On absolute Nörlund summability factor of Legendre series

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1. Let $f(x)$ be a Lebesgue integrable function in the interval $[-1, 1]$. The Legendre series associated with this function is

$$(1.1) \quad \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) C_n(x) \int_{-1}^1 f(y) C_n(y) dy \equiv \sum V_n(x),$$

where $C_n(x)$ is the n -th Legendre polynomial.

Let $\{s_n\}$ be the sequence of partial sums of a series $\sum u_n$. Let t_n be the Nörlund mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$, real or complex. Let

$$P_n = p_0 + p_1 + p_2 + \dots + p_n, \quad P_n \neq 0.$$

The series $\sum u_n$ is said to be *absolutely summable by Nörlund means*, if the series

$$\sum |t_n - t_{n-1}|$$

is convergent.

In the special case when $p_n = 1/(n+1)$, t_n reduces to the familiar harmonic mean.

2. For Fourier series of a function $F(x)$ integrable (L) in $(-\pi, \pi)$ and periodic with period 2π , Varshney [6] has proved the following theorem on the absolute harmonic summability of the series:

$$(2.1) \quad \sum_1^{\infty} \frac{a_n \cos nt + b_n \sin nt}{\log(n+1)},$$

a_n, b_n are Fourier coefficients of $F(t)$.

THEOREM A. If

$$\tfrac{1}{2} \{F(x+t) + F(x-t)\}$$

is of bounded variation in $(0, \pi)$, then the series (2.1) is absolutely summable by harmonic means.

In a recent paper Pandey [3] has applied the method of absolute harmonic summability to the series

$$(2.2) \quad \sum_{n=1}^{\infty} \frac{V_n(x)}{\log(n+1)}.$$

He has established the following

THEOREM B. *If $f(x)$ is of bounded variation in $[-1, 1]$, then the series (2.2) is absolutely summable by harmonic means at a point x of the interval $(-1, 1)$.*

In the present paper we extend Theorem B to the general Nörlund means and prove the following

THEOREM. *If $f(x)$ is of bounded variation in $[-1, 1]$, then the series*

$$\sum_{n=1}^{\infty} \frac{(n+1)p_n}{P_n} \cdot V_n(x)$$

is absolutely summable by Nörlund means at a point x of the interval $(-1, 1)$, where the sequence $\{p_n\}$ is real, non-negative and non-increasing such that,

- (1) *the sequence $\{(n+1)p_n/P_n\}$ is of bounded variation,*
- (2) *the sequence $\{p_n - p_{n+1}\}$ is non-increasing.*

3. To prove the theorem we need the following lemmas:

LEMMA 1 [1]. *If $\{p_n\}$ is non-negative and non-increasing sequence, then for $0 < t < \pi$ and for every n, a, b we have*

$$\left| \sum_{k=a}^b p_k e^{i(n-k)t} \right| \leq AP_T,$$

where A is an absolute constant and $T = [1/t]$, and $[m]$ denotes the integral part in m .

LEMMA 2. *If $\{p_n\}$ is positive and non-increasing, then the sequence $\left\{ \frac{P_n - P_k}{P_{n-k}} \right\}$ is non-increasing and bounded for $k < n$.*

Proof. We have

$$\frac{P_n - P_{k+1}}{P_{n-k-1}} = \frac{(P_n - P_k) - p_{k+1}}{P_{n-k} - p_{n-k}}$$

and

$$\frac{P_n - P_k}{P_{n-k}} \leq \frac{(n-k)p_{k+1}}{(n-k)p_{n-k}} = \frac{p_{k+1}}{p_{n-k}}.$$

In view of the relation

$$A/B \geq \frac{A-a}{B-b} \quad \text{if} \quad A/B \leq a/b,$$

A, B, a, b are positive and $A > a, B > b$, the lemma is proved.

LEMMA 3. If $\{p_n - p_{n+1}\}$ is non-increasing, then the sequence $\left\{\frac{p_k - p_n}{n-k}\right\}$ is non-increasing for $k < n$.

Proof. We have

$$\frac{p_{k+1} - p_n}{n-k-1} = \frac{(p_k - p_n) - (p_k - p_{k+1})}{(n-k) - 1}$$

and

$$\frac{p_k - p_n}{n-k} = \frac{\sum_{v=k}^{n-1} (p_v - p_{v+1})}{n-k} \leq \frac{(n-k)(p_k - p_{k+1})}{n-k} = \frac{(p_k - p_{k+1})}{1},$$

the lemma follows on the lines of Lemma 2.

LEMMA 4. If the sequence $\left\{\frac{(n+1)p_n}{P_n}\right\}$ is of bounded variation, then

$$\sum_{k=m}^{n-2} \left| \Delta \left\{ \left(\frac{p_k - p_n}{n-k} \right) P_k \frac{(n-k+1)p_{n-k}}{P_{n-k}} \right\} \right| = O \left[\frac{P_{n-1} p_m}{m} \right],$$

where m is the integral part of $n/2$.

Proof. Since $\left\{\frac{p_k - p_n}{n-k}\right\}$ is non-increasing, we have

$$\begin{aligned} & \sum_{k=m}^{n-2} \left| \Delta \left\{ \left(\frac{p_k - p_n}{n-k} \right) P_k \frac{(n-k+1)p_{n-k}}{P_{n-k}} \right\} \right| \\ &= \left[\sum_{k=m}^{n-2} \frac{(n-k+1)p_{n-k}}{P_{n-k}} P_k \left| \Delta \left(\frac{p_{k+1} - p_n}{n-k-1} \right) \right| \right] + \\ &+ \left[\sum_{k=m}^{n-2} \frac{(n-k+1)p_{n-k}}{P_{n-k}} p_{k+1} \frac{p_{k+1} - p_n}{n-k-1} \right] + \\ &+ \left[\sum_{k=m}^{n-2} \frac{(p_{k+1} - p_n)}{n-k-1} P_k \left| \Delta \left(\frac{(n-k+1)p_{n-k}}{P_{n-k}} \right) \right| \right] \end{aligned}$$

$$\begin{aligned}
&= O\left[P_{n-1}\left|\sum_{k=m}^{n-2}\Delta\left(\frac{p_k-p_n}{n-k}\right)\right|\right]+O\left[\frac{p_m-p_n}{n-m}\sum_{k=m}^{n-2}p_{k+1}\right]+ \\
&\quad +O\left[P_{n-1}\frac{p_m-p_n}{n-m}\cdot\sum_{k=m}^{n-2}\left|\Delta\left(\frac{(n-k+1)p_{n-k}}{P_{n-k}}\right)\right|\right] \\
&= O\left[P_{n-1}\left\{\frac{p_m-p_n}{n-m}-\frac{p_{n-1}-p_n}{1}\right\}\right]+ \\
&\quad +O\left[\frac{p_m-p_n}{n-m}(P_{n-1}-P_m)\right]+O\left[P_{n-1}\frac{p_{m+1}-p_n}{n-m-1}\right] \\
&= O\left[P_{n-1}\cdot\frac{p_m}{m}\right].
\end{aligned}$$

This lemma has been proved by Singh [5].

LEMMA 5 ([2], p. 70). For $0 \leq \theta \leq \pi$

$$|C_{n+1}(\cos \theta) - C_{n-1}(\cos \theta)| \leq m \sqrt{\frac{\sin \theta}{n}},$$

m is independent of n and θ .

LEMMA 6. For $\varepsilon \leq \theta \leq \pi - \varepsilon$, $0 < \varepsilon < \pi/2$; $\varepsilon_1 \leq \Phi \leq \pi - \varepsilon_1$, $0 < \varepsilon_1 < \pi/2$,

$$\begin{aligned}
&[C_{n-k+1}(\cos \Phi) - C_{n-k-1}(\cos \Phi)]C_{n-k}(\cos \theta) \\
&= \frac{2}{\pi(n-k)} \sqrt{\frac{\sin \Phi}{\sin \theta}} [\sin(n-k+\frac{1}{2})(\Phi - \theta) + \cos(n-k+\frac{1}{2})(\theta + \Phi)] + \\
&\quad + O(n-k)^{-2}.
\end{aligned}$$

The proof of the lemma follows from the relation ([4], p. 208)

$$C_n(\cos \Phi) = \sqrt{\frac{2}{n\pi \sin \Phi}} \cos\left\{\left(n+\frac{1}{2}\right)\Phi - \frac{\pi}{4}\right\} + O(n^{-3/2}), \quad \varepsilon \leq \Phi \leq \pi - \varepsilon.$$

4. Proof of the theorem. We have

$$\begin{aligned}
t_n - t_{n-1} &= \sum_{v=0}^{n-1} \left(\frac{P_v}{P_n} - \frac{P_{v-1}}{P_{n-1}}\right) V_{n-v}(x) \frac{(n-v+1)p_{n-v}}{P_{n-v}} \\
&= \frac{1}{P_n P_{n-1}} \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{(n-v+1)p_{n-v}}{P_{n-v}} C_{n-v}(x) \int_{-1}^1 f(y) C_{n-v}(y) dy
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2P_n P_{n-1}} \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{(n-v+1)p_{n-v}}{P_{n-v}} c_{n-v}(x) \times \\
&\quad \times \int_{-1}^1 f(y) \left\{ \frac{d}{dy} C_{n-v+1}(y) - \frac{d}{dy} C_{n-v-1}(y) \right\} dy.
\end{aligned}$$

Now in view of the relation

$$C_n(1) = 1, \quad C_n(-1) = (-1)^n,$$

in order to prove the theorem we have to show that

$$\begin{aligned}
(4.1) \quad \sum_n \frac{1}{P_n P_{n-1}} \left| \int_{-1}^1 \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{(n-v+1)p_{n-v}}{P_{n-v}} \cdot C_{n-v}(x) \times \right. \\
\left. \times \{C_{n-v+1}(y) - C_{n-v-1}(y)\} df(y) \right| < \infty
\end{aligned}$$

we set $x = \cos \theta$, $y = \cos \Phi$ and the above expression becomes

$$\begin{aligned}
(4.2) \quad \sum_n \frac{1}{P_n P_{n-1}} \left| \int_0^\pi \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{(n-v+1)p_{n-v}}{P_{n-v}} C_{n-v}(\cos \theta) \times \right. \\
\left. \times \{C_{n-v+1}(\cos \Phi) - C_{n-v-1}(\cos \Phi)\} df(\cos \Phi) \right| \\
= \sum_n \frac{1}{P_n P_{n-1}} \left| \int_0^{\frac{c}{\sqrt{n}}} + \int_{\frac{c}{\sqrt{n}}}^{\theta - \frac{cp_n}{P_n^2}} + \int_{\theta - \frac{cp_n}{P_n^2}}^{\theta + \frac{cp_n}{P_n^2}} + \int_{\theta + \frac{cp_n}{P_n^2}}^{\pi - \frac{c}{\sqrt{n}}} + \int_{\pi - \frac{c}{\sqrt{n}}}^\pi \right| \\
\leq J_1 + J_2 + J_3 + J_4 + J_5
\end{aligned}$$

say. We have

$$\begin{aligned}
(4.3) \quad J_1 &\leq \sum_n \frac{1}{P_n P_{n-1}} \left| \int_0^{\frac{c}{\sqrt{n}}} \sum_{v=0}^{n-1} P_n p_v \frac{m \sqrt{\sin \Phi}}{(n-v)} df(\cos \varphi) \right| \\
&\leq \sum_n \frac{1}{P_n P_{n-1}} \frac{m}{n^{1/4}} \sum_{v=0}^{n-1} \frac{P_n p_v}{n-v} \\
&= \sum_n \frac{1}{P_n P_{n-1}} \frac{m}{n^{1/4}} \frac{1}{n+1} \sum_{v=0}^{n-1} \left(\frac{1}{n-v} + \frac{1}{v+1} \right) (v+1) P_n p_v \\
&= O(1).
\end{aligned}$$

We now consider

$$J_3 \leq \sum_1^T + \sum_{T+1}^\infty = J_{3.1} + J_{3.2}, \quad \text{where} \quad T = \left\lfloor \frac{1}{\theta + \Phi} \right\rfloor,$$

$$(4.4) \quad J_{3.1} \leq \sum_1^T \frac{\sin \Phi}{P_n P_{n-1}} P_n P_{n-1} \sum_{v=0}^{n-1} \frac{1}{(v+1)(n-v)} = O(1).$$

Also

$$J_{3.2} = \sum_{T+1}^\infty \frac{1}{P_n P_{n-1}} \left| \int_{\theta - c p_n / P_n^2}^{\theta + c p_n / P_n^2} \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{(n-v+1) p_{n-v}}{P_{n-v}} \times \right.$$

$$\times \left[\frac{2}{\pi(n-v)} \sqrt{\frac{\sin \Phi}{\sin \theta}} \{ \sin(n-v+\frac{1}{2})(\Phi - \theta) + \cos(n-v+\frac{1}{2})(\theta + \Phi) \} + \right.$$

$$\left. \left. + O(n-v)^{-2} \right] df(\cos \Phi) \right|$$

$$\leq J_{3.2.1} + J_{3.2.2} + J_{3.2.3}$$

say. Now

$$J_{3.2.2} \leq J_{3.2.2.1} + J_{3.2.2.2},$$

where

$$J_{3.2.2.1} = \sum_{T+1}^\infty \frac{1}{P_n P_{n-1}} \left| \int_{\theta - c p_n / P_n^2}^{\theta + c p_n / P_n^2} \sum_{n=0}^{[n/2]-1} \left(P_n - \frac{P_v P_n}{p_v} \right) p_v \frac{(n-v+1) p_{n-v}}{P_{n-v}} \times \right.$$

$$\times \frac{2}{\pi(n-v)} \sqrt{\frac{\sin \Phi}{\sin \theta}} \{ \cos(n-v+\frac{1}{2})(\theta + \Phi) \} df(\cos \Phi) \left| \right.$$

Now by the Abel's transformation and the hypothesis

$$J_{3.2.2.1} = O \left\{ \sum_{T+1}^\infty \frac{1}{P_n P_{n-1}} \left[P_T \left(\frac{P_n - p_n P_{[n/2]-1}}{P_{[n/2]-1}} \right) \frac{(n - [n/2] + 2) p_{n - [n/2] + 1}}{(n - [n/2] + 1) P_{n - [n/2] + 1}} \right] \right\} +$$

$$+ O \left[\sum_{T+1}^\infty \frac{1}{P_n P_{n-1}} P_T \sum_{v=0}^{[n/2]-2} \left| \Delta \left\{ (P_n - p_n P_v / p_v) \frac{(n-v+1) p_{n-v}}{P_{n-v}} \right\} \right| \right]$$

$$= O \left[\sum_{T+1}^\infty \frac{1}{P_n P_{n-1}} \{ P_T p_{n - [n/2] + 1} \} \right] +$$

$$\begin{aligned}
& + O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left\{ P_T \sum_{v=0}^{[n/2]-2} \left(P_n - \frac{p_n P_v}{p_v} \right) \frac{(n-v+1)}{n-v} \left| \Delta \left(\frac{p_{n-v}}{P_{n-v}} \right) \right| \right\} \right] + \\
& + O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left\{ P_T p_n \sum_{v=0}^{[n/2]-2} \frac{(n-v+1) p_{n-v}}{(n-v) P_{n-v}} \left(\frac{P_{v+1}}{p_{v+1}} - \frac{P_v}{p_v} \right) \right\} \right] + \\
& + O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left\{ P_T \sum_{v=0}^{[n/2]-2} \frac{p_{n-v}}{P_{n-v}} \left(P_n - \frac{p_n P_v}{p_v} \right) \frac{1}{(n-v)(n-v+1)} \right\} \right] \\
& = O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} (P_T p_m) \right] + O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left\{ P_T P_n \frac{p_{n-[n/2]+1}}{P_{n-[n/2]+1}} \right\} \right] + \\
& + O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left\{ P_T p_n \frac{p_{n-[n/2]+2}}{P_{n-[n/2]+2}} \frac{P_{[n/2]-1}}{p_{[n/2]-1}} \right\} \right] \\
& = O \left[\sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} P_T p_{[n/2]} \right]
\end{aligned}$$

since

$$P_n \leq 2P_{[n/2]},$$

therefore

$$(4.5) \quad J_{3 \cdot 2 \cdot 2 \cdot 1} = O \left[P_T \sum_{T+1}^{\infty} \frac{p_{[n/2]}}{P_{[n/2]} P_{[n/2]-1}} \right] = O(1).$$

We now consider $J_{3 \cdot 2 \cdot 2 \cdot 2}$. Since

$$P_n p_k - P_k p_n = (P_n - P_k) p_k + (p_k - p_n) P_k$$

we have

$$\begin{aligned}
J_{3 \cdot 2 \cdot 2 \cdot 2} & \leq \sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\theta - c p_n / P_n^2}^{\theta + c p_n / P_n^2} \sum_{v=[n/2]}^{n-1} (P_n - P_v) p_v \frac{(n-v+1) p_{n-v}}{P_{n-v}} \frac{2}{\pi(n-v)} \times \right. \\
& \times \left. \sqrt{\frac{\sin \Phi}{\sin \theta}} \cos(n-v+\tfrac{1}{2})(\theta+\Phi) df(\cos \Phi) \right| + \\
& + \sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\theta - c p_n / P_n^2}^{\theta + c p_n / P_n^2} \sum_{[n/2]}^{n-1} (p_v - p_n) P_v \frac{1}{(n-v)} \frac{(n-v+1) p_{n-v}}{P_{n-v}} \times \right. \\
& \times \left. \cos(n-v+\tfrac{1}{2})(\theta+\Phi) df(\cos \Phi) \right| \\
& \leq J'_{3 \cdot 2 \cdot 2 \cdot 2} + J''_{3 \cdot 2 \cdot 2 \cdot 2}.
\end{aligned}$$

Also as $\left\{\frac{P_n - P_v}{P_{n-v}}\right\}$ is non-increasing and bounded sequence for $v < n$ therefore by Abel transformation we have

$$\begin{aligned}
 & \left| \sum_{v=[n/2]}^{n-1} \left(\frac{P_n - P_v}{P_{n-v}} \right) \left(\frac{n-v+1}{n-v} p_v \right) p_{n-v} \cos(n-v+\tfrac{1}{2})(\theta+\Phi) \right| \\
 &= O \left[P_T \sum_{v=[n/2]}^{n-1} \left| \Delta \left\{ \left(\frac{P_n - P_v}{P_{n-v}} \right) \left(\frac{n-v+1}{n-v} p_v \right) \right\} \right| \right] + O[P_T p_{n-1} p_n] + \\
 & \quad + O \left[P_T p_{[n/2]} \frac{P_n - P_{[n/2]}}{P_{n-[n/2]}} \right] \\
 &= O \left[P_T \sum_{v=[n/2]}^{n-2} p_v \frac{(n-v+1)}{(n-v)} \left| \Delta \left(\frac{P_n - P_v}{P_{n-v}} \right) \right| \right] + \\
 & \quad + O \left[P_T \sum_{v=[n/2]}^{n-2} \frac{P_n - P_v}{P_{n-v}} \frac{(n-v+1)}{(n-v)} (p_v - p_{v+1}) \right] + \\
 & \quad + O \left[P_T \sum_{v=[n/2]}^{n-2} p_v \frac{P_n - P_v}{P_{n-v}(n-v)(n-v+1)} \right] + O(P_T p_n p_{n-1}) + \\
 & \quad + O(P_T p_{[n/2]}) \\
 &= O \left[P_T p_{[n/2]} \left\{ \frac{P_n - P_{[n/2]}}{P_{n-[n/2]}} - \frac{P_n - P_{n-1}}{P_1} \right\} \right] + \\
 & \quad + O \{ P_T (p_{[n/2]} - p_{[n/2]-1}) \} + O(P_T p_{[n/2]}) + O(P_T p_{[n/2]}) \\
 &= O(P_T p_{[n/2]}) .
 \end{aligned}$$

Therefore,

$$(4.6) \quad J'_{3 \cdot 2 \cdot 2 \cdot 2} = O \sum_{T+1}^{\infty} P_T \frac{p_{[n/2]}}{P_n P_{n-1}} = O(1) .$$

By Abel's transformation and Lemma 4 we get for the inner sum of $J''_{3 \cdot 2 \cdot 2 \cdot 2}$

$$\begin{aligned}
 & \left| \sum_{v=[n/2]}^{n-1} \frac{(p_v - p_n) P_v (n-v+1) p_{n-v}}{(n-v) P_{n-v}} \cos(n-v+\tfrac{1}{2})(\theta+\Phi) \right| \\
 &= O \left[\frac{P_{n-1} p_{[n/2]}}{t_{[n/2]}} \right] + O \left[\frac{1}{t} P_{n-1} (p_{n-1} - p_n) \right] + O \left[\frac{P_{[n/2]} p_{[n/2]}}{t_{[n/2]}} \right] \\
 &= O \left[\frac{P_{n-1} p_{[n/2]}}{t_{[n/2]}} \right] + O \left[\frac{1}{t} P_{n-1} (p_{n-1} - p_n) \right] ,
 \end{aligned}$$

where

$$t \equiv (\theta + \Phi)$$

now

$$(4.7) \quad J_{3 \cdot 2 \cdot 2}'' = O \sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \frac{P_{n-1} p_{[n/2]}}{t_{[n/2]}} + O \sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \frac{p_{n-1} - p_n}{t} P_{n-1} \\ = O(1).$$

Thus we see that

$$(4.8) \quad J_{3 \cdot 2 \cdot 2} = O(1).$$

Also

$$(4.9) \quad J_{3 \cdot 2 \cdot 1} = \sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\theta - c p_n / P_n^2}^{\theta + c p_n / P_n^2} \sum_{v=0}^{n-1} P_n p_v |\Phi - \theta| |df(\cos \Phi)| \right| \\ = \sum_{T+1}^{\infty} \frac{1}{P_n P_{n-1}} \sum_{v=0}^{n-1} P_n p_v \frac{c p_n}{P_n^2} = O(1)$$

and obviously we get

$$(4.10) \quad J_{3 \cdot 2 \cdot 3} = O(1),$$

and thus

$$(4.11) \quad J_3 = O(1).$$

We shall now consider

$$J_2 = \sum_1^e + \sum_{e+1}^{\infty} \leq J_{2 \cdot 1} + J_{2 \cdot 2}$$

say. Where e is any positive integer.

Now

$$(4.12) \quad J_{2 \cdot 1} = O(1)$$

as in the case of $J_{3 \cdot 1}$.

Now to calculate $J_{2 \cdot 2}$ we use the following asymptotic expression for $C_n(\cos \Phi)$, for $c/\sqrt{n} \leq \Phi \leq \pi - c/\sqrt{n}$

$$C_n(\cos \Phi) = \sqrt{\frac{2}{4\pi \sin \Phi}} \cos \left\{ \left(n + \frac{1}{2} \right) \Phi - \frac{\pi}{4} \right\} + (n \sin \Phi)^{-1} O(1),$$

and therefore we get

$$C_{n-v}(\cos \theta) \{c_{n-v+1}(\cos \Phi) - c_{n-v-1}(\cos \Phi)\} \\ = \frac{2}{\pi(n-v)} \cdot \sqrt{\frac{\sin \Phi}{\sin \theta}} \{ \cos(n-v+\frac{1}{2})(\theta + \Phi) + \sin(n-v+\frac{1}{2})(\Phi - \theta) \} + \\ + O(n-v)^{-2} \sin^{-3/2} \Phi.$$

Thus

$$\begin{aligned}
 J_{2.2} &= \sum_{e+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\frac{c}{\sqrt{n}}}^{\theta - c p_n / P_n^{\frac{1}{2}}} \sum_{v=0}^{n-1} (P_n p_v - P_v p_n) \frac{(n-v+1) p_{n-v}}{P_{n-v}} \times \right. \\
 &\quad \times \left[\frac{2}{\pi(n-v)} \sqrt{\frac{\sin \Phi}{\sin \theta}} \{ \cos(n-v+\frac{1}{2})(\theta+\Phi) + \sin(n-v+\frac{1}{2})(\Phi-\theta) \} + \right. \\
 &\quad \left. \left. + O(n-v)^{-2} \sin^{-3/2} \Phi \right] df(\cos \Phi) \right| \\
 &= \sum_{e+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \int_{\frac{c}{\sqrt{n}}}^{\theta - c p_n / P_n^{\frac{1}{2}}} \left\{ \sum_{v=0}^{[n/2]-1} + \sum_{[n/2]}^{n-1} \right\} \dots \right| \\
 &\leq J_{2.2.1} + J_{2.2.2}.
 \end{aligned}$$

Also

$$J_{2.2.1} \leq J_{2.2.1.1} + J_{2.2.1.2} + J_{2.2.1.3}.$$

Proceeding as in $J_{3.2.2.1}$ we can show that

$$(4.13) \quad J_{2.2.1.1} = O(1).$$

Now to calculate $J_{2.2.1.2}$ we choose $e = \left[\frac{1}{\theta - \Phi} \right]$ and following the procedure adopted in $J_{3.2.2.1}$ we get

$$(4.14) \quad J_{2.2.1.2} = O(1)$$

and easily

$$(4.15) \quad J_{2.2.1.3} = O(1).$$

Thus we see that

$$(4.16) \quad J_{2.2.1} = O(1),$$

$J_{2.2.2}$ can also be treated in a similar manner. J_4 and J_5 can be disposed off in exactly the same way as J_2 and J_1 , respectively.

This completes the proof of the theorem.

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