and $h_n(A_{qn})$ are disjoint. This is a contradiction. Clearly $h$ is onto $B(1)/G$ and $h$ is a homeomorphism on $\mathrm{Int}\, C$. To show $h$ is a homeomorphism it suffices to show $h$ is continuous at each point of $S$. Suppose $p$ is in $S$, $\{b_i\}$ is a sequence of points in $C$, and $\{b_i\}$ converges to $p$. If $U$ is an open set in $B(1)/G$ that contains $h(p)$, then there exists a positive integer $k$ such that $h_k(A_{pk})$ is contained in $U$. Since $\{b_i\}$ converges to $p$, all but finitely many of the $b_i$'s belong to $A_{pi}$. Therefore all but finitely many of the $h_i(b_i)$'s belong to $h_i(A_{pi})$. By the way the $h_i$'s are constructed, if $j > i$ and $h_i(b_i) \in h_i(A_{pi})$, then $h_j(b_i) \in h_i(A_{pi})$, and hence all but finitely many of the $h(b_i)$'s belong to $U$. Therefore $h$ is continuous at $p$ and $h$ is a homeomorphism.

If $p$ is an element in $\bigcup_{i=1}^{\infty} a_i$ then $\bigcup_{i=1}^{\infty} h_i(A_{pi})$ is a one point set and hence the projection from $B(1)$ to $C$ takes no non-degenerate element of $G$ to a point in $\bigcup_{i=1}^{\infty} a_i$.

It follows that there exists a pseudo-isotopy $H$ from $R^3$ onto $R^3$ such that

(1) $H\colon R^3 \times I \to R^3$,

(2) for each $t \in I$, $H(R^3 \times \{t\})$ is $R^3$,

(3) if $x \in R^3$ then $H(x, 0) = x$,

(4) if $0 \leqslant t \leqslant 1$ then $H|R^3 \times \{t\}$ is a homeomorphism,

(5) $H(x, 1) = H(y, 1)$ if and only if $x$ and $y$ are in the same element of $G$ or $x = y$,

(6) $H(B(1), 1) = C$.

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# On subdirect embeddings in categories

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**§ 1.** In his paper [4] Suliński considers categories satisfying certain natural, although strong, additional conditions, and asks whether every object of such a category could be subdirectly embedded in a direct product of subdirectly irreducible objects. Such a theorem for universal algebras was proved by Birkhoff [1]. In the proof of this theorem it is implicitly assumed that the lattice of all congruence-relations of any universal algebra is a so-called algebraic lattice (1). However, the notion of congruence-relation cannot be formulated in a category-theoretical manner; it is possible to consider factor-objects instead of congruence-relations. Among the factor-objects one can define a partial ordering. Thus the condition that the congruence-relations form an algebraic lattice means that the factor-objects form a lattice and its dual lattice is algebraic.

After the preliminaries we consider a category satisfying weaker conditions than those of [4]. We assume, that every epimorphism is normal, but we do not suppose that every map has a kernel. (Related investigations are made in [5], where every map has a kernel, but an epimorphism need not be normal. There the possibility of dualization is also discussed.)

In § 3 we prove that an object of such a category can be subdirectly embedded in a direct product of subdirectly irreducible objects if the dual lattice of that of all factor-objects is algebraic. In § 4 we show that this condition is independent of all the conditions assumed by Suliński [4]; moreover, in the category $\mathcal{A}^*$, which is dual to the category of all abelian groups $\mathcal{A}$, there are objects which cannot be subdirectly embedded in a direct product of subdirectly irreducible objects.

**§ 2.** Let $\mathcal{C}$ be a category whose objects and maps will be denoted by small Latin and Greek letters, respectively. By definition, the following axioms hold:

(1) Algebraic lattices are sometimes called compactly generated lattices.

(C$_1$) *If* $\alpha$: $a\to b$ *and* $\beta$: $b\to c$ *are maps, then there is a uniquely defined map* $\alpha\beta$: $a\to c$ *which is called the product of the maps* $\alpha$ *and* $\beta$.

(C$_2$) *If* $\alpha$: $a\to b$, $\beta$: $b\to c$, $\gamma$: $c\to d$ *are maps, then* $(\alpha\beta)\gamma=\alpha(\beta\gamma)$ *holds.*

(C$_3$) *For each object* $a\in\mathfrak{C}$ *there is a map* $\varepsilon_a$: $a\to a$ *such that for any* $\alpha$: $b\to a$ *and* $\beta$: $a\to c$ *we have* $\alpha\varepsilon_a=\alpha$, $\varepsilon_a\beta=\beta$.

In this paper we adopt the notions and notations of Kuroš–Livšits–Šulgeifer–Tsalenko [3], and Suliński [4], but we shortly recall the fundamental concepts, which will be necessary later. Moreover, we shall require the validity of some additional axioms for the category $\mathfrak{C}$. So we assume that

(C$_4$) $\mathfrak{C}$ *possesses zero objects.*

An *epimorphism* $\nu$ is called *normal* if, for any map $\alpha$ satisfying the condition that for every map $\gamma$ with $\gamma\nu=\omega$ also $\gamma\alpha=\omega$ holds, there exists a map $\alpha'$ such that $\alpha=\nu\alpha'$. We suppose that

(C$_5$) *Every epimorphism of* $\mathfrak{C}$ *is normal.*

According to (C$_5$) we shall say briefly "epimorphism" instead of "normal epimorphism". Let us remark that in the category of groups every epimorphism is normal.

If the map $\alpha$: $a\to b$ can be represented in the form $\alpha=\nu\mu$, where $\nu$: $a\to l$ is an epimorphism and $\mu$: $l\to b$ is a monomorphism, then the subobject $(l,\mu)$ of $b$ will be called the *image of* $\alpha$ *with the epimorphism* $\nu$. We assume that

(C$_6$) *Every map has an image* (²).

Let $a_i$, $i\in I$ be a family of objects of $\mathfrak{C}$. An object $g\in\mathfrak{C}$ is said to be a *direct product* of the objects $a_i$, $i\in I$ if there are such maps $\pi_i$: $g\to a_i$ that for each object $h\in\mathfrak{C}$ and for any system $\alpha_i$: $h\to a_i$, $i\in I$, of maps there is a unique map $\gamma$: $h\to g$, the so-called *canonical map*, such that $\gamma\pi_i=\alpha_i$ for all $i\in I$. This product will be denoted by $g=\prod_{i\in I}a_i(\pi_i)$. We suppose that

(C$_7$) *Every family of objects has a direct product.*

Consider all pairs $(\beta,b)$ where $\beta$: $a\to b$ is an epimorphism. We shall say that $(\beta_1,b_1)\leqslant(\beta_2,b_2)$ if there is such an epimorphism $\beta'$ that $\beta_2\beta'=\beta_1$. The pairs $(\beta_1,b_1)$ and $(\beta_2,b_2)$ are said to be equivalent if both of $(\beta_1,b_1)\leqslant(\beta_2,b_2)$ and $(\beta_2,b_2)\leqslant(\beta_1,b_1)$ hold. The equivalence classes are called the *factor-objects* of the object $a$ and the factor-object determined by the pair $(\beta,b)$ will also be denoted by $(\beta,b)$. We assume that

(²) Instead of (C$_5$) and (C$_6$) it would be sufficient to suppose the existence of normal images (cf. [3] § 10) and that the product of two normal epimorphisms is again a normal one.



(C$_8$) *The class of all factor-objects of an object is a set; moreover, it forms a complete lattice.*

An object $a\in\mathfrak{C}$ is said to be *subdirectly embedded* in a direct product $g=\prod_{i\in I}a_i(\pi_i)$ if there is such a monomorphism $\gamma$: $a\to g$ that all maps $\alpha_i=\gamma\pi_i$: $a\to a_i$, $i\in I$ are epimorphisms. An object $a\in\mathfrak{C}$ is said to be *subdirectly irreducible* if the union of all its proper factor-objects (i.e. $\neq(\varepsilon_a,a)$) is again a proper factor-object. The Lemma of § 3 will give reasons for the definition of subdirect irreducibility.

In the theory of lattices a well-known concept is that of algebraic lattice. (The lattice of all congruence-relations of a universal algebra is algebraic, and any algebraic lattice is isomorphic to the lattice of the congruence-relations of a suitable universal algebra. Cf. Grätzer--Schmidt [2].) In this paper we shall need the dual concept. An element $k$ of a complete lattice $L$ is *co-compact* if $k\geqslant\bigcap_{i\in I}l_i$ implies $k\geqslant\bigcap_{j\in F}l_j$ for some finite subset $F\subseteq I$. The lattice $L$ is said to be *co-algebraic* if $L$ is complete and every element of $L$ is an intersection of co-compact elements.

Sometimes we shall refer to the following condition.

(A) *The lattice* $L_a$ *of all factor-objects of the object is a co-algebraic.*

In § 4 we shall show that (A) is independent of all the axioms assumed in paper [4] of Suliński. In [4] it is assumed in addition to (C$_1$)–(C$_8$) that

(C$_9$) *Every family of objects has a free product.*

(C$_{10}$) *Every map has a kernel.*

(C$_{11}$) *An image of an ideal by an epimorphism is always an ideal.*

We remark that in [4] instead of (C$_8$) we use another, but equivalent axiom by considering ideals instead of (normal) factor-objects.

**§ 3.** First of all we prove the following auxiliary:

LEMMA. *If an object* $a\in\mathfrak{C}$ *can be subdirectly embedded in a direct product* $g=\prod_{i\in I}a_i(\pi_i)$, *then the union of the factor-objects* $(\alpha_i,a_i)=(\gamma\pi_i,a_i)$ $i\in I$ *is* $(\varepsilon_a,a)$. *Conversely, if there are factor-objects* $(\alpha_i,a_i)$, $i\in I$ *of an object* $a\in\mathfrak{C}$ *satisfying* $\bigcup_{i\in I}(\alpha_i,a_i)=(\varepsilon_a,a)$, *then* $a$ *can be subdirectly embedded in a direct product* $g=\prod_{i\in I}a_i(\pi_i)$.

Proof. Let $a$ be embedded in a direct product $g=\prod_{i\in I}a_i(\pi_i)$ by the monomorphism $\gamma$, and denote $\bigcup_{i\in I}(\gamma\pi_i,a_i)$ by $(\alpha_0,a_0)$. Since $(\alpha_0,a_0)$ is the union of all $(\alpha_i,a_i)$ $(\alpha_i=\gamma\pi_i)$, there are epimorphisms $\beta_i$: $a_0\to a_i$ with $\alpha_0\beta_i=\alpha_i$, $i\in I$, and therefore there exists a canonical map $\delta$: $a_0\to g$ into the direct product $g=\prod_{i\in I}a_i(\pi_i)$. Moreover, by the uniqueness of $\gamma$ and $\delta$

we have $\gamma=\alpha_0\delta$. Since $\gamma$ is a monomorphism, the first factor $\alpha_0$ is also a monomorphism. Thus $\alpha_0$ is a monomorphism and a (normal) epimorphism, whence it is an equivalence.

Conversely, assume that there exists a family $(\alpha_i, a_i)$, $i\in I$, of factor-objects of $a$ such that $\bigcup_{i\in I}(\alpha_i, a_i)=(\varepsilon_a, a)$, and consider a direct product $g=\prod_{i\in I} a_i(\pi_i)$. By definition, there exists a canonical map $\gamma\colon a\to g$ with $\gamma\pi_i=\alpha_i$ for each $i\in I$. We have to prove that $\gamma$ is a monomorphism. Since every epimorphism is normal, it is sufficient to show [3] $\operatorname{Ker}\gamma=(0,\omega)$. For this aim, let us consider a map $\delta\colon d\to a$ with $\delta\gamma=\omega$ and the image $(a_0, \alpha')$ of $\gamma$ with the epimorphism $\alpha_0$ (i.e. $\alpha_0\alpha'=\gamma$).

$$\begin{tikzcd} & & a_0 \\ d \arrow[r, "\delta"] & a \arrow[ur, "\alpha_0"] \arrow[r, "\gamma"] \arrow[dr, "\alpha_i"'] & g \arrow[u, "\alpha'"'] \arrow[d, "\pi_i"] \\ & & a_i \end{tikzcd}$$

Taking into account $\gamma=\alpha_0\alpha'$, we have $\delta\alpha_0\alpha'=\delta\gamma\pi_i=\omega$, and since $\alpha'$ is a monomorphism, $\delta\alpha_0=\omega$ follows.

On the other hand, $\alpha_i=\gamma\pi_i=\alpha_0(\alpha'\pi_i)$ is valid where both $\alpha_i$ and $\alpha_0$ are epimorphisms. Therefore $\alpha'\pi_i$ is also an epimorphism for all $i\in I$. Thus $(\alpha_0, a_0)\geqslant\bigcup_{i\in I}(\alpha_i, a_i)=(\varepsilon_a, a)$. Hence $\alpha_0$ is an equivalence, and so $\delta\alpha_0=\omega$ implies $\delta=\omega$, and $\operatorname{Ker}\gamma=(0,\omega)$ is proved.

Now we are going to prove our main result.

THEOREM 1. *If the object $a\in\mathfrak{C}$ fulfils condition* (A), *then $a$ can be subdirectly embedded in a direct product of subdirectly irreducible objects.*

Proof. Let $(\varkappa, k)\neq(\varepsilon_a, a)$ be a co-compact element of the lattice $L_a$ of all factor-objects of $a$. Consider the set $S(k)=\{(\lambda_j, l_j)\mid j\in J\}$ of all factor-objects $(\lambda_j, l_j)$ of $a$ for which $(\varkappa, k)\cup(\lambda_j, l_j)>(\varkappa, k)$ holds. Let $(\lambda_1, l_1)>(\lambda_2, l_2)>\dots$ be a descending chain of factor-objects chosen from $S(k)$, and denote $\bigcap_n(\lambda_n, l_n)$ by $(\lambda_0, l_0)$. We show $(\varkappa, k)\cup(\lambda_0, l_0)>(\varkappa, k)$. Otherwise there would be $(\varkappa, k)\geqslant(\lambda_0, l_0)=\bigcap_n(\lambda_n, l_n)$ and since $(\varkappa, k)$ is a co-compact element, for a finite index $N$ a relation $(\varkappa, k)\geqslant(\lambda_N, l_N)$ would hold, which contradicts the choice of $S(k)$. Making use of the dual statement of Zorn's lemma, we obtain the existence of a minimal element $(\bar\lambda, \bar l)$ of $S(k)$.

To any co-compact element $(\varkappa_i, k_i)$, $i\in I$ of $L_a$, consider a minimal element $(\alpha_i, a_i)$ of $S(k_i)$. Now we shall show $\bigcup_{i\in I}(\alpha_i, a_i)=(\varepsilon_a, a)$. Suppose

(3) Cf. footnote (2).



the contrary case, i.e. $\bigcup_{i\in I}(\alpha_i, a_i)=(\alpha', a')\neq(\varepsilon_a, a)$. Since $L_a$ is co-algebraic, $(\alpha', a')$ is an intersection $\bigcap_{t\in T}(\varkappa_t, k_t)$ of co-compact elements $(\varkappa_t, k_t)\neq(\varepsilon_a, a)$, $t\in I$. A minimal element $(\alpha_t, a_t)$ belonging to $(\varkappa_t, k_t)$ occurs in the union-representation $(\alpha', a')=\bigcup_{i\in I}(\alpha_i, a_i)$. Thus we get

$$(\varkappa_t, k_t)\geqslant(\alpha', a')\geqslant(\alpha_t, a_t)\,,$$

from which

$$(\varkappa_t, k_t)\cup(\alpha_t, a_t)=(\varkappa_t, k_t)$$

follows, contradicting the choice of $(\alpha_t, a_t)$. Thus $\bigcup_{i\in I}(\alpha_i, a_i)=(\varepsilon_a, a)$ holds, and according to the Lemma, $a$ can be subdirectly embedded in a direct product $g=\prod_{i\in I} a_i(\pi_i)$.

Finally, we have to prove that each object $a_i$ is subdirectly irreducible, i.e. for any object $a_i$, $i\in I$, the union of all proper factor-objects $(\gamma_s, c_s)$ of $a_i$ differs from $(\varepsilon_{a_i}, a_i)$. Any factor-object $(\gamma_s, c_s)$ of $a_i$ is also a factor-object (4) $(\delta_s, c_s)$ of $a$, by $\delta_s=\alpha_i\gamma_s$. Thus $(\delta_s, c_s)<(\alpha_i, a_i)$ implies $(\delta_s, c_s)\cup(\varkappa_i, k_i)=(\varkappa_i, k_i)$, i.e. $(\delta_s, c_s)\leqslant(k_i, \varkappa_i)$. Hence, for the union of all proper factor-objects $\bigcup_{s\in S}(\delta_s, c_s)=(\delta_0, c_0)$ we have $(\delta_0, c_0)\leqslant(k_i, \varkappa_i)$; therefore $(\delta_0, c_0)<(\alpha_i, a_i)$ holds, and so $(\gamma_0, c_0)$, which is just the union of all proper factor-objects, is again a proper factor-object of $a_i$. Thus $a_i$ is subdirectly irreducible for each $i\in I$, and the theorem is proved.

§ **4.** In this section we prove

THEOREM 2. *There is a category satisfying conditions* $(C_1)$–$(C_{11})$ *in which not every element can be subdirectly embedded in a direct product of subdirectly irreducible objects. Hence axioms* $(C_1)$–$(C_{11})$ *do not imply the validity of condition* (A).

Proof. Consider the category $\mathcal{A}^*$ dual to that of abelian groups $\mathcal{A}$. Obviously $\mathcal{A}^*$ satisfies conditions $(C_1)$–$(C_{11})$; moreover, in $\mathcal{A}^*$ every monomorphism is a normal one. A factor-object of an object $a$ belonging to $\mathcal{A}^*$ is just a subgroup of $a$ regarded as an object of $\mathcal{A}$. Consider the infinite cyclic group $C(\infty)$. The lattice $L$ of all subgroups fails to be co-algebraic. Any proper subgroups of $C(\infty)$ is an infinite cyclic group generated by an integer $n$. Let $p$ a prime number with $(n, p)=1$. Now $nC(\infty)>0=\bigcap_k p^kC(\infty)$, but for any finite index $k$ $nC(\infty)\ngeqslant p^kC(\infty)$. Hence the subgroup $nC(\infty)$ is not a co-compact element of $L$ for all $n$. Thus $L$ is not co-algebraic.

A subdirect embedding of an object $a$ of $\mathcal{A}^*$ means that there is a monomorphism $\gamma\colon a\to\prod_{i\in I} a_i(\pi_i)$ such that $\gamma\pi_i=\alpha_i$ is a (normal) epi-

(4) Cf. footnote (2).

morphism for every $i \in I$. Equivalently, this can be expressed in $\mathcal{A}$ as follows: there is an epimorphism $\gamma^*$ mapping $a$ into a free product $g^* = \sum_{i \in I} a_i(\pi_i^*)$ such that $\pi_i^* \gamma^* = \alpha_i^*$ is a (normal) monomorphism for each $i \in I$. In this case the object $a$ is called a *transfree image* of the objects $a_i$, $i \in I$. (This concept is introduced and discussed in [5].) The object $a_i$ is said to be *transfreely irreducible* if the union of all its proper ideals is again a proper one. The notion of transfree irreducibility is dual to that of subdirect irreducibility.

In particular, if $C(\infty) \in \mathcal{A}$ is a transfree image of objects $a_i$, then every $a_i$ can be regarded as a subgroup of $C(\infty)$, and so each $a_i$ is isomorphic to $C(\infty)$. Since the union of all proper subgroups of $C(\infty)$ is $C(\infty)$ itself, the components $a_i$ cannot be transfreely irreducible. Dualizing, we find that $C(\infty)$, as an object of $\mathcal{A}^*$, cannot be subdirectly embedded in a direct product of subdirectly irreducible objects, and the theorem is proved.

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# Generalized connected functions

by

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**1. Introduction.** A function $f\colon S \to T$ is said to be *connected* if it maps every connected set in $S$ onto a connected set in $T$. Every continuous function is connected and the question as to when a connected function is continuous has been studied by many authors; for example, [2]–[5]. In this article $S$ will denote a regular topological $\mathrm{T}_1$-space with a base $\mathfrak{B}$ for the open sets such that every $U \in \mathfrak{B}$ is connected. The generalized connected function studied here will be a function $f$ taking $S$ to a $\mathrm{T}_1$-space $T$ such that $f(\overline{U})$ is connected in $T$ for every $U \in \mathfrak{B}$. Such functions will be called functions *connected with respect to* $\mathfrak{B}$ or, simply, *connected* ($\mathfrak{B}$) functions. These functions have been studied in [1] for a domain restricted to euclidean space and for a range which is separable metric.

In this article some theorems on conditions implying continuity of connected ($\mathfrak{B}$) functions are presented as well as a sufficient condition as to when a connected ($\mathfrak{B}$) function is a connected function. In Section 3 it is shown that Theorem 2.1 is a generalization of the well known result in functional analysis (a linear functional $f$ is continuous if and only if the null space of $f$ is closed). It is shown that a linear functional is continuous if and only if it is connected. Finally, in Theorem 4.1, a condition is given as to when a certain type of function is a homeomorphism.

It is clear that a connected function on $S$ is a connected ($\mathfrak{B}$) function and if $f$ is a connected ($\mathfrak{B}$) function on $S$, then it can be easily shown that $f$ takes all connected, open sets onto connected sets. In particular, it follows that $f(U)$ is connected for each $U \in \mathfrak{B}$. An example of a function which is connected ($\mathfrak{B}$) with respect to a certain base ($\mathfrak{B}$), but which is not connected, is provided in [1]. Another interesting example is given in Section 3 below.

**2. Continuity of connected ($\mathfrak{B}$) functions.** The following theorem gives a necessary and sufficient condition under which a connected ($\mathfrak{B}$) function is continuous. This is a generalization of Theorem 3 of [1] and of Theorem C of [3]. In particular, if $f$ is real valued, then $f$ is continuous