ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XI, 1 (1969)

E. NEUMAN (Wrocław)

# ON THE CALCULATION OF ELLIPTIC INTEGRALS OF THE SECOND AND THIRD KINDS

**1. Introduction.** Fettis [4] has given a method of computation of elliptic integrals in Legendre's form

$$\Pi(\varphi, a^2, k) = \int_0^{\varphi} \frac{dt}{(1-a^2\sin^2 t)\sqrt{1-k^2\sin^2 t}} \tag{1}$$

$$\left(0 \leqslant \varphi \leqslant \frac{\pi}{2}, \quad -\infty < a^2 < \infty, \quad 0 \leqslant k^2 < 1\right).$$

The method of Fettis does not include the case $k^2 = a^2$ (see [4], eqs. (6) and (33)). This paper contains an algorithm (being somewhat similar to the Fettis method) for finding the value of the integral (1) in this particular case. For $k^2 = a^2$ (see [2], eq. 434.01):

$$\Pi(\varphi, k^2, k) = \left[E(\varphi, k) - \frac{k^2\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right]\frac{1}{1-k^2}, \tag{1'}$$

where

$$E(\varphi, k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2 t}\, dt \tag{2}$$

is the elliptic integral of the second kind in Legendre's form. Let us notice that the function

$$F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}} \tag{3}$$

is the elliptic integral of the first kind in Legendre's form. In equations (2) and (3) the variation domain of $\varphi$ and $k^2$ is the same as in (1). First of all, the calculation of the integral (1') reduces to finding the value of the integral (2). The algorithm given in section 3 allows the calculation of both integral (1) (in the case $k^2 = a^2$) and integral (2).

**2. The method of calculation of the integral (2).** If

$$k_1 = \frac{2\sqrt{k}}{1+k} \quad \text{and} \quad \varphi_1 = \frac{1}{2}[\varphi + \arcsin(k\sin\varphi)]$$

then the following equations hold (see [3]):

$$F(\varphi, k) = \frac{2}{1+k}F(\varphi_1, k_1),$$

$$E(\varphi, k) = (1+k)E(\varphi_1, k_1) + (1-k)F(\varphi_1, k_1) - k\sin\varphi.$$

It is called Landen's transformation. It can be applied repeatedly. Defining the quantities $k_0, k_1, \ldots, \varphi_0, \varphi_1, \ldots$ by the formulae

$$k_0 = k, \quad \varphi_0 = \varphi, \tag{4}$$

$$k_{n+1} = \frac{2\sqrt{k_n}}{1+k_n} \quad (n = 0, 1, \ldots, p-1), \tag{5}$$

$$\varphi_{n+1} = \tfrac{1}{2}[\varphi_n + \arcsin(k_n \sin\varphi_n)] \tag{6}$$

and having calculated the integrals $F(\varphi_p, k_p)$ and $E(\varphi_p, k_p)$ we are able to calculate the integrals $F(\varphi, k)$ and $E(\varphi, k)$ from the formulae

$$F(\varphi_n, k_n) = \frac{2}{1+k_n}F(\varphi_{n+1}, k_{n+1}), \tag{7}$$

$$E(\varphi_n, k_n) = (1+k_n)E(\varphi_{n+1}, k_{n+1}) + (1-k_n)F(\varphi_{n+1}, k_{n+1}) - k_n\sin\varphi_n \tag{8}$$

($n = p-1, p-2, \ldots, 0$). It is easy to show the sequence (5) is fast convergent to 1 for $0 \leqslant k^2 < 1$ (so that $1-k_{n+1} \sim \frac{1}{8}(1-k_n)^2$). For sufficiently large $n$ we obtain (making use of (3) and (2)) the following approximate relations

$$F(\varphi_n, k_n) \approx \log\left(\frac{1}{\cos\varphi_n} + \operatorname{tg}\varphi_n\right), \tag{9}$$

$$E(\varphi_n, k_n) \approx \sin\varphi_n. \tag{10}$$

We want the integral (2) in the form

$$E(\varphi, k) = A_n E(\varphi_n, k_n) + B_n F(\varphi_n, k_n) - C_n. \tag{11}$$

After taking into consideration (9) and (10) the last formula takes the form

$$E(\varphi, k) \approx A_n \sin\varphi_n + B_n \log\left(\frac{1}{\cos\varphi_n} + \operatorname{tg}\varphi_n\right) - C_n. \tag{12}$$

In order to obtain $A_n, B_n, C_n$ we put (7) and (8) into (11), and have

$$\begin{aligned}
E(\varphi, k) &= A_n[(1+k_n)E(\varphi_{n+1}, k_{n+1})+(1-k_n)F(\varphi_{n+1}, k_{n+1})-k_n\sin\varphi_n]+\\
&\quad +B_n\frac{2}{1+k_n}F(\varphi_{n+1}, k_{n+1})-C_n\\
&= A_n(1+k_n)E(\varphi_{n+1}, k_{n+1})+\left[A_n(1-k_n)+\frac{2B_n}{1+k_n}\right]\times\\
&\quad \times F(\varphi_{n+1}, k_{n+1})-(C_n+A_nk_n\sin\varphi_n).
\end{aligned}$$

In virtue of the last formula $A_n, B_n, C_n$ satisfy the following recurrence relations

$$\begin{aligned}
A_{n+1} &= A_n(1+k_n),\\
B_{n+1} &= A_n(1-k_n)+\frac{2B_n}{1+k_n},\\
C_{n+1} &= C_n+A_nk_n\sin\varphi_n,
\end{aligned} \tag{13}$$

with starting values

$$A_0 = 1,\quad B_0 = C_0 = 0. \tag{14}$$

**3. Algorithm. Examples.** For a computation of the approximate value of the integral (1′) it is necessary to find first the value of the integral $E(\varphi, k)$ from eq. (12). For that purpose we make use of formulae (4) to (6) and then of (13) and (14). The algorithm is to be found in this volume. It is written in the ALGOL language in the form a procedure declaration.

### References


[1] В. М. Беляков, Р. И. Кравцова и М. Г. Раппопорт, *Таблицы эллиптических интегралов*, т. 1, Москва 1962.

[2] P. F. Byrd and M. D. Friedman, *Handbook of elliptic integrals for engineers and physicists*, Berlin 1954.

[3] W. G. Fair, and Y. L. Luke, *Rational approximations to the incomplete elliptic integrals of the first and second kinds*, Math. of Comput. 21 (1967), pp. 418-422.

[4] H. E. Fettis, *Calculation of elliptic integrals of the third kind by means of Gauss transformation*, Math. of Comput. 19 (1965), pp. 97-104.


DEPT. OF APPLIED MATHEMATICS
UNIVERSITY OF WROCŁAW

*Received on 28. 3. 1968*

E. NEUMAN (Wrocław)

## O OBLICZANIU CAŁEK ELIPTYCZNYCH DRUGIEGO I TRZECIEGO RODZAJU

STRESZCZENIE

W pracy podano metodę obliczania całki (1), gdy $k^2 = a^2$ (ten przypadek nie jest objęty metodą podaną przez Fettisa [4]). Na mocy (1′) zadanie sprowadza się m. in. do obliczenia całki (2). Stosując transformację Landena dla całki (2), uzyskano przybliżony wzór (12). Występujące w nim wielkości $A_n, B_n, C_n$ oblicza się z wzorów (14) i (13), a $\varphi_n$ i $k_n$ z wzorów (4), (5), (6).

---

Э. НЕЙМАН (Вроцлав)

## О ВЫЧИСЛЕНИИ ЭЛЛИПТИЧЕСКИХ ИНТЕГРАЛОВ ВТОРОГО И ТРЕТЬЕГО РОДА

РЕЗЮМЕ

В работе дан метод вычисления интеграла (1) в случае $k^2 = a^2$ (этот случай не рассмотрен Феттисом в [4]). После (1′) задача сведена к вычислению интеграла (2). С помощью трансформации Ландена для интеграла (2) получена приближенная формула (12). Величины $A_n, B_n, C_n$ в этой формуле вычисляются по формулам (14) и (13), а $\varphi_n$ и $k_n$ по формулам (4), (5), (6).

---