Proof of Corollary. If $f(0) \neq 0$ we set $g(x) = Ax^n + Bx^m + f(x)$ and apply Theorem with $A = B = 1$ if $f(1) \neq -2$, with $A = -B = 1$ if $f(1) = -2$.

The inequality for $|g_0|$ follows, even with $\|f\|+3$ replaced by $\|f\|+2$.

If $f(0) = 0$ we set $g(x) = Ax^n + Bx^m + f(x) + 1$ and apply Theorem with $A = B = 1$ if $f(1) \neq -3$, with $A = -B = 1$ if $f(1) = -3$.

If $f(x) \not\equiv 0$ we have $|f(x)+1| = |f|$, $\|f(x)+1\| = f+1$, which implies the inequality for $|g_0|$. If $f(x) \equiv 0$, $|f| = -\infty$ we set $g_0(x) = x$.

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Reçu par la Rédaction le 30. 4. 1969

ACTA ARITHMETICA
XVI (1970)

On a generalization of a theorem of Borel

by

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1. Let τ be a real number between 0 and 1. A classical theorem of Borel asserts that if we put

$$\tau = \sum_{k=1}^{\infty} \varepsilon_k(\tau) 2^{-k} \qquad (\varepsilon_k = 0 \text{ or } 1)$$

then we have for almost all τ

$$\sum_{k=1}^{n} \varepsilon_k(\tau) \sim \frac{n}{2}.$$

An analogous result holds, of course, for expansions with respect to an arbitrary basis, for instance, for decimal expansions.

Now let a be an irrational number with the regular continued fraction expansion

$$a = \{0;\ a_1, a_2, \ldots\} \tag{1.1}$$

and put

$$D_n = \frac{(-1)^n}{\zeta_{n+1} B_n + B_{n-1}} = B_n a - A_n, \tag{1.2}$$

where A_n/B_n are the convergents of a and $\zeta_n = \{a_n; a_{n+1}, \ldots\}$.

It is well known [3] that each τ with $D_1 < \tau < 1 - D_1$ can be represented in the form

$$\tau = \sum_{k=0}^{\infty} C_{k+1}(\tau) D_k \tag{1.3}$$

where $C_1(\tau) < a_1$, $0 \leqslant C_{k+1}(\tau) \leqslant a_{k+1}$ and $C_{k+1}(\tau) = a_{k+1} \Rightarrow C_k(\tau) = 0$. We have uniqueness if in addition we do not allow $C_{k+2i} = a_{k+2i}$ for some k and $i = 1, 2, \ldots$

For some purposes of the metrical theory of inhomogeneous diophantine approximation it would be of interest to know something about the behavior of the $C_k(\tau)$'s for almost all τ's, where a is a given irrational number. In the present paper we prove such a theorem, which gives as a special case that if a is a quadratic irrationality, then for almost all τ the asymptotic density for each digit in (1.3) exists and is independent of τ.

The main result of the present paper is the following

THEOREM 1.1. *We have for almost all* τ:

$$\begin{cases} \sum\limits_{\substack{k=0\\ C_{k+1}=0}}^{n} 1 - \sum\limits_{k=0}^{n} B_k|D_k - D_{k+1}| = o(n), \\ \sum\limits_{\substack{k=0\\ C_{k+1}=r}}^{n} 1 - \sum\limits_{\substack{k=0\\ a_{k+1}>r}}^{n} B_k|D_k| - \sum\limits_{\substack{k=0\\ a_{k+1}=r}}^{n} B_{k-1}|D_k| = o(n) \qquad (r>0). \end{cases} \tag{1.4}$$

Statement (1.4) has particular interest when the partial quotients of a repeat periodically (i.e. a is a quadratic irrationality). We obtain

THEOREM 1.2. *Let a be a quadratic irrationality. Then for almost all* τ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{k=0\\ C_{k+1}=r}}^{n} 1 = k_r(a), \tag{1.5}$$

where the limit $k_r(a)$ depends only on a and r, but not on τ. The value of $k_r(a)$ can be determined.

As a further problem we mention the convergence of

$$P\Big[\Big\{\sum_{\substack{k=1\\ C_{k+1}=r}}^{n} 1 - nc_1\Big\}\{nc_2\}^{-1/2} \leqslant x\Big]$$

to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy$$

and the law of iterated logarithm. We intend to return to these in a forthcoming paper.

The following sections contain the proofs of Theorem 1.1 and 1.2.

2. Let a be an irrational number between 0 and 1 with regular continued fraction expansion $a = \{0; a_1, a_2, \ldots\}$; A_n, B_n, D_n and ζ_{n+1} $(n = 0, 1, 2, \ldots)$ defined by (1.2). Denote by $P(A)$ the measure of the set of numbers τ for which the property A holds.

LEMMA 2.1.

$$P(C_{k+1}(\tau)=r) = \begin{cases} B_k|D_k - D_{k+1}| & \text{if} \quad r=0, \\ B_k|D_k| & \text{if} \quad 0<r<a_{k+1}, \\ B_{k-1}|D_k| & \text{if} \quad r=a_{k+1}. \end{cases} \tag{2.1}$$

Proof. Each τ can be written in the form (1.3). The set of τ's with

$$\tau = C_1D_0 + C_2D_1 + \ldots + C_{k+1}D_k + \ldots, \tag{2.2}$$

where $C_1, \ldots, C_{k+1}$ are given is the set of τ's between

$$C_1D_0 + \ldots + C_{k+1}D_k + (a_{k+2}-1)D_{k+1} + a_{k+4}D_{k+3} + \ldots \tag{2.3}$$

and

$$C_1D_0 + \ldots + C_{k+1}D_k + a_{k+3}D_{k+2} + a_{k+5}D_{k+4} + \ldots \tag{2.4}$$

if $C_{k+1} \geqslant 1$. This is an interval of length

$$|D_{k+1} - (D_k - D_{k+1})| = |D_k|. \tag{2.5}$$

If $C_{k+1} = 0$, then the length of the corresponding interval is equal to

$$|D_k - D_{k+1}| \tag{2.6}$$

since then in (2.3) the term $(a_{k+2}-1)D_{k+1}$ is replaced by $a_{k+2}D_{k+1}$.

To each set $C_1, \ldots, C_{k+1}$ there corresponds such an interval and obviously different intervals are disjoint.

If we want to fix only C_{k+1}, we have to let $C_1, \ldots, C_k$ run over all possibilities. There are B_k possibilities if $C_{k+1} < a_{k+1}$ and B_{k-1} otherwise since $C_{k+1} = a_{k+1} \Rightarrow C_k = 0$. This proves our Lemma 2.1.

Using the same reasoning we obtain a similar formula for the measure $P(C_{k+1} = r_1, C_{l+1} = r_2)$ which shows that we have

LEMMA 2.2. *There is a* q, $0<q<1$, *such that*

$$\frac{P(C_{k+1} = r_1, C_{l+1} = r_2)}{P(C_{k+1} = r_1)P(C_{l+1} = r_2)} = 1 + O(q^{|k-l|}). \tag{2.7}$$

3. Now we are in a position to complete our proofs.

Proof of Theorem 1.1. Let τ be a random variable uniformly distributed in $(D_1, 1-D_1)$. Define the random variable $\xi_1, \xi_2, \ldots$ by

$$\xi_{k+1} = \xi_{k+1}(r) = \begin{cases} 1 & \text{if} \quad C_{k+1}(\tau) = r, \\ 0 & \text{otherwise}; \end{cases} \tag{3.1}$$

put

$$\zeta_n = \zeta_n(r) = \sum_{k=0}^{n} \xi_{k+1};$$

$$M(\zeta_n) = \begin{cases} \sum_{k=0}^{n} B_k|D_k - D_{k+1}| & \text{if} \quad r = 0, \\ \sum_{\substack{k=1 \\ a_k > r}}^{n} B_k|D_k| + \sum_{\substack{k=0 \\ a_k = r}}^{n} B_{k-1}|D_k| & \text{if} \quad r > 0. \end{cases}$$

Equation (2.7), Čebyšev's inequality and a direct application of Borel–Cantelli's lemma give that almost everywhere in τ

(3.2) $$\zeta_{m^2} - M(\zeta_{m^2}) = o(m^2).$$

An even stronger statement holds almost everywhere, namely

(3.3) $$\zeta_n - M(\zeta_n) = o(n).$$

This follows by repeating the argument of Khintchine [1], pp. 89–95, which proves Theorem 1.1.

Proof of Theorem 1.2. If α is a quadratic irrationality, then the sequence $a_1, a_2, \ldots$ is periodic. Suppose that

$$a_{km+l} = a_{km'+l}, \quad m, m' \geqslant m_0,$$

which is equivalent to

(3.4) $$\zeta_{km+l} = \zeta_{km'+l}; \quad m, m' \geqslant m_0.$$

Since, as well known (see Perron [2], p. 27)

$$\frac{B_{n-1}}{B_n} = \{0; a_n, a_{n-1}, \ldots, a_1\}$$

and, therefore, the limits

$$\lim_{m\to\infty} \frac{B_{km+l-1}}{B_{km+l}}, \quad l = 0, 1, \ldots, k-1$$

exist, the limits

$$\lim_{m\to\infty} B_{km+l}|D_{km+l} - D_{km+l+1}|,$$

$$\lim_{m\to\infty} B_{km+l}|D_{km+l}|$$

and

$$\lim_{m\to\infty} B_{kml-1}|D_{km+l}|$$

exist for $l = 0, 1, \ldots, k-1$. By Theorem 1.1 we obtain Theorem 1.2.

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Reçu par la Rédaction le 30. 4. 1969

ACTA ARITHMETICA
XVI (1970)

An effective p-adic analogue of a theorem of Thue II
The greatest prime factor of a binary form

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I. Introduction. In part I of this paper [4] there appeared various numbers, which, it was asserted, could be effectively determined, but which in fact were not explicitly calculated. The purpose of the present paper is to derive appropriate values for these numbers, and thereby to obtain explicit statements of the principal results of [4]. As in [4], $f(x, y)$ will signify a binary form with integer coefficients and degree $n \geqslant 3$, irreducible over the rationals, and m will signify a non-zero integer. By $p_1, \ldots, p_s$ we shall denote a set of $s \geqslant 0$ prime numbers, and we shall use $\mathfrak{m}$ to denote the largest integer, comprised solely of powers of $p_1, \ldots, p_s$, which divides m. We denote by $\mathfrak{F}$ any number not less than the maximum of the absolute values of the coefficients of $f(x, y)$, and we suppose that $\mathfrak{F} \geqslant 2$. We write P for the maximum of $p_1, \ldots, p_s$; if no primes $p_1, \ldots, p_s$ are specified, we take $P = 2$. Finally, we signify by $\varkappa$ any number satisfying $\varkappa > n(s+1)+1$. Then we shall establish the following explicit form of Theorem 1 of [4].

THEOREM 1. *All solutions of the equation $f(x, y) = m$ in integers x, y, with $(x, y, p_1 \ldots p_s) = 1$, satisfy*

$$\max(|x|, |y|) < \exp\{2^{\nu^2} P^{26n^6\nu} \mathfrak{F}^{2n^3\nu} + (\log(|m|/\mathfrak{m}))^{\varkappa}\},$$

where $\nu = 64n(s+1)\varkappa^2/(\varkappa - n(s+1) - 1)$.

It will be observed that when $s = 0$, that is when no primes $p_1, \ldots, p_s$ are specified, Theorem 1 reduces to a slightly weaker form of the main result of Baker's paper [2]. On the other hand, if m is comprised solely of powers of $p_1, \ldots, p_s$ so that $|m|/\mathfrak{m} = 1$, then Theorem 1 implies that all solutions of the equation $f(x, y) = m$ in integers x, y with $(x, y, p_1 \ldots p_s) = 1$, satisfy

$$\max(|x|, |y|) < \exp\{2^{\nu^2} P^{26n^6\nu} \mathfrak{F}^{2n^3\nu}\}. \tag{1}$$

The interest of this result lies in the fact that the number on the right does not depend on the exponents to which $p_1, \ldots, p_s$ divide m. In partic-