Metrizable subsets of Moore spaces

by

C. W. Proctor* (Houston)

In [10], Younglove proved that if a space satisfies Axioms 0 and 1 of [8], then it contains a dense, metrizable, inner limiting subset. Fitzpatrick [4] has shown that a normal Moore space which is not a counterexample of Type D [2] has a dense, metrizable subset. Sufficient conditions are given in this paper for a Moore space to have a dense, metrizable subset. Fitzpatrick [3] pointed out that Mary Ellen Estill Rudin's example [9] of a non-separable Moore space in which every collection of mutually exclusive domains is countable, has no dense, metrizable subset. Theorems 1 and 2 show how close Moore spaces come to having dense, metrizable subsets.

A Moore space is one which satisfies Axiom 0 and the first three parts of Axiom 1 of [8]. Suppose S is a Moore space with development $G_1, G_2, G_3, \ldots$ The development is said to satisfy Axiom C at a point P of S if and only if for each region R containing P there is a positive integer n_0 such that if R_1 and R_2 are interesecting regions of G_{n_0} containing P in their sum, then $R_1 \cup R_2$ is a subset of R. A subset M of S is said to be *m-dense in S* with respect to the development $G_1, G_2, G_3, \ldots$ if and only if m is a positive integer with the property that for each point P in S and each positive integer n there are regions $R_1, R_2, \ldots, R_m$ in G_n such that P is contained in R_1, R_i intersects R_{i+1} for $1 \leqslant i \leqslant m-1$ whenever $m > 1$, and R_m intersects M. A collection E of subsets of S is said to be a *discrete collection* if and only if the closures of elements of E are mutually exclusive and the closure of the sum is the sum of the closures of any subcollection of elements of E. The space S is *collectionwise normal* with respect to a discrete collection E of subsets of S if and only if there is a collection G of mutually exclusive domains covering E^* such that each element of G intersects only one element of E. The statement that a subset M of S can be approximated by a class ξ of subsets of S means that for each open covering G of M there is a subset N of G^* belonging to ξ such that for each point in M there is an element in G containing it and a point of N. A subset K of the space S is said to be *screenable in* S if and only

* Supported by National Science Foundation Grant GZ 614.

if for each collection H of domains of S covering K there are countably many collections $H_1, H_2, H_3, \ldots$ of mutually exclusive domains of the space S such that their sum is a refinement of H covering K.

LEMMA 1. *If S is a topological space and G is an open covering of S, there is a discrete subcollection H of G such that for each element R of G the collection G contains an element which intersects both R and H^*.*

Proof. Let Ψ denote the family of subcollections of G which contains the collection C if and only if no element of G intersects two elements of C. Define a partial order $\leqslant$ on Ψ such that $C' \leqslant C''$ if and only if C' and C'' are elements of Ψ and C' is a subcollection of C''. Suppose Ψ' is a subfamily of Ψ such that any two elements from Ψ' compare. The subcollection $(\Psi')^*$ of G is an element of Ψ such that $C \leqslant (\Psi')^*$ for each C in Ψ'. Using Zorn's Lemma, Ψ contains an element H such that $H \leqslant C$ for no C in Ψ except for $C = H$. The subcollection H of G is the desired collection.

THEOREM 1. *If S is a Moore space, then for each development $G_1, G_2, G_3, \ldots$ for the space there is a metrizable, inner limiting set M which is 3-dense in S with respect to the development. Moreover, S has a development which satisfies Axiom C at each point of M.*

Proof. Let H_1 denote a discrete subcollection of G_1 such that for each region R of G_1 there is an element of G_1 which intersects both R and H_1^*. Let P_1 denote a function from H_1 into S such that $P_1(R_1)$ is a point of R_1 for each R_1 in H_1. For each R_1 in H_1, define G_{2,R_1} to be the collection which consists of all the regions of G_2 whose closures lie in R_1. Let H_{2,R_1} denote a discrete subcollection of G_{2,R_1} such that $P_1(R_1)$ is contained in an element of H_{2,R_1} and each element of G_{2,R_1} has an element of G_{2,R_1} that intersects it and H^*_{2,R_1}. Defining $G_{2,S-\overline{H_1^*}}$ to denote the regions of G_2 whose closures are subsets of $S-H_1^*$, $G_{2,S-\overline{H_1^*}}$ has a subcollection $H_{2,S-\overline{H_1^*}}$ such that each region of $G_{2,S-\overline{H_1^*}}$ has an element of $G_{2,S-\overline{H_1^*}}$ intersecting it and $H^*_{2,S-\overline{H_1^*}}$. The subcollection $H_2 = \bigcup_{R_1 \in H_1} H_{2,R_1} \cup H_{2,S-\overline{H_1^*}}$ of G_2 is discrete. Since each region intersects an element of $H_1 \cup \{S-\overline{H_1^*}\}$, each region has an element in G_2 intersecting it and H_2^*. Define a function P_2 from H_2 into S such that $P_2(R_2)$ is a point of R_2 for each R_2 in H_2 and $P_1(H_1)$ is a subset of $P_2(H_2)$. Two sequences are being constructed, $H_1, H_2, H_3, \ldots$ and $P_1, P_2, P_3, \ldots$, such that H_1, H_2, P_1, and P_2 are as defined above and (1) for each region R the collection G_n contains an element which intersects both R and H_n^*, (2) if R_n and R_i are elements of H_n and H_i respectively such that $\bar{R}_n$ intersects $\bar{R}_i$ and $i < n$, then $\bar{R}_n$ is a subset of R_i, and (3) P_n is a function from H_n into S such that $P_n(R_n)$ is a point of R_n for each R_n in H_n and $P_{n-1}(H_{n-1})$ is a subset of $P_n(H_n)$ for each positive integer $n > 1$. Define M to be the inner limiting set $\bigcap_{i \geqslant 1}^{\infty} (\bigcup_{n \geqslant i}^{\infty} H_n^*)$.

Notice that $\bigcup_n P_n(H_n)$ is a subset of M; thus, M is 3-dense in S with respect to the development $G_1, G_2, G_3, \ldots$ The sequence of discrete collections $H_1, H_2, H_3, \ldots$ forms a base at each point of M, so M is perfectly screenable. Bing has proven in [1] that perfectly screenable Moore spaces are metrizable; thus, M is metrizable.

We now show that S a development which satisfies Axiom C at each point of M. Let $G'_1, G'_2, G'_3, \ldots$ denote a sequence of collections of regions such that (1) G'_1 is a subcollection of G_1, (2) G'_{n+1} is a subcollection of $G'_n \cap G_{n+1}$, (3) R is a region in G_1 not in G'_1 if and only if R intersects $H_1^* \cap H_2^*$ and is not a subset of an element of H_1, and (4) R is a region in $G'_n \cap G_{n+1}$ not in G'_{n+1} if and only if R intersects $H_n^* \cap H_{n+1}^*$ and is not a subset of an element of H_n. Since $H_n^* \cap H_{n+1}^*$ is a closed subset of H_n^*, G'_n covers S for each positive integer n. Since G'_{n+1} is a subcollection of both G_{n+1} and G'_n, S has $G'_1, G'_2, G'_3, \ldots$ as a development. Suppose P is a point of M and D is a domain containing P. There is an integer m_0 such that each region in G_{m_0} which contains P lies in D. Pick $n_0 \geqslant m_0$ such that P is in $H_{n_0}^*$. Choose elements $R_{n_0,P}$ of H_{n_0} and $R_{n_0+1,P}$ of H_{n_0+1} such that they each contain P. Notice that $\bar{R}_{n_0+1,P}$ is a subset of $R_{n_0,P}$. There is an integer $k > n_0$ such that each region in G_k which contains P has its closure lying in $R_{n_0+1,P}$. Now if R' and R'' are intersecting regions in G'_k which have P in their sum, then either R' or R'', say R', is a subset of $R_{n_0+1,P}$. The region R'' intersects $R_{n_0+1,P}$; thus, R'' intersects $H_{n_0}^* \cap H_{n_0+1}^*$ which shows that R'' is a subset of an element of H_{n_0}. Since R'' intersects $R_{n_0,P}$, R'' must be a subset of $R_{n_0,P}$; thus, $R' \cup R''$ is a subset of D which proves that $G'_1, G'_2, G'_3, \ldots$ satisfies Axiom C at each point of M.

THEOREM 2. *If S is a Moore space, then for each development $G_1, G_2, G_3, \ldots$ for the space there is a metrizable, inner limiting subset M which is 2-dense in S with respect to the develepment. Moreover, S has a development which satisfies Axiom C at each point of M if S is normal.*

Proof. Let H_1 denote a maximal collection of mutually exclusive regions from G_1. Define P_1 to be a function from H_1 into S such that $P_1(R_1)$ is a point of R_1 for each R_1 in H_1. For each R_1 in H_1, let G_{2,R_1} denote the subcollection of G_2 which contains all the regions which are subsets of R_1. Let H_{2,R_1} denote a maximal collection of mutually exclusive regions from G_{2,R_1} such that $P_1(R_1)$ is a point of some element of H_{2,R_1}. Now $H_2 = \bigcup_{R_1 \in H_1} H_{2,R_1}$ is a maximal collection of mutually exclusive regions from G_2. Define P_2 to be a function from H_2 into S such that $P_2(R_2)$ is a point of R_2 for each R_2 in H_2 and $P_1(H_1)$ is a subset of $P_2(H_2)$. In general, G_{n+1,R_n} denotes the subcollection of G_{n+1} which contains all of the regions of G_{n+1} which are subsets of R_n for each R_n in H_n. The collection H_{n+1,R_n} is a maximal collection of mutually exclusive regions from G_{n+1,R_n} so that $P_n(R_n)$ is a point of some element of H_{n+1,R_n}. The collection H_{n+1}

is defined to be the maximal subcollection $\bigcup_{R_n \in H_n} H_{n+1,R_n}$ of mutually exclusive regions of G_{n+1}. The function P_{n+1} is defined from H_{n+1} into S such that $P_{n+1}(R_{n+1})$ is a point of R_{n+1} for each R_{n+1} in H_{n+1} and $P_n(H_n)$ is a subset of $P_{n+1}(H_{n+1})$.

Letting $M = \bigcap_n H_n^*$, M contains $\bigcup_n P_n(H_n)$ as a subset. Since $\bigcup_n P_n(H_n)$ is 2-dense in S with respect to the development, the set M is 2-dense in S with respect to the development $G_1, G_2, G_3, \ldots$ Notice that $\bigcup_n H_n$ forms a base at each point of M such that if P is a point of M and P is a limit point of H_n^*, then P is a limit point of the element of H_n which contains P. This shows that M is perfectly screenable, so M is metrizable.

Suppose that S is normal. Let $D_{n,1}, D_{n,2}, D_{n,3}, \ldots$ denote a sequence of domains such that $\bigcap_i D_{n,i} = S - H_n^*$. Using normality, define $O_{n,1}$, $O_{n,2}, O_{n,3}, \ldots$ such that (1) $O_{n,i}$ is open for each positive integer i, (2) $S - H_n^*$ is a subset of $O_{n,1}$ and $\bar{O}_{n,1}$ is a subset of $D_{n,1}$, and (3) $S - H_n^*$ is a subset of $O_{n,i}$ and $\bar{O}_{n,i}$ is a subset of the set $O_{n,i-1} \cap \bigcap_{j=1}^{i} D_{n,j}$ for each integer $i > 1$. For each positive integer i, define $H_{n,i}$ to be the collection containing O if and only if there is an element D of H_n such that $O = D \cap (S - \bar{O}_{n,i})$. Suppose K is any subcollection of $H_{n,i}$ for some pair of integers n and i. Suppose P is a limit point of K^*. Now P is a limit point of $S - \bar{O}_{n,i}$ since K^* is a subset of $S - \bar{O}_{n,1}$. Due to $S - \bar{O}_{n,i}$ being a subset of $S - O_{n,i}$ which is closed, P is a point of $S - O_{n,i}$. This shows that P is contained in an element R_P of H_n since $S - O_{n,i}$ is a subset of H_n^*. The open set R_P intersects only one element of K. This proves that $H_{n,i}$ is a discrete collection for each pair of integers n and i. Notice that $\bigcup_i H_{n,i}$ is a refinement of H_n covering H_n^* and that $\overline{H_{n,i}^*}$ is a subset of H_n^* for each i. For each positive integer n, there is a development $G_{1,n}, G_{2,n}$, $G_{3,n}, \ldots$ for the space such that each element of $G_{i,n}$ which intersects $H_{n,1}^*$ is a subset of some element of H_{n-1}. Let $G'_1, G'_2, G'_3, \ldots$ denote a development for S such that G'_i consists of all regions R in G_i such that R is a subset of some region of $G_{i,n}$ for each $n = 1, 2, \ldots, i$. Now suppose that A is a point of M and that D is a domain containing A. There is an inetger n_0 such that each region in G_{n_0} which contains A is a subset of D. Choose a region R_A from H_{n_0} which contains A. There is an integer i_0 such that there is an open set Q in H_{n_0+1,i_0} containing A. There is an integer k such that each region in G'_k which contains A is a subset of Q. Letting R' and R'' be any pair of intersecting regions from $G'_{k+i_0+n_0}$ such that A is contained in R', then R' is a subset of Q and R'' intersects Q. The region R'' is a subset of some region of G_{i_0,n_0+1} which shows that R'' is a subset of some element of H_{n_0}. Due to R'' intersecting R_A which is an element of H_{n_0}, R'' is a subset of R_A. Now we have that the sum of R' and R'' is a subset of R_A; thus, their sum is a subset of D. This proves that $G'_1, G'_2, G'_3, \ldots$ satisfies Axiom C at each point of M.

Grace and Heath [5] have shown that each connected, locally connected, locally peripherally separable, pointwise paracompact Moore space is separable. In comparison, the following theorem is now offered.

THEOREM 3. *Each locally connected, locally peripherally separable, Moore space contains a dense, metrizable, inner limiting subset.*

Proof. Suppose S is a locally connected, locally peripherally separable, Moore space. Let $G_1, G_2, G_3, \ldots$ denote a development for S such that the regions in each G_n are peripherally separable. Let H_1 denote a maximal collection of mutually exclusive elements from G_1. Define P_1 to be a function from H_1 into S such that $P_1(R_1)$ is a point of R_1 for each R_1 in H_1. Assuming that $H_1, H_2, \ldots, H_n$ and $P_1, P_2, \ldots, P_n$ have been described, let R_n be any element of H_n. Define $R_{n,0}$ to be a region in G_{n+1} that contains $P_n(R_n)$ such that its closure is a subset of R_n. Choose a countable dense subset $\{Q_{n,1}, Q_{n,2}, Q_{n,3}, \ldots\}$ of the boundary of R_n if the boundary of R_n exists. For each positive integer i, define $\{R_{n,i,1}, R_{n,i,2}, R_{n,i,3}, \ldots, R_{n,i,i}\}$ to be mutually exclusive regions from G_{n+1} such that (1) $\bar{R}_{n,1,1}$ is a subset of R_n which does not intersect $R_{n,0}$ and $R_{n,1,1}$ is a subset of some element of G_{n+1} which contains $Q_{n,1}$ and (2) $\bigcup_{j=1}^{i} \bar{R}_{n,i,j}$ is a subset of the set $R_n - (\bar{R}_{n,0} \cup \bigcup_{k=1}^{i-1} \bigcup_{j=1}^{k} \bar{R}_{n,k,j})$ and $R_{n,i,j}$ is a subset of some element of G_{i+n} which contains $Q_{n,j}$ for each positive integer $j \leqslant i$. Letting G_{n+1,R_n} denote the regions of G_{n+1} which are subsets of R_n, define H_{n+1,R_n} to be a maximal collection of mutually exclusive elements from G_{n+1,R_n} such that $R_{n,0}$ is in H_{n+1,R_n} and $R_{n,i,j}$ is in H_{n+1,R_n} for each positive integer i and each positive integer $j \leqslant i$. If the boundary of R_n does not exist, let H_{n+1,R_n} denote a maximal collection of mutually exclusive elements from G_{n+1,R_n} so that $R_{n,0}$ is in H_{n+1,R_n}. Now $H_{n+1} = \bigcup_{R_n \in H_n} H_{n+1,R_n}$ is a maximal collection of mutually exclusive elements from G_{n+1}. Define P_{n+1} to be a function from H_{n+1} into S such that $P_{n+1}(R_{n+1})$ is a point of R_{n+1} for each R_{n+1} in H_{n+1} and $P_n(H_n)$ is a subset of $P_{n+1}(H_{n+1})$.

Define $M = \bigcap_n H_n^*$. Notice that $P_n(H_n)$ is a subset of M for each positive integer n. Suppose Q is a point of $S - M$ and D is a domain containing Q. There is an integer n_0 such that Q is not contained in $H_{n_0}^*$. Choose a connected domain D' which is a subset of D and contains Q. Since H_{n_0} is a maximal collection of mutually exclusive elements from G_{n_0}, D' intersects an element R_{n_0} of H_{n_0}. Due to D' being connected, D' must contain a boundary point of R_{n_0}. This shows that D' must contain a point Q_{n_0,j_0} of the countabe dense subset $\{Q_{n_0,1}, Q_{n_0,2}, Q_{n_0,3}, \ldots\}$ of the boundary of R_{n_0} which was used to define H_{n_0+1}. There is a positive integer i_0 such that if R is a region in G_{i_0} containing Q_{n_0,j_0} then R is a subset of D'. Now R_{n_0,i_0,j_0} is a subset of some element of $G_{i_0+n_0}$ which contains Q_{n_0,j_0}; thus, R_{n_0,i_0,j_0} is a subset of D'. Since R_{n_0,i_0,j_0} contains a point of M, the set M is dense in S.

Defining $H'_n = \{R_n \cap M\colon R_n \in H_n\}$, $H'_1, H'_2, H'_3, \ldots$ is a sequence of discrete collections of open sets in the space M such that $\bigcup_n H'_n$ forms a base for M. This proves that M is perfectly screenable, so M is metrizable.

LEMMA 2. *If S is a Moore space and if M is a subset of S, then there are countably many discrete subsets $K_1, K_2, K_3, \ldots$ of S such that their sum is a dense subset of M.*

Proof. For each positive integer n, let a_n denote a well-ordering of the elements of G_n which intersect M. Let β_n denote a well-ordering of a subset K_n of M such that (1) the first term of β_n is a point of the first term of a_n and (2) for each proper initial segment β'_n of β_n the first term x of β_n that has each term of β'_n preceding it has the property that x is not a point of a region in G_n which contains a term of β'_n and x is a point of the first term of a_n whose common part with M is not a subset of $\{R\colon R$ is a region in G_n containing a term of $\beta'_n\}^*$. The set K_n is a discrete subset of S since no region in G_n contains two points of K_n. Due to the fact that each point of M has a region of G_n which contains it and a point of K_n, the set $\bigcup_n K_n$ is dense in M.

Theorems 4 and 5 give conditions under which normal Moore spaces have dense, metrizable subsets.

THEOREM 4. *If S is a locally connected, normal Moore space which has a base G with the property that S is collectionwise normal with respect to each discrete subset that is contained in the boundary of some element of G, then S contains a dense, metrizable subset.*

Proof. Let $G_1, G_2, G_3, \ldots$ denote a development for the space S such that G_n is a subcollection of G for each positive integer n. Define H_1 to be a maximal collection of mutually exclusive elements from G_1. Choose any element R_1 from H_1. Suppose that R_1 has a boundary. There are countably many discrete subsets $K_{1,1}, K_{1,2}, K_{1,3}, \ldots$ of the boundary of R_1 such that their sum is dense in the boundary of R_1. The space is collectionwise normal with respect to $K_{1,i}$ for each positive integer i, so let $J_{1,i,j}$ denote a subcollection of mutually exclusive regions of G_j covering $K_{1,i}$ for each pair of positive integers i and j so that each element of $J_{1,i,j}$ contains one and only one point of $K_{1,i}$. For each pair of positive integers i and j, define $H_{2,R_1,i,j}$ to be a subcollection of mutually exclusive regions of G_2 such that (1) each element of $H_{2,R_1,i,j}$ is a subset of R_1 and a subset of some element of $J_{1,i,j}$ and (2) each element of $J_{1,i,j}$ contains an element of $H_{2,R_1,i,j}$ as a subset. If R_1 does not have a boundary, define $H_{2,R_1,i,j}$ to be any collection of mutually exclusive regions from G_2 such that $H^*_{2,R_1,i,j}$ is a subset of R_1. Now the collection $H_{2,i,j,0} = \bigcup_{R_1 \in H_1} H_{2,R_1,i,j}$ contains mutually exclusive regions from G_2 for each pair of integers i and j. Defining G_{2,R_1} to be the collection of all regions in G_2 which are subsets of R_1 for each R_1 in H_1, let H_{2,R_1} denote a maximal subcollection of mutually exclusive regions of G_{2,R_1}. The collection H_2 is defined to be the maximal subcollection $\bigcup_{R_1 \in H_1} H_{2,R_1}$ of mutually exclusive regions of G_2. A function P_2 is defined from $\bigcup_{i,j} H_{2,i,j,0}$ into S such that $P_2(R_2)$ is a point of R_2 for each R_2 in $\bigcup_{i,j} H_{2,i,j,0}$. In general, let R_n denote an element of H_n. If R_n does not have a boundary, then define $H_{n+1,R_n,i,j}$ to be any collection of mutually exclusive regions from G_{n+1} such that $H^*_{n+1,R_n,i,j}$ is a subset of R_n. If R_n has a boundary, then there are discrete sets $K_{n,1}, K_{n,2}, K_{n,3}, \ldots$ such that their sum is a dense subset of the boundary of R_n. For each pair of positive integers i and j, define $J_{n,i,j}$ to be a subcollection of mutually exclusive regions of G_j covering $K_{n,i}$ such that each element in $J_{n,i,j}$ contains only one point of $K_{n,i}$. In this case, define $H_{n+1,R_n,i,j}$ to be a subcollection of mutually exclusive regions of G_{n+1} such that (1) each element of $H_{n+1,R_n,i,j}$ is a subset of R_n and a subset of some element of $J_{n,i,j}$ and (2) each element of $J_{n,i,j}$ contains an element of $H_{n+1,R_n,i,j}$ as a subset. Let $H_{n+1,i,j,0}$ denote the collection $\bigcup_{R_n \in H_n} H_{n+1,R_n,i,j}$ of mutually exclusive regions from G_{n+1}. For each $m = 2, \ldots, n$ and each pair of positive integers i and j, we define $H_{m,i,j,n+1-m}$ to be a collection of mutually exclusive regions from G_{n+1} covering $P_m(H_{m,i,j,0})$ such that each region of $H_{m,i,j,n+1-m}$ contains only one point of $P_m(H_{m,i,j,0})$ and is a subset of only one element of $H_{m,i,j,n-m}$. By letting G_{n+1,R_n} denote the collection of regions of G_{n+1} which are subsets of R_n, we define H_{n+1,R_n} to be a maximal subcollection of mutually exclusive regions of G_{n+1,R_n}. The collection H_{n+1} is the maximal subcollection $\bigcup_{R_n \in H_n} H_{n+1,R_n}$ of mutually exclusive regions of G_{n+1}. Define a function P_{n+1} from the collection $\bigcup_{i,j} H_{n+1,i,j,0}$ into S such that $P_{n+1}(R_{n+1})$ is a point of R_{n+1} for each R_{n+1} in $\bigcup_{i,j} H_{n+1,i,j,0}$.

We now show that the set $M = \bigcap_{n=2}^{\infty} (H^*_n \cup \bigcup_{i,j} \bigcup_{m=2}^{n} H^*_{m,i,j,n-m})$ is a dense, metrizable subset of S. There are countably many elements in the family $\{H_n\colon n$ is a positive integer$\} \cup \{H_{m,i,j,n-m}\colon m, i, j, n$ are positive integers with $2 \leqslant m \leqslant n\}$ of collections of mutually exclusive regions such that their sum forms a base at each point of M. This shows that M is screenable. Due to S being a normal Moore space, M is also a normal Moore space; thus, M is metrizable. Suppose P is a point of $S-M$. There is an integer n_0 such that P is not a point of $H^*_{n_0}$. Letting D denote any domain which contains P, there is a connected domain D' which is a subset of D and contains P. Since H_{n_0} is a maximal subcollection of mutually exclusive regions of G_{n_0}, there is a R_{n_0} in H_{n_0} which intersects D'. Due to D' being connected, D' must contain a point of the boundary of R_n, so D' contains a point Q of K_{n_0,i_0} for some integer i_0 where K_{n_0,i_0} is a discrete subset of the boundary of R_{n_0} which was used to define $H_{n_0+1,R_{n_0},i_0,j}$ for each positive integer j. Now there is an integer j_0 such that each region of G_{j_0} which contains Q is a subset of D. The element R' of J_{n_0,i_0,j_0} which contains Q is in G_{j_0}, so R' is a subset of D. The region R' contains a point of the subset $P_{n_0+1}(H_{n_0+1,R_{n_0},i_0,j_0})$ of M. This proves that M is dense in S.

Under the hypothesis of Theorem 4, one can say that S has a dense, metrizable, inner limiting subset. We have shown that S has a dense, metrizable subset. In [3], Fitzpatrick proved that a normal Moore space which has a dense, metrizable subset possesses a development which satisfies Axiom C at each point of a dense subset of the space. The set of all points where this development satisfies Axiom C is an inner limiting set.

The author wishes to thank B. Fitzpatrick for bringing this to his attention.

The following theorem generalizes a theorem by Fitzpatrick of [4].

THEOREM 5. *A normal Moore space S has a dense metrizable subset if and only if each discrete subset of S can be approximated by subsets of S which are screenable in S.*

Proof. Suppose S is a Moore space with a dense, metrizable subset M. Letting K denote any discrete subset of S and G any open covering of K, the set $M \cap G^*$ is screenable in S and dense in G^*; thus, each discrete subset of S can be approximated by subsets of S which are screenable in S.

Suppose S is a normal Moore space such that each discrete subset of S can be approximated by subsets of S which are screenable in S. For each positive integer n, let K_n be a discrete subset of S such that $\bigcup_n K_n$ is dense in S. For each pair of positive integers n and m, let $M_{n,m}$ denote a subset of S such that (1) $M_{n,m}$ is screenable in S and (2) for each point P in K_n there is an open set in G_m containing both P and a point of $M_{n,m}$. Now the subset $M = \bigcup_{n,m} M_{n,m}$ of S is screenable. Since M is also normal, M is metrizable. Due to $\bigcup_n K_n$ being a subset of $\overline{M}$, the set M is dense in S.

References

[1] R. H. Bing, *Metrization of topological spaces*, Canad. Journ. Math. 3 (1951), pp. 175-186.

[2] — *A translation of the normal Moore space conjecture*, Proc. Amer. Math. Soc. 16 (1965), pp. 612-619.

[3] B. Fitzpatrick, *On dense subspaces of Moore spaces*, ibidem 16 (1965), pp. 1324-1328.

[4] — *On dense subspaces of Moore spaces II*, Fund. Math. 61 (1967), pp. 91-92.

[5] E. E. Grace and R. W. Heath, *Separability and metrizability in pointwise paracompact Moore spaces*, Duke Math. Journ. 31 (1964), pp. 603-610.

[6] F. B. Jones, *Concerning normal and completely normal spaces*, Bull. Amer. Math. Soc. 43 (1937), pp. 671-677.

[7] J. L. Kelley, *General topology*, Princeton 1955.

[8] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications 13, New York 1962.

[9] Mary Ellen Estill Rudin, *Concerning abstract spaces*, Duke Math. Journ. 17 (1950), pp. 317-327.

[10] J. N. Younglove, *Concerning dense metric subspaces of certain non-metric spaces*, Fund. Math. 48 (1960), pp. 15-25.

Reçu par la Rédaction le 10. 6. 1968