of its kind.* In order to have an idea as to how close it is to the correct estimate we compare it in the case $\delta = 1$ with the result of Geronimus (Theorem B) which is known to be precise.

In the special case $\delta = 1$ our theorem gives the estimate

$$|c_k| \leqslant \frac{1}{4\left[\frac{n}{2k} + \frac{3}{2}\right]} \cot \frac{\pi}{2\left[\frac{n}{2k} + \frac{3}{2}\right]} \int_0^{2\pi} |F(\theta)|\, d\theta$$

since $C_1 = \pi/4$. Thus for $k > n/3$ we get

$$|c_k| \leqslant \tfrac{1}{8}\int_0^{2\pi} |F(\theta)|\, d\theta$$

which agrees with estimate (1.9) of Geronimus.

If $k \leqslant n/3$ but $> n/5$, then we obtain

$$|c_k| \leqslant .144 \int_0^{2\pi} |F(\theta)|\, d\theta$$

which differs from the precise estimate (1.10)

$$|c_k| \leqslant .142 \int_0^{2\pi} |F(\theta)|\, d\theta$$

by 1.4 per cent. Thus our estimate (1.13) though not best possible, appears to be fairly good.

Our next theorem gives estimate for linear combinations of any two coefficients of a polynomial $P_n(z)$ bounded by M on the unit circle.

THEOREM 2. *If $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $|P_n(z)| \leqslant M$ for $|z| \leqslant 1$ and $0 \leqslant u < v \leqslant n$, then for any real $\lambda > 0$*

$$|a_u| + \lambda |a_v| \leqslant \frac{M}{2\pi} \int_0^{2\pi} |1 + \lambda e^{i\theta}|\, d\theta\, . \tag{1.14}$$

The result is best possible.

If u and v satisfy a separation condition like $0 \leqslant 2u < v \leqslant n$, then the bound in (1.14) can be considerably improved.

THEOREM 3. *If $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $|P_n(z)| \leqslant M$ for $|z| \leqslant 1$, then for any real $\lambda > 0$*

$$|a_u| + \lambda |a_v| \leqslant M\left(\lambda + \frac{1}{4\lambda}\right) \text{ or } M\left(1 + \frac{\lambda^2}{4}\right) \tag{1.15}$$

according as $0 \leqslant 2u < v \leqslant n$ or $0 \leqslant u < 2v - n \leqslant n$ respectively.

The case $\lambda = \frac{1}{2}$ is particularly interesting, for then the right-hand side in (1.15) reduces to M when $0 \leqslant 2u < v \leqslant n$. For $u = 0$, $v = 1$ we have the stronger

THEOREM 4. *If $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $\operatorname{Re} P_n(z) \leqslant A$ for $|z| \leqslant 1$, and a_0 is real, then*

$$a_0 + \tfrac{1}{2}|a_1| \leqslant A\, . \tag{1.16}$$

In an attempt to generalize Theorem F by finding estimates for the coefficients in terms of

$$\left(\tfrac{1}{2} \int_{-1}^{1} |P_n(x)|^\delta dx\right)^{1/\delta}$$

we have obtained the following

THEOREM 5. *If* $P_n(x) = \sum_{\nu=0}^{n} A_\nu x^\nu$ *is a polynomial of degree* n, *then for every* $\delta \geqslant 1$, $\nu = n, n-1$,

$$(1.17) \qquad |A_\nu| \leqslant 2^{\nu+1}\left(\frac{2}{\pi}C_\delta\right)^{1/\delta}\left(\frac{1}{2}\int_{-1}^{1}|P_n(x)|^\delta(1-x^2)^{(\delta-1)/2}dx\right)^{1/\delta},$$

where C_δ *is given by* (1.11).

The following corollary is immediate.

COROLLARY. *If* $P_n(x) = \sum_{\nu=0}^{n} A_\nu x^\nu$ *is a polynomial of degree* n, *then for every* $\delta \geqslant 1$, $p > 1$, $q = p/(p-1)$ *and* $\nu = n, n-1$,

$$(1.18) \quad |A_\nu| \leqslant 2^{\nu+1}\left(\frac{2}{\pi}C_\delta\right)^{1/\delta}\left(\frac{1}{2}\int_{-1}^{1}(1-x^2)^{(\delta-1)q/2}dx\right)^{1/\delta q}\left(\frac{1}{2}\int_{-1}^{1}|P_n(x)|^{\delta p}dx\right)^{1/\delta p}.$$

2. For one of our results we shall need the following

LEMMA 1. *Let* π_n *denote the linear space of polynomials*

$$P_n(z) = a_0 + a_1 z + \ldots + a_n z^n$$

of fixed degree n *with complex coefficients, normed by* $\|P_n\| = \max_{0\leqslant\theta\leqslant 2\pi}|P_n(e^{i\theta})|$. *Define the linear functional* L *on* π_n *as*

$$(2.1) \qquad L\colon P_n \to l_0 a_0 + \ldots + l_n a_n,$$

where the l_ν *are complex numbers. If the norm of the functional is* N, *then for every polynomial* $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$

$$(2.2) \qquad \int_0^{2\pi}\varphi\left(\frac{\left|\sum_{\nu=0}^{n} l_\nu a_\nu e^{i\nu\theta}\right|}{N}\right)d\theta \leqslant \int_0^{2\pi}\varphi\left(\left|\sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}\right|\right)d\theta,$$

where $\varphi(t)$ *is a non-decreasing convex function of* t.

This was proved by Shapiro ([11], Theorem 8) for the case $\varphi(t) = t$.

Proof of Lemma 1. According to a theorem of Shapiro ([11], Theorem 3) L can be represented in the form

$$(2.3) \qquad L[P_n(z)] = \sum_{k=1}^{r} u_k P_n(z_k),$$

for all $P_n \in \pi_n$, where $z_1, \ldots, z_r$ are distinct numbers of modulus 1 and $\sum_{k=1}^{r} |u_k| = N$. Let ξ be any number of modulus one, and apply this to the polynomial $P_n(\xi z)$. We get

$$\Big|\sum_{\nu=0}^{n} l_\nu a_\nu \xi^\nu\Big| = \Big|\sum_{\nu=1}^{r} u_\nu P_n(z_\nu \xi)\Big| \leqslant \sum_{\nu=1}^{r} |u_\nu|\,|P_n(z_\nu \xi)|\,.$$

Hence if $\varphi(t)$ is a non-decreasing convex function of t, then by Jensen's inequality ([6], pp. 150-151)

$$\varphi\left(\frac{\Big|\sum_{\nu=1}^{n} l_\nu a_\nu \xi^\nu\Big|}{N}\right) \leqslant \varphi\left\{\frac{\sum_{\nu=1}^{r} |u_\nu|\,|P_n(z_\nu \xi)|}{\sum_{\nu=1}^{r} |u_\nu|}\right\} \leqslant \frac{\sum_{\nu=1}^{r} |u_\nu| \varphi\{|P_n(z_\nu \xi)|\}}{\sum_{\nu=1}^{r} |u_\nu|}\,.$$

Setting $\xi = e^{i\theta}$ and integrating both sides with respect to θ from 0 to 2π we get the result.

3. Proof of Theorem 1. According to a theorem of Rahman ([8], Theorem 7) if $P_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n such that $|P_n(z)| \leqslant M$ for $|z| \leqslant 1$, then for $\nu < n/2$

$$|a_\nu| + |a_{n-\nu}| \leqslant \frac{2M}{l} \cot \frac{\pi}{2l}\,, \tag{3.1}$$

where $l = \left[\frac{n}{2(n-2\nu)} + \frac{3}{2}\right]$.

Since $|a_\nu + a_{n-\nu}| \leqslant |a_\nu| + |a_{n-\nu}|$ the norm of the functional

$$L\colon\ P_n \to 0\cdot a_0 + \ldots + 0\cdot a_{\nu-1} + 1\cdot a_\nu + 0\cdot a_{\nu+1} + \ldots + 0\cdot a_{n-\nu-1} + \\ + 1\cdot a_{n-\nu} + 0\cdot a_{n-\nu-1} + \ldots + 0\cdot a_n$$

does not exceed $\frac{2}{l}\cot\frac{\pi}{2l}$. Applying the lemma with $\varphi(t) = t^\delta$ we get for every $\delta \geqslant 1$

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |a_\nu e^{i\nu\theta} + a_{n-\nu} e^{i(n-\nu)\theta}|^\delta d\theta\right)^{1/\delta} \leqslant \frac{2}{l}\cot\frac{\pi}{2l}\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^\delta d\theta\right)^{1/\delta}.$$

But by Theorem C

$$|a_\nu| + |a_{n-\nu}| \leqslant 2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi} |a_\nu e^{i\nu\theta} + a_{n-\nu} e^{i(n-\nu)\theta}|^\delta d\theta\right)^{1/\delta}.$$

Therefore

$$(3.2)\qquad |a_\nu| + |a_{n-\nu}| \leqslant 2(C_\delta)^{1/\delta}\frac{2}{l}\cot\frac{\pi}{2l}\left(\frac{1}{2\pi}\int_0^{2\pi}|P_n(e^{i\theta})|^\delta d\theta\right)^{1/\delta}.$$

If $F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a real trigonometric polynomial of degree n, then $e^{in\theta}F(\theta)$ is a polynomial $\sum_{\nu=0}^{2n} b_\nu e^{i\nu\theta}$ of degree $2n$. On applying (3.2) to the polynomial $e^{in\theta}F(\theta) = \sum_{\nu=0}^{2n} b_\nu e^{i\nu\theta}$ and noting that $|b_\nu| = |b_{2n-\nu}|$ for $0 \leqslant \nu \leqslant n$ we shall get the desired result.

Proof of Theorem 2. If $0 \leqslant u < v \leqslant n$, $0 \leqslant a < 2\pi$ and λ any positive real number, then

$$|a_u e^{iau} + \lambda a_v e^{iav}| = \left|\frac{1}{2\pi i}\int_{|z|=1} P_n(ze^{ia})\left[\frac{1}{z^{u+1}} + \frac{\lambda}{z^{v+1}}\right]dz\right|$$

$$\leqslant \frac{M}{2\pi}\int_0^{2\pi}|1 + \lambda e^{i(v-u)\theta}|\,d\theta$$

$$= \frac{M}{2\pi}\int_0^{2\pi}|1 + \lambda e^{i\theta}|\,d\theta,$$

and by choosing a such that $|a_u e^{iau} + \lambda a_v e^{iav}| = |a_u| + \lambda|a_v|$ we shall get the result.

In order to prove that the estimate is best possible let $f(\theta) = e^{i\tan^{-1}\left(\frac{\lambda\sin\theta}{1+\lambda\cos\theta}\right)}$ for $0 \leqslant \theta < 2\pi$ (we choose any fixed branch of $\tan^{-1}\left(\frac{\lambda\sin\theta}{1+\lambda\cos\theta}\right)$) have the Fourier series representation

$$f(\theta) = \sum_{-\infty}^{\infty} b_\nu e^{i\nu\theta}.$$

Then

$$b_0 + \lambda b_1 = \frac{1}{2\pi}\int_0^{2\pi}\exp\left[\tan^{-1}\left(\frac{\lambda\sin\theta}{1+\lambda\cos\theta}\right)\right](1+\lambda e^{-i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}|1+\lambda e^{i\theta}|\,d\theta.$$

For every $n > 0$, we have by a classical result of Fejér ([12], p. 440, Ex. 9)

$$\left|\sum_{-n}^{n}\frac{(n+1)-|\nu|}{n+1}b_\nu e^{i\nu\theta}\right| \leqslant 1$$

and so the Laurent polynomial

$$L(e^{i\theta}) = \sum_{-n}^{n} \frac{(n+1)-|\nu|}{n+1} b_\nu e^{i\nu\theta} = \sum_{-n}^{n} a_\nu e^{i\nu\theta}$$

is such that $|L(e^{i\theta})| \leqslant 1$ for $0 \leqslant \theta < 2\pi$ and

$$|a_0| + \lambda |a_1| = |b_0| + \frac{\lambda n}{n+1} |b_1| > \frac{1}{2\pi} \int_0^{2\pi} |1 + \lambda e^{i\theta}| \, d\theta - \varepsilon ,$$

where ε can be made arbitrary small by choosing n sufficiently large. Thus $z^n L(z)$ is the polynomial which we wanted to construct.

Proof of Theorem 3. It is known ([1], pp. 140) that if $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ is regular and $|f(z)| \leqslant M$ in $|z| \leqslant 1$, then

$$|a_1| \leqslant M - \frac{|a_0|^2}{M} . \tag{3.3}$$

Now for $0 \leqslant 2u < v \leqslant n$, putting $P(z) = z^{v-2u} P_n(z)$, $w = e^{2\pi i/(v-u)}$ and $\xi = z^{v-u}$, the function

$$\begin{aligned} \frac{P(z) + P(wz) + \ldots + P(w^{v-u-1} z)}{(v-u) z^{v-u}} &= a_u + a_v z^{v-u} + a_{2v-u} z^{2(v-u)} + \ldots \\ &= a_u + a_v \xi + a_{2v-u} \xi^2 + \ldots \end{aligned}$$

of ξ is regular and in absolute value $\leqslant M$ in $|\xi| \leqslant 1$. Hence by (3.3)

$$|a_v| \leqslant M - \frac{|a_u|^2}{M}$$

provided $0 \leqslant 2u < v \leqslant n$. Thus for $\lambda > 0$ and $0 \leqslant 2u < v \leqslant n$,

$$|a_u| + \lambda |a_v| \leqslant \lambda M + |a_u| - \frac{\lambda |a_u|^2}{M} .$$

Whatever be the value of $|a_u|$ the right-hand side of the last inequality is $\leqslant M(\lambda + 1/4\lambda)$ and so the first part of the theorem follows. For the second part we may consider $z^n P_n(1/z)$ instead of $P_n(z)$.

In order to prove (1.16) we note that the function $A - P_n(z)$ is regular for $|z| < 1$, where its real part is positive (we assume that $P_n(z)$ is not a constant; clearly (1.16) is true even if $P_n(z)$ is a constant). It is well known (see for example [12], pp. 194-195) that if $f(z)$ is regular for $|z| < 1$,

$\mathrm{Re}\,\{f(z)\} > 0$, and $f(0) = a > 0$, then $|f'(0)| \leqslant 2a$. Applying this result to the function $A - P_n(z)$ we get

$$|a_1| \leqslant 2(A - a_0) \quad \text{or} \quad a_0 + \tfrac{1}{2}|a_1| \leqslant A\,.$$

Proof of Theorem 5. We denote by

$$V_0(x), V_1(x), \ldots$$

the polynomials defined by

$$V_\nu(t) = \frac{\sin(\nu+1)t}{\sin t}\,,$$

where $x = \cos t$.

The polynomial $P_n(x)$ has a unique expansion

$$P_n(x) = \sum_{\nu=0}^{n} d_\nu V^\nu(x)\,.$$

We put

$$G(t) = P_n(\cos t)\sin t = \sum_{\nu=0}^{n} d_\nu \sin(\nu+1)t$$

$$= \sum_{\nu=0}^{n} d_\nu \frac{e^{i(\nu+1)t} - e^{-i(\nu+1)t}}{2i}$$

$$= \sum_{\nu=-n-1}^{n+1} c_\nu e^{i\nu t}\,,$$

where $c_0 = 0$, $-c_{-\nu} = c_\nu = \dfrac{1}{2i} d_{\nu-1}$ for $\nu \geqslant 1$.

If s and m are two integers of which the first is positive, then ([4], pp. 383)

$$\sum_{p=0}^{s-1} e^{-\frac{2\pi i p m}{s}} G\left(t + \frac{2\pi p}{s}\right) = s \sum_{\nu=m(\mathrm{mod}\, s)} c_\nu e^{i\nu t}\,. \tag{3.4}$$

Putting $s = 2N$ and $m = -N$ for all N for which $N > (n+1)/3$ the left-hand side reduces to $2Nc_N(e^{iNt} - e^{-iNt})$.

Clearly, for any complex λ,

$$\int_0^{2\pi} |1 + e^{i(t - \arg\lambda)}|^\delta dt = \int_0^{2\pi} |1 + e^{it}|^\delta dt$$

so that

$$|\lambda|\left(\frac{1}{2\pi}\int_0^{2\pi}|1+e^{it}|^\delta dt\right)^{1/\delta}=\left(\frac{1}{2\pi}\int_0^{2\pi}|\lambda+\bar{\lambda}e^{it}|^\delta dt\right)^{1/\delta}.$$

Thus for every $\delta\geqslant 1$ and $0\leqslant\gamma<2\pi$,

$$|ae^{-i\gamma}+\bar{b}e^{i\gamma}|\left(\frac{1}{2\pi}\int_0^{2\pi}|1+e^{it}|^\delta dt\right)^{1/\delta}$$

$$=\left(\frac{1}{2\pi}\int_0^{2\pi}|(ae^{-i\gamma}+\bar{b}e^{i\gamma})+(be^{-i\gamma}+\bar{a}e^{i\gamma})e^{it}|^\delta dt\right)^{1/\delta}$$

$$=\left(\frac{1}{2\pi}\int_0^{2\pi}|e^{-i\gamma}(a+be^{it})+e^{i\gamma}(\bar{b}+\bar{a}e^{it})|^\delta dt\right)^{1/\delta}$$

$$\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|a+be^{it}|^\delta dt\right)^{1/\delta}+\left(\frac{1}{2\pi}\int_0^{2\pi}|b+ae^{-it}|^\delta dt\right)^{1/\delta}$$

$$=2\left(\frac{1}{2\pi}\int_0^{2\pi}|a+be^{it}|^\delta dt\right)^{1/\delta}.$$

Choosing γ such that $|ae^{-i\gamma}+\bar{b}e^{i\gamma}|=|a|+|b|$ we get

$$|a|+|b|\leqslant 2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi}|a+be^{it}|^\delta dt\right)^{1/\delta} \tag{3.5}$$

$$=2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi}|ae^{iNt}+be^{-iNt}|^\delta dt\right)^{1/\delta}.$$

Putting $a=2Nc_N$ and $b=-2Nc_N$ in (3.5) and making use of (3.4) we get

$$4N|c_N|\leqslant 2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi}|2Nc_N(e^{iNt}-e^{-iNt})|^\delta dt\right)^{1/\delta}$$

$$=2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{p=0}^{2N-1}e^{\pi ip}G\left(t+\frac{\pi p}{N}\right)\right|^\delta dt\right)^{1/\delta}$$

$$\leqslant 2(C_\delta)^{1/\delta}2N\left(\frac{1}{2\pi}\int_0^{2\pi}|G(t)|^\delta dt\right)^{1/\delta}$$

by Minkowski inequality. Hence

$$|d_{N-1}| = 2|c_N| \leqslant 2(C_\delta)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi} |G(t)|^\delta dt\right)^{1/\delta}$$

$$= 2\left(\frac{2}{\pi}C_\delta\right)^{1/\delta}\left(\frac{1}{2}\int_{-1}^{1} |P_n(x)|^\delta (1-x^2)^{(\delta-1)/2} dx\right)^{1/\delta}.$$

In particular

$$|A_n| = 2^n|d_n| \leqslant 2^{n+1}\left(\frac{2}{\pi}C_\delta\right)^{1/\delta}\left(\frac{1}{2}\int_{-1}^{1} |P_n(x)|^\delta (1-x^2)^{(\delta-1)/2} dx\right)^{1/\delta},$$

$$|A_{n-1}| = 2^{n-1}|d_{n-1}| \leqslant 2^{n}\left(\frac{2}{\pi}C_\delta\right)^{1/\delta}\left(\frac{1}{2}\int_{-1}^{1} |P_n(x)|^\delta (1-x^2)^{(\delta-1)/2} dx\right)^{1/\delta}.$$

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