Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups

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§ 0. Introduction. In this paper we give necessary and sufficient conditions in order that two algebraic systems of the type mentioned in the title have homomorphic endomorphism semigroups. An analogous result is obtained for semigroups of some special endomorphisms of ordered sets (those endomorphisms have to preserve some suprema and infima of certain subsets of a given ordered set, e.g., one may consider only complete endomorphisms, or the smallest element of the ordered set, if it exists, may be fixed under all endomorphisms considered). The typical result is as follows: every homomorphism (excluding some trivial ones) is an isomorphism induced by an isomorphism or an anti-isomorphism of the ordered sets. In particular, every two algebraic systems of the type mentioned in the title are isomorphic or anti-isomorphic if and only if they have isomorphic endomorphism semigroups.

It is well known that groups of automorphisms cannot characterize the above-mentioned algebraic systems up to an isomorphism: two non-isomorphic Boolean algebras may have isomorphic—in fact, trivial—automorphism groups [13].

Every ordered set, semilattice, distributive lattice or Boolean algebra may be faithfully represented as an inclusion-ordered set of subsets, meet- or join-semilattice of subsets, ring or field of sets [1] (this fact is infimately connected with the other one: all those algebraic systems are unary relation algebras in the sense of [9], whence they must be representable). This representability is essential for our proofs.

We begin with some definitions and then formulate the main results. The rest of the paper is devoted to proofs and some corollaries to those results.

§ 1. Main definitions. Let ϱ be a binary relation over a set A. A transformation f of A is called an *endomorphism* of the algebraic system (A, ϱ) if $(a_1, a_2) \in \varrho \rightarrow \big(f(a_1), f(a_2)\big) \in \varrho$ for all $a_1, a_2 \in A$.

If $\mathfrak{A}$ is an algebraic system, then $|\mathfrak{A}|$ denotes the cardinality of $\mathfrak{A}$, i.e., the cardinality of the set A of all elements of $\mathfrak{A}$. If $|\mathfrak{A}| = 1$, $\mathfrak{A}$ is called *degenerate*. A homomorphism onto a degenerate algebraic system is called *degenerate*. If $\mathfrak{A}$ is a Boolean algebra and $|\mathfrak{A}| = 2$, $\mathfrak{A}$ is called *trivial*.

$\mathfrak{P}(A)$ is the set of all subsets of a set A, $\mathfrak{P}_0(A) = \mathfrak{P}(A)\setminus\{\emptyset\}$, $\mathfrak{P}_\omega(A)$ is the set of all non-empty finite subsets of A.

$\mathcal{E}(\mathfrak{A})$ denotes the endomorphism semigroup of an algebraic system $\mathfrak{A}$ (the elements of $\mathcal{E}(\mathfrak{A})$ are all endomorphisms of $\mathfrak{A}$, the operation is natural); $\mathcal{ES}(\mathfrak{A})$ is the surmorphism (i.e., endomorphism onto) semigroup of $\mathfrak{A}$, $\mathcal{A}(\mathfrak{A})$ is the automorphism group of $\mathfrak{A}$. Clearly, $\mathcal{A}(\mathfrak{A}) \subset \mathcal{ES}(\mathfrak{A}) \subset \mathcal{E}(\mathfrak{A})$. If $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ are isomorphic, then $\mathcal{A}(\mathfrak{A})$ and $\mathcal{A}(\mathfrak{B})$ are isomorphic (since $\mathcal{A}(\mathfrak{A})$ is the maximal subgroup of $\mathcal{E}(\mathfrak{A})$ containing the identity of $\mathcal{E}(\mathfrak{A})$). The converse is not generally true.

Now let $\mathfrak{A} = (A, \leqslant)$ be an ordered set, $\mathfrak{M}$ and $\mathfrak{N}$ subsets of $\mathfrak{P}_0(A)$ and φ an isomorphism of $\mathfrak{A}$ onto some inclusion ordered set of subsets. φ is called an $(\mathfrak{M}, \mathfrak{N})$*-representation* (cf. [2]) if, for every $\mathfrak{a} \in \mathfrak{M}$ for which the supremum $\bigvee \mathfrak{a}$ exists, $\varphi(\bigvee \mathfrak{a}) = \bigcup \varphi(\mathfrak{a})$; and, for every $\mathfrak{a} \in \mathfrak{N}$ for which the infimum $\bigwedge \mathfrak{a}$ exists, $\varphi(\bigwedge \mathfrak{a}) = \bigcap \varphi(\mathfrak{a})$. $\mathfrak{A}$ is called $(\mathfrak{M}, \mathfrak{N})$*-representable* if there exists an $(\mathfrak{M}, \mathfrak{N})$-representation of $\mathfrak{A}$.

An endomorphism $\varphi \in \mathcal{E}(\mathfrak{A})$ is called an $(\mathfrak{M}, \mathfrak{N})$*-endomorphism* if it preserves the suprema from $\mathfrak{M}$ and the infima from $\mathfrak{N}$, that is, for every $\mathfrak{a} \in \mathfrak{M}$ for which the supremum $\bigvee \mathfrak{a}$ exists, the supremum $\bigvee \varphi(\mathfrak{a})$ also exists and $\varphi(\bigvee \mathfrak{a}) = \bigvee \varphi(\mathfrak{a})$; and for every $\mathfrak{a} \in \mathfrak{N}$ for which the infimum $\bigwedge \mathfrak{a}$ exists, the infimum $\bigwedge \varphi(\mathfrak{a})$ also exists and $\varphi(\bigwedge \mathfrak{a}) = \bigwedge \varphi(\mathfrak{a})$.

The pair $(\mathfrak{M}, \mathfrak{N})$ is called *permissible* if the product of every two $(\mathfrak{M}, \mathfrak{N})$-endomorphisms of $\mathfrak{A}$ is an $(\mathfrak{M}, \mathfrak{N})$-endomorphism. In this case we may consider the semigroup $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ of all $(\mathfrak{M}, \mathfrak{N})$-endomorphisms of $\mathfrak{A}$.

The necessary and sufficient conditions for the $(\mathfrak{M}, \mathfrak{N})$-representability of $\mathfrak{A}$ are well known [2]. An evident sufficient condition for the permissibility of $(\mathfrak{M}, \mathfrak{N})$ is the following: $\varphi(\mathfrak{M}) \subset \mathfrak{M}$ and $\varphi(\mathfrak{N}) \subset \mathfrak{N}$ for every $(\mathfrak{M}, \mathfrak{N})$-endomorphism φ. Hence, $(\mathfrak{M}, \mathfrak{N})$ is permissible if $\mathfrak{M}$ and $\mathfrak{N}$ are either empty, or equal to $\mathfrak{P}_\omega(A)$, or equal to $\mathfrak{P}_0(A)$.

Clearly, $\mathcal{E}_{\emptyset,\emptyset}(\mathfrak{A}) = \mathcal{E}(\mathfrak{A})$. If $(\mathfrak{M}, \mathfrak{N})$ equals $(\mathfrak{P}_\omega(A), \emptyset)$, $(\emptyset, \mathfrak{P}_\omega(A))$, $(\mathfrak{P}_0(A), \emptyset)$, $(\emptyset, \mathfrak{P}_0(A))$, $(\mathfrak{P}_\omega(A), \mathfrak{P}_\omega(A))$, $(\mathfrak{P}_\omega(A), \mathfrak{P}_0(A))$, $(\mathfrak{P}_0(A), \mathfrak{P}_\omega(A))$ or $(\mathfrak{P}_0(A), \mathfrak{P}_0(A))$, then the semigroup $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ is denoted respectively by $\mathcal{E}_\vee(\mathfrak{A})$, $\mathcal{E}_\wedge(\mathfrak{A})$, $\mathcal{E}_\bigvee(\mathfrak{A})$, $\mathcal{E}_\bigwedge(\mathfrak{A})$, $\mathcal{E}_{\vee\wedge}(\mathfrak{A})$, $\mathcal{E}_{\vee\bigwedge}(\mathfrak{A})$, $\mathcal{E}_{\bigvee\wedge}(\mathfrak{A})$ or $\mathcal{E}_{\bigvee\bigwedge}(\mathfrak{A})$.

Let $\mathfrak{A} = (A, \leqslant)$ be an ordered set. The smallest equivalence relation over A containing $\leqslant$ (i.e., the equivalence closure of $\leqslant$) is called the *connectivity relation* of $\mathfrak{A}$ and is denoted by $\varkappa_\mathfrak{A}$. $\mathfrak{A}$ is called *connected* if $\varkappa_\mathfrak{A} = A \times A$. Clearly, $a_0 \equiv a(\varkappa_\mathfrak{A})$ if and only if there exists a sequence $a_0, a_1, \ldots, a_n = a$ of elements of A such that a_i and a_{i+1} are comparable for every $i = 0, \ldots, n-1$.

Let $\mathcal{T}(B)$ be the full transformation semigroup over B (i.e., let $\mathcal{T}(B)$ be the semigroup of all transformations of A) and let f be a mapping of A onto B such that $\varkappa_\mathfrak{A}$ is the kernel equivalence of f (the latter means that $f(a_1) = f(a_2) \leftrightarrow a_1 \equiv a_2(\varkappa_\mathfrak{A})$ for every $a_1, a_2 \in A$). If $\varphi \in \mathcal{E}(\mathfrak{A})$, then φ induces a transformation $\bar{\varphi}$ of the quotient set $A/\varkappa_\mathfrak{A}$, since $a_1 \equiv a_2(\varkappa_\mathfrak{A}) \rightarrow \varphi(a_1) \equiv \varphi(a_2)(\varkappa_\mathfrak{A})$. Therefore, every $\varphi \in \mathcal{E}(\mathfrak{A})$ defines a transformation $\bar{\varphi} \in \mathcal{T}(B)$: $\bar{\varphi}(b_1) = b_2$ means that $\varphi(a_1) = a_2$ for some a_1 and a_2 such that $f(a_i) = b_i$ $(i = 1, 2)$. The mapping $\bar{f}$: $\mathcal{E}(\mathfrak{A}) \rightarrow \mathcal{T}(B)$ is a homomorphism. Moreover, it is a surmorphism. In effect, let $\psi \in \mathcal{T}(B)$. For every $b \in B$ choose an element $a_b \in A$ such that $f(a_b) = b$ and define $\varphi_\psi(a) = a_{\psi(f(a))}$ for all $a \in A$. Then $\varphi_\psi \in \mathcal{E}(\mathfrak{A})$ and $\bar{\varphi}_\psi = \psi$. Clearly, $\varphi_\psi \in \mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ for every permissible pair $(\mathfrak{M}, \mathfrak{N})$, whence $\bar{f}$ induces a surmorphism of $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(A)$ onto $\mathcal{T}(B)$. The homomorphism induced by $\bar{f}$ on any subsemigroup of $\mathcal{E}(\mathfrak{A})$ is called *disconnecting*. Endomorphisms φ_ψ of $\mathfrak{A}$ are called *disconnecting endomorphisms*. If $\mathfrak{A}$ is connected, then every disconnecting homomorphism is degenerate.

A *left zero semigroup* is any semigroup satisfying the identity $xy = x$. *Right zero semigroups* are defined dually. Left zero semigroups and right zero semigroups constitute the class of *singular* semigroups.

If $\mathfrak{A}$ is a semilattice or lattice, then $\mathfrak{A}^\cdot = (A, \leqslant_\mathfrak{A})$ where $\leqslant_\mathfrak{A}$ is the natural order of $\mathfrak{A}$. If $\mathfrak{A}^\cdot$ is linearly ordered, then $\mathfrak{A}$ is called *linear*. A *linear bi-semilattice* is an algebraic system $\mathfrak{A} = (A, o, o)$ with two equal binary operations o such that (A, o) is a linear semilattice. In this case we define $\mathfrak{A}^\cdot = (A, o)^\cdot$. A *singular chain* is any algebraic system $\mathfrak{A} = (A, o, t)$ with two binary operations o and t such that one of the operations is a singular semigroup multiplication and the other operation is a linear semilattice multiplication. If $\mathfrak{A}$ is a singular chain, then $\mathfrak{A}^\cdot$ is the set A ordered with the natural semilattice order of $\mathfrak{A}$. If $\mathfrak{A}$ is an ordered set, then, by definition, $\mathfrak{A}^\cdot = \mathfrak{A}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebraic systems for which $\mathfrak{A}^\cdot$ and $\mathfrak{B}^\cdot$ are defined. A bijection f of A onto B is called an *order (anti-)isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$ if f is an (anti-)isomorphism of $\mathfrak{A}^\cdot$ onto $\mathfrak{B}^\cdot$. Clearly, if $\mathfrak{A}$ and $\mathfrak{B}$ are both ordered sets, semilattices or lattices, then order (anti-)isomorphisms are precisely the ordinary (anti-)isomorphisms.

Let f be a bijection of a set A onto a set B and $\varphi \in \mathcal{T}(A)$. Define $\pi_f(\varphi) \in \mathcal{T}(B)$ as follows: $\pi_f(\varphi)\big(f(a)\big) = f\big(\varphi(a)\big)$. Then π_f is an isomorphism of $\mathcal{T}(A)$ onto $\mathcal{T}(B)$. If $\Phi \subset \mathcal{T}(A)$, $\Psi \subset \mathcal{T}(B)$ and π is an isomorphism of Φ onto Ψ which is a restriction of π_f to Φ, we say that π is *induced by* f.

An *ordered set with zero* is an algebraic system $\mathfrak{A} = (A, \leqslant, 0)$ where $(A, \leqslant)$ is an ordered set with the smallest element 0 (0 is considered as a nullary operation). Define $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A}) = \mathcal{E}(\mathfrak{A}) \cap \mathcal{E}_{\mathfrak{M},\mathfrak{N}}\big((A, \leqslant)\big)$ (i.e., $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ are precisely those endomorphisms from $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}\big((A, \leqslant)\big)$ for which 0 is a fixed point). An *ordered set with identity* is a system $(A, \leqslant, 1)$ where $(A, \leqslant)$ is an ordered set with the largest element 1.

Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{M} \subset \mathfrak{P}_0(A)$. $\mathfrak{A}$ is called $\mathfrak{M}$-*representable* if the corresponding lattice $(A, \wedge, \vee)$ is $(\mathfrak{M}, \mathfrak{M})$-representable. $\mathfrak{M}$-*endomorphisms* of $\mathfrak{A}$ are endomorphisms of $\mathfrak{A}$ which are $(\mathfrak{M}, \mathfrak{M})$-endomorphisms of this lattice. If the set $\mathcal{E}_{\mathfrak{M}}(\mathfrak{A})$ of all $\mathfrak{M}$-endomorphisms of $\mathfrak{A}$ is a subsemigroup of $\mathcal{E}(\mathfrak{A})$, $\mathfrak{M}$ is called *permissible*.

A transformation is called n-*valued* if its range has cardinality n; c_a denotes the constant (i.e., the 1-valued transformation) taking the value a.

Let $\mathfrak{A}$ be an ordered set. A subsemigroup $\Phi \subset \mathcal{E}(\mathfrak{A})$ is called *sufficient* if it satisfies the following two conditions:

1) every constant from $\mathcal{E}(\mathfrak{A})$ belongs to Φ;

2) if $\{a_1, a_2\}$ and $\{a_3, a_4\}$ are two ordered subsets of $\mathfrak{A}$, then every one-to-one homomorphism of $\{a_1, a_2\}$ onto $\{a_3, a_4\}$ which is a restriction of some $\varphi \in \mathcal{E}(\mathfrak{A})$ is also a restriction of some $\psi \in \Phi$.

Remark. There exists a one-to-one homomorphism of $\{a_1, a_2\}$ onto $\{a_3, a_4\}$ which is a restriction of some $\varphi \in \mathcal{E}(\mathfrak{A})$ if and only if either $a_1 \not\equiv a_2(\varkappa_{\mathfrak{A}})$ or a_3 and a_4 are comparable (provided $a_1 \neq a_2$, $a_3 \neq a_4$).

Let $\mathfrak{A}$ be an ordered set with zero. A subsemigroup $\Phi \subset \mathcal{E}(\mathfrak{A})$ is called *sufficient* if it satisfies the following two conditions:

1) for every $a_1 \neq 0$ and $a_2 \in A$ there exists an n-valued $\varphi \in \Phi$ such that $n \leqslant 2$ and $\varphi(a_1) = a_2$;

2) if $a_1, a_2 \in A$ and $a_1 \nleqslant a_2$, then $\varphi(a_1) \neq 0 = \varphi(a_2)$ for some $\varphi \in \Phi$.

The first condition implies $c_0 \in \Phi$.

Let $\mathfrak{A}$ be a Boolean algebra. A subsemigroup $\Phi \subset \mathcal{E}(\mathfrak{A})$ is called *sufficient* if it satisfies the following three conditions:

1) for every distinct $a_1, a_2 \in A$ there exists a two-valued $\varphi \in \Phi$ such that $\varphi(a_1) \neq \varphi(a_2)$;

2) if $\varphi_i \in \Phi$, φ_i is two-valued for $1 \leqslant i \leqslant 4$ and $\varphi_1 \neq \varphi_2$, then $\{\varphi_1 \circ \varphi, \varphi_2 \circ \varphi\} = \{\varphi_3, \varphi_4\}$ for some $\varphi \in \Phi$;

3) if $|A| > 2$, then the ranges of 4-valued endomorphisms from Φ cover A.

§ 2. Main results.

THEOREM 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ordered sets, Φ and Ψ sufficient semigroups of endomorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and π a homomorphism of Φ onto Ψ. Then one of the following three cases holds:*

1) *$\mathfrak{B}$ and π are degenerate;*

2) *$\mathfrak{B}$ is trivially ordered and π is disconnecting;*

3) *π is an isomorphism induced by a uniquely defined isomorphism or anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.*

THEOREM 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be ordered sets with zeros, Φ and Ψ sufficient semigroups of endomorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, π be an isomorphism of Φ onto Ψ. Then π is induced by a uniquely defined isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.*

THEOREM 3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, Φ and Ψ sufficient semigroups of endomorphisms of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and π a homomorphism of Φ onto Ψ. Then one of the following two cases holds:*

1) *$\mathfrak{B}$ is either degenerate or trivial, and π is degenerate;*

2) *π is an isomorphism induced by a uniquely defined isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.*

COROLLARY 1. *Let $(\mathfrak{M}, \mathfrak{N})$ and $(\mathfrak{P}, \mathfrak{Q})$ be permissible pairs of sets of non-empty subsets of ordered sets $\mathfrak{A}$ and $\mathfrak{B}$ respectively, $\mathfrak{A}$ being $(\mathfrak{M}, \mathfrak{N})$-representable and $\mathfrak{B}$ — $(\mathfrak{P}, \mathfrak{Q})$-representable. Every isomorphism π between $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ and $\mathcal{E}_{\mathfrak{P},\mathfrak{Q}}(\mathfrak{B})$ is induced either by a uniquely defined isomorphism f of $\mathfrak{A}$ onto $\mathfrak{B}$ (in which case $\mathcal{E}_{\mathfrak{P},\mathfrak{Q}}(\mathfrak{B}) = \mathcal{E}_{f(\mathfrak{M}),f(\mathfrak{N})}(\mathfrak{B})$) or by a uniquely defined anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ (in which case $\mathcal{E}_{\mathfrak{P},\mathfrak{Q}}(\mathfrak{B}) = \mathcal{E}_{f(\mathfrak{N}),f(\mathfrak{M})}(\mathfrak{B})$).*

COROLLARY 2. *Let $(\mathfrak{M}, \mathfrak{N})$ and $(\mathfrak{P}, \mathfrak{Q})$ be permissible pairs for ordered sets with zeros $\mathfrak{A}$ and $\mathfrak{B}$ respectively, $\mathfrak{A}$ being $(\mathfrak{M}, \mathfrak{N})$-representable and $\mathfrak{B}$ — $(\mathfrak{P}, \mathfrak{Q})$-representable. Every isomorphism between $\mathcal{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ and $\mathcal{E}_{\mathfrak{P},\mathfrak{Q}}(\mathfrak{B})$ is induced by a uniquely defined isomorphism f between $\mathfrak{A}$ and $\mathfrak{B}$ and $\mathcal{E}_{\mathfrak{P},\mathfrak{Q}}(\mathfrak{B}) = \mathcal{E}_{f(\mathfrak{M}),f(\mathfrak{N})}(\mathfrak{B})$.*

COROLLARY 3. *Let $\mathfrak{M}$ be a permissible set of non-empty subsets of a Boolean algebra $\mathfrak{A}$, $\mathfrak{N}$ a permissible set of non-empty subsets for a Boolean algebra $\mathfrak{B}$, $\mathfrak{A}$ being $\mathfrak{M}$-representable and $\mathfrak{B}$ — $\mathfrak{N}$-representable. If $\mathfrak{A}$ and $\mathfrak{B}$ are not degenerate, then every isomorphism between $\mathcal{E}_{\mathfrak{M}}(\mathfrak{A})$ and $\mathcal{E}_{\mathfrak{N}}(\mathfrak{B})$ is induced by a uniquely defined isomorphism f between $\mathfrak{A}$ and $\mathfrak{B}$, in which case $\mathcal{E}_{\mathfrak{N}}(\mathfrak{B}) = \mathcal{E}_{f(\mathfrak{M})}(\mathfrak{B})$.*

THEOREM 4. *Let $\mathfrak{A}$ be an ordered set and $\mathfrak{B} = (B, \varrho)$ be a non-empty set with a reflexive binary relation. There exists a homomorphism π of $\mathcal{E}(\mathfrak{A})$ onto $\mathcal{E}(\mathfrak{B})$ if and only if one of the following three cases holds:*

1) *$\mathfrak{B}$ and π are degenerate;*

2) *$\mathfrak{B}$ is a trivially quasi-ordered set (which means that ϱ is either the identical, or universal binary relation), $\mathcal{E}(\mathfrak{B}) = \mathcal{T}(\mathfrak{B})$ and $\mathfrak{B}$ has the same cardinality as $\mathfrak{A}/\varkappa_{\mathfrak{A}}$ (any homomorphism of $\mathcal{E}(\mathfrak{A})$ onto $\mathcal{E}(\mathfrak{B})$ is disconnecting in this case);*

3) *$\mathfrak{B}$ is an ordered set isomorphic or anti-isomorphic with $\mathfrak{A}$.*

THEOREM 5. *Let $\mathfrak{A}$ be a semilattice and $\mathfrak{B} = (B, o)$ a set with an idempotent binary operation o. There exists a homomorphism π of $\mathcal{E}(\mathfrak{A})$ onto $\mathcal{E}(\mathfrak{B})$ if and only if one of the following three cases holds:*

1) $\mathfrak{B}$ *and* π *are degenerate;*

2) $\mathfrak{B}$ *is a semilattice and* π *is induced by an isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$;

3) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are anti-isomorphic linear semilattices and* π *is an isomorphism induced by an anti-isomorphism between* $\mathfrak{A}$ *and* $\mathfrak{B}$.

THEOREM 6. *Let* $\mathfrak{A}$ *be a distributive lattice and* $\mathfrak{B} = (B, o, t)$ *an algebra with two idempotent binary operations o and t. There exists a homomorphism* π *of* $\mathcal{E}(\mathfrak{A})$ *onto* $\mathcal{E}(\mathfrak{B})$ *if and only if one of the following four cases holds:*

1) $\mathfrak{B}$ *and* π *are degenerate;*

2) $\mathfrak{B}$ *is a distributive lattice isomorphic or anti-isomorphic with* $\mathfrak{A}$ *and* π *is induced by an isomorphism or anti-isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{B}$;

3) $\mathfrak{B}$ *is a linear bi-semilattice order-isomorphic or order-antiisomorphic with* $\mathfrak{A}$ *and* π *is an isomorphism induced by a uniquely defined order-isomorphism or order anti-isomorphism between* $\mathfrak{A}$ *and* $\mathfrak{B}$;

4) $\mathfrak{B}$ *is a singular chain order-isomorphic or order-anti-isomorphic with* $\mathfrak{A}$ *and* π *is an isomorphism induced by a uniquely defined order-isomorphism or order-anti-isomorphism between* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Remark. Theorems 5–7 could be strengthened (one may consider some subsemigroups $\Phi \subset \mathcal{E}(\mathfrak{A})$ and $\Psi \subset \mathcal{E}(\mathfrak{B})$ instead of semigroups $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ themselves. These subsemigroups Φ and Ψ should be "sufficient" in a sense that they have to contain "enough" endomorphisms with "small" ranges; it is sufficient to consider only n-valued endomorphisms with $n \leqslant 4$). We state these Theorems in their present form to simplify the proofs.

§ 3. Proofs. Proof of Theorem 1. Let $\mathfrak{A}$, $\mathfrak{B}$, Φ and Ψ be as in Theorem 1. Clearly, the constant c_a is a left zero of Φ for every $a \in A$. If $\varphi \in \Phi$ is a left zero of Φ, then $\varphi = \varphi \circ c_a = c_{\varphi(a)}$, which shows that every left zero of Φ is a constant. Let π be a homomorphism of Φ onto Ψ. Then $\pi(c_a)$ is a left zero of Ψ for every $a \in A$, whence $\pi(c_a) = c_b$ for some $b \in B$. Define $b = f(a)$ (i.e., $\pi(c_a) = c_{f(a)}$) for every $a \in A$. Then f is a mapping of A into B. If $b \in B$ then $c_b \in \Psi$ and the counter-image of c_b under π is a right ideal of Φ. Every right ideal of Φ meets the smallest ideal of Φ containing precisely all left zeros of Φ. Hence, $c_b = \pi(c_a)$ for some $a \in A$. It follows that f is an onto-mapping.

For every $\varphi \in \Phi$, $\varphi(a_1) = a_2 \leftrightarrow \varphi \circ c_{a_1} = c_{a_2} \rightarrow \pi(\varphi) \circ c_{f(a_1)} = c_{f(a_2)}$ $\rightarrow \pi(\varphi)\big(f(a_1)\big) = f(a_2)$, whence

$$\pi(\varphi)\big(f(a)\big) = f\big(\varphi(a)\big)\,. \tag{1}$$

Let ε_f be the kernel equivalence of f, i.e., $a_1 \equiv a_2(\varepsilon_f) \leftrightarrow f(a_1) = f(a_2)$. If $a_1 \equiv a_2(\varepsilon_f)$ then, for every $\varphi \in \Phi$, $f\big(\varphi(a_1)\big) = \pi(\varphi)\big(f(a_1)\big) = \pi(\varphi)\big(f(a_2)\big)$ $= f\big(\varphi(a_2)\big)$, whence $\varphi(a_1) \equiv \varphi(a_2)(\varepsilon_f)$.

Case 1.1. Let $\varepsilon_f = A \times A$. Then f is degenerate, which means that $\mathfrak{B}$ and π are degenerate.

Now let $\varepsilon_f \neq A \times A$, which means that $a \not\equiv a_0(\varepsilon_f)$ for some $a, a_0 \in A$.

Case 1.2. Let $a_1 \equiv a_2(\varepsilon_f)$ and $a_1 < a_2$ for some distinct $a_1, a_2 \in A$. If $a_3 < a_4$, then there exists a $\varphi \in \Phi$ such that $a_3 = \varphi(a_1)$ and $a_4 = \varphi(a_2)$. Hence, $a_3 \equiv a_4(\varepsilon_f)$. It follows that $\varkappa_{\mathfrak{A}} \subset \varepsilon_f$. Therefore $\varkappa_{\mathfrak{A}} \neq A \times A$, i.e., $\mathfrak{A}$ is not connected. Let $a \not\equiv a_0(\varkappa_{\mathfrak{A}})$ for some $a, a_0 \in A$. Then for every two-element trivially ordered subsystem $\{a_3, a_4\}$ of $\mathfrak{A}$ there exists a $\varphi \in \Phi$ inducing an isomorphism between $\{a, a_0\}$ and $\{a_3, a_4\}$, whence $a_0 \equiv a(\varepsilon_f)$ implies that $a_3 \equiv a_4(\varepsilon_f)$. Hence, if $a \equiv a_0(\varepsilon_f)$, then $a_3 \equiv a_4(\varepsilon_f)$ for every $a_3, a_4 \in A$, which means that $\varepsilon_f = A \times A$—a contradiction. Therefore $a \not\equiv a_0(\varepsilon_f)$, that is, $\varkappa_{\mathfrak{A}} = \varepsilon_f$. It means that π is a disconnecting homomorphism of Φ into $\mathcal{C}(B)$.

Now let b_1, b_2 be distinct elements of B, $f(a) = b_1$, $f(a_0) = b_2$ for some $a, a_0 \in A$. Then $a \not\equiv a_0(\varkappa_A)$, whence the subsystem $\{a, a_0\}$ of $\mathfrak{A}$ is trivially ordered. This subsystem has two automorphisms which are restrictions of some endomorphisms $\varphi_1, \varphi_2 \in \Phi$. It follows that the restrictions of $\pi(\varphi_1)$ and of $\pi(\varphi_2)$ on $\{b_1, b_2\}$ are two different automorphisms of the subsystem $\{b_1, b_2\}$. Hence, b_1 and b_2 are not comparable. It follows that $\mathfrak{B}$ is trivially ordered.

Case 1.3. Let $a_1 \equiv a_2(\varepsilon_f)$ imply that a_1 and a_2 are not comparable or that $a_1 = a_2$. There exist comparable distinct elements $a_3, a_4 \in A$ and $\varphi \in \Phi$ such that $\varphi(a_1) = a_3$, $\varphi(a_2) = a_4$, whence, $a_3 \equiv a_4(\varepsilon_f)$ if $a_1 \equiv a_2(\varepsilon_f)$ for non-comparable a_1, a_2. It follows that $a_1 \equiv a_2(\varepsilon_f) \leftrightarrow a_1 = a_2$, i.e., f is a bijection of A onto B. Together with (1) it implies that π is an isomorphism induced by f.

Let π be induced by another bijection g. Then $c_{f(a)} = \pi(c_a) = c_{g(a)}$ whence, $f = g$.

Let $\{b_1, b_2\}$ be a two-element trivially ordered subsystem of $\mathfrak{B}$. Then $\{b_1, b_2\}$ possesses two automorphisms which are induced by some transformations from Ψ. Hence, $\{f^{-1}(b_1), f^{-1}(b_2)\}$ has two distinct one-to-one endomorphisms (it may be proved that these endomorphisms are automorphisms), whence $\{f^{-1}(b_1), f^{-1}(b_2)\}$ is a trivially ordered subsystem of $\mathfrak{A}$. In exactly the same way we may prove that if a_1, a_2 are incomparable in $\mathfrak{A}$, then $f(a_1)$, $f(a_2)$ are incomparable in $\mathfrak{B}$.

Now let $a_1 < a_2$ for some $a_1, a_2 \in A$. Then $f(a_1)$ and $f(a_2)$ are comparable in $\mathfrak{B}$. Two subcases are possible:

Subcase 1.3.1. Let $f(a_1) < f(a_2)$. If $a_3 < a_4$ for some $a_3, a_4 \in A$, then there exists a $\varphi \in \Phi$ such that $\varphi(a_1) = a_3$, $\varphi(a_2) = a_4$, whence $f(a_3)$ $= \pi(\varphi)\big(f(a_1)\big) < \pi(\varphi)\big(f(a_2)\big) = f(a_4)$. If $f(a_5) < f(a_6)$ for some $a_5, a_6 \in A$, then a_5 and a_6 are comparable in $\mathfrak{A}$ and $a_6 < a_5$ is impossible, whence, $a_5 < a_6$. Therefore f is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Subcase 1.3.2. Let $f(a_2) < f(a_1)$. In the same way as in the previous subcase we may prove that f is an anti-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 1 is proved. To prove Corollary 1 we mention that if $\mathfrak{A}$, $\mathfrak{B}$, $(\mathfrak{M}, \mathfrak{N})$, $(\mathfrak{P}, \mathfrak{Q})$ are as in Corollary 1, then $\mathfrak{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ and $\mathfrak{E}_{\mathfrak{P},\mathfrak{Q}}(\mathfrak{B})$ are sufficient semigroups of endomorphisms (evidently, the constants belong to $\mathfrak{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$; to check the other property of sufficient semigroups consider two-valued endomorphisms from $\mathfrak{E}_{\mathfrak{M},\mathfrak{N}}(\mathfrak{A})$ and apply the $(\mathfrak{M}, \mathfrak{N})$-representability criterion from [2]).

Theorem 1 strengthens Gluskin's pioneer result on endomorphism semigroups of ordered sets [5].

2. Proof of Theorem 2. Let $\mathfrak{A}$, $\mathfrak{B}$, Φ, Ψ and π be as in Theorem 2. If Φ is degenerate, then Ψ is, and Theorem 2 is obviously true.

Let $\mathfrak{A}$ be non-degenerate. $H_{\mathfrak{A}}$ denotes the set of all two-valued endomorphisms from Φ. If $d \in H_{\mathfrak{A}}$ then $d \circ \varphi \circ d$ equals either d or c_0 for every $\varphi \in \Phi$. Conversely, let $d \circ \Phi \circ d \subset \{c_0, d\}$. Suppose $d \neq c_0$. Then $d(a) = a_0 \neq 0$ for some $a \in A$. Let $d_a \in H_{\mathfrak{A}}$ and $d_a(a_0) = a$. Then $d \circ d_a \circ d(a) = a_0 \neq 0$, whence $d = d \circ d_a \circ d$. Clearly, $d \circ d_a \circ d$ is two-valued. Therefore $h \in \Phi$ is two-valued if and only if $h \neq c_0$ and $h \circ \Phi \circ h \subset \{c_0, h\}$. Since c_0 is zero of Φ we have $\pi(c_0) = c_0$ and $\pi(H_{\mathfrak{A}}) = H_{\mathfrak{B}}$.

If $d \in H_{\mathfrak{A}}$, then $\bar{d}$ denotes the non-zero value of d.

Let $d_1, d_2 \in H_{\mathfrak{A}}$ and $\bar{d}_1 = \bar{d}_2$. Then $\varphi \circ d_1 = c_0 \leftrightarrow \varphi \circ d_2 = c_0$ for all $\varphi \in \Phi$. It follows that $\varphi \circ \pi(d_1) = c_0 \leftrightarrow \varphi \circ \pi(d_2) = c_0$ for all $\varphi \in \Psi$, whence $\varphi\big(\overline{\pi(d_1)}\big) = 0 \leftrightarrow \varphi\big(\overline{\pi(d_2)}\big) = 0$. Hence, $\overline{\pi(d_1)} = \overline{\pi(d_2)}$. In exactly the same way we may prove the converse part of the equivalence $\bar{d}_1 = \bar{d}_2 \leftrightarrow \overline{\pi(d_1)} = \overline{\pi(d_2)}$, which means that if we define $f(0) = 0$ and $f(\bar{d}) = \overline{\pi(d)}$ for every $d \in H_{\mathfrak{A}}$, we obtain a bijection f between $\mathfrak{A}$ and $\mathfrak{B}$.

Let $a \neq 0$ and $\varphi \in \Phi$. Then $\varphi(a) = 0 \leftrightarrow \varphi \circ d = c_0$ where $\bar{d} = a$. The latter means that $\pi(\varphi) \circ \pi(d) = c_0$, i.e., $\pi(\varphi)\big(\overline{\pi(d)}\big) = 0$, which means that $\pi(\varphi)\big(f(a)\big) = 0$. If $\varphi(a) = a_0 \neq 0$ and $\bar{d} = a$, $\bar{d}_0 = a_0$ for $d, d_0 \in H_{\mathfrak{A}}$, then $\varphi(a) = a_0 \leftrightarrow \pi(\varphi) \circ \pi(d) = \pi(d_0) \leftrightarrow \pi(\varphi)\big(f(a)\big) = f(a_0)$. Therefore π is induced by f.

Let π be induced by another bijection g between A and B. Then $g(a) = \overline{\pi(d)}$ for $d \in H_{\mathfrak{A}}$ such that $\bar{d} = a$. Hence, $g = f$.

Now $a_1 \leqslant a_2$ for $a_1, a_2 \in A$ means that $\varphi(a_2) = 0 \to \varphi(a_1) = 0$ for all $\varphi \in \Phi$. Equivalently, $\pi(\varphi)\big(f(a_2)\big) = 0 \to \pi(\varphi)\big(f(a_1)\big) = 0$, i.e., $f(a_1) \leqslant f(a_2)$. Therefore f is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 2 is proved.

Corollary 2 follows from Theorem 2 in precisely the same way as Corollary 1 from Theorem 1.

A very special corollary of Theorem 2 (for some special sufficient semigroups of endomorphisms of ordered sets with identities) has been found by E. S. Ljapin [8].

3. Proof of Theorem 3. Let $\mathfrak{A}$, $\mathfrak{B}$, Φ, Ψ and π be as in Theorem 3. Clearly, two-valued endomorphisms from Φ are homomorphisms of A onto $\{0, 1\} \subset A$. Let $H_{\mathfrak{A}}$ be the set of all two-valued endomorphisms from Φ. Every element of $H_{\mathfrak{A}}$ is a right zero of Φ; if φ is a right zero of Φ, then $\varphi = d \circ \varphi$ for $d \in H_{\mathfrak{A}}$. Clearly, $d \circ \varphi$ is two-valued, i.e., $H_{\mathfrak{A}}$ is the set of all right zeros of Φ. Analogously, $H_{\mathfrak{B}}$ is the set of all right zeros of Ψ.

If Ψ is degenerate, then $\mathfrak{B}$ is either degenerate or trivial.

Let $|\mathfrak{B}| > 2$. Then π is not degenerate. Let $F_{\mathfrak{A}}$ be the set of all 4-valued endomorphisms from Φ. If $d \in F_{\mathfrak{A}}$, then the range of d is $\{0, a, a', 1\}$ for some $a \in A\backslash\{0, 1\}$. Let $h \in H_{\mathfrak{A}}$. Then $\pi(h)$ is a right zero of Ψ, whence, $\pi(h) \in H_{\mathfrak{B}}$. Conversely, if $\psi \in H_{\mathfrak{B}}$, then the counter-image of ψ under π is a left ideal of Φ which meets the smallest ideal $H_{\mathfrak{A}}$ of Φ. Therefore $\pi(h) = \psi$ for some $h \in H_{\mathfrak{A}}$. This means that $\pi(H_{\mathfrak{A}}) = H_{\mathfrak{B}}$.

Let $\pi(h_1) = \pi(h_2)$ for some distinct $h_1, h_2 \in H_{\mathfrak{A}}$. Then for every $h_3, h_4 \in H_{\mathfrak{A}}$ there exists a $\varphi \in \Phi$ such that $\{h_1, h_2\} \circ \varphi = \{h_3, h_4\}$. Since $\pi(h_1 \circ \varphi) = \pi(h_2 \circ \varphi)$, we infer that $\pi(h_3) = \pi(h_4)$ and $|\pi(H_{\mathfrak{A}})| = 1$—a contradiction. It follows that $\pi(h_1) = \pi(h_2) \to h_1 = h_2$.

Now let $d \in F_{\mathfrak{A}}$, a be a value of d and $a \notin \{0, 1\}$. There exist $h_1, h_2 \in H_{\mathfrak{A}}$ such that $h_1(a) \neq h_2(a)$. It follows that $h_1 \circ d \neq h_2 \circ d$, whence, $|H_{\mathfrak{A}} \circ d| = 2$. If $|H_{\mathfrak{A}} \circ d| = 2$ for some $d \in \Phi$ then, clearly, $d \notin H_{\mathfrak{A}}$. If the range of d contains more that 4 elements, then it contains elements a_1 and a_2 such that $0 < a_1 < a_2 < 1$. There exist $h_i \in H_{\mathfrak{A}}$, $i = 1, 2, 3$, such that $h_1(a_1) = 1$, $h_2(a_1) = 0$, $h_2(a_2) = 1$, $h_3(a_2) = 0$, whence, $h_1 \circ d$, $h_2 \circ d$ and $h_3 \circ d$ are three distinct elements of $H_{\mathfrak{A}} \circ d$—a contradiction. Therefore, $d \in F_{\mathfrak{A}}$. Now if $d \in F_{\mathfrak{A}}$, then $|H_{\mathfrak{A}} \circ d| = 2$ and $|H_{\mathfrak{B}} \circ \pi(d)| = 2$ (we use the fact that π is one-to-one on $H_{\mathfrak{A}}$). It follows that $\pi(d) \in H_{\mathfrak{B}}$. Conversely, let $\pi(h) \in F_{\mathfrak{B}}$. It means that $|H_{\mathfrak{A}} \circ h| = |H_{\mathfrak{B}} \circ \pi(h)| = 2$, whence, $h \in F_{\mathfrak{A}}$. Let $h_1, h_2 \in F_{\mathfrak{A}}$ and $\pi(h_1) = \pi(h_2)$. It follows that $\pi(d \circ h_1) = \pi(d) \circ \pi(h_1) = \pi(d) \circ \pi(h_2) = \pi(d \circ h_2)$ for all $d \in H_{\mathfrak{A}}$. Therefore $d \circ h_1 = d \circ h_2$ for all $d \in H_{\mathfrak{A}}$. Let $a \in A$. Then $d\big(h_1(a)\big) = d\big(h_2(a)\big)$ for all $d \in H_{\mathfrak{A}}$, which means that $h_1(a) = h_2(a)$ for all $a \in A$ and $h_1 = h_2$. Hence, π is one-to-one on $F_{\mathfrak{A}}$ and $\pi(F_{\mathfrak{A}}) = F_{\mathfrak{B}}$.

Define a bijection f of A onto B. By definition, $f(0) = 0$, $f(1) = 1$. Let $a \in A\backslash\{0, 1\}$. Consider $h \in F_{\mathfrak{A}}$ having a as a value. Let b and b' be values of $\pi(h) \in F_{\mathfrak{B}}$, $b \notin \{0, 1\}$. For every $d_i \in H_{\mathfrak{A}}$, $i = 1, 2$ $d_1(a) = d_2(a) \leftrightarrow d_1 \circ h = d_2 \circ h \leftrightarrow \pi(d_1) \circ \pi(h) = \pi(d_2) \circ \pi(h) \leftrightarrow \pi(d_1)(b) = \pi(d_2)(b)$. Let $d(a) = 0$ for some $d \in H_{\mathfrak{A}}$. Then either $\pi(d)(b) = 0$ or $\pi(d)(b') = 0$. In the first case define $f(a) = b$, in the second case $f(a) = b'$. Hence, $d(a) = 0 \leftrightarrow \pi(d)\big(f(a)\big) = 0$. This definition does not depend on the choice of d. Let $d_1 \in H_{\mathfrak{A}}$ and $d_1(a) = 0$. Then $d(a) = d_1(a)$, which means that $\pi(d)(b) = \pi(d_1)(b)$. Hence, $\pi(d)(b) = 0 \leftrightarrow \pi(d_1)(b) = 0$. Evidently, f is a bijection.

Now let $\varphi(a_1) = a_2$ for some $a_1, a_2 \in A$. If $a_1 \in \{0, 1\}$ then, clearly, $\pi(\varphi)\big(f(a_1)\big) = f(a_2)$. Let $a_1 \notin \{0, 1\}$. For every $d \in H_{\mathfrak{A}}$ $d(a_2) = 0 \leftrightarrow d \circ \varphi(a_1) = 0 \leftrightarrow \pi(d \circ \varphi)\big(f(a_1)\big) = 0 \leftrightarrow \pi(d)\Big(\pi(\varphi)\big(f(a_1)\big)\Big) = 0$. But $d(a_2) = 0 \leftrightarrow \pi(d)\big(f(a_2)\big)$

$= 0$, whence $\pi(d)\big(\pi(\varphi)(f(a_1))\big) = \big(\pi(d)(f(a_2))\big)$ for all $d \in H_{\mathfrak{A}}$. It follows that $\pi(\varphi)\big(f(a_1)\big) = f(a_2)$. Therefore f induces π, and π is an isomorphism.

Let $a_1 \leqslant a_2$ for some $a_1, a_2 \in A$. Then $\varphi(a_2) = 0 \to \varphi(a_1) = 0$ for all $\varphi \in \Phi$. Therefore $\psi\big(f(a_2)\big) = 0 \to \psi\big(f(a_1)\big) = 0$ for all $\psi \in \Psi$. It follows that $d\big(f(a_1) \wedge f(a_2)'\big) = d\big(f(a_1)\big) \wedge d\big(f(a_2)\big)' = 0$ for all $d \in H_{\mathfrak{B}}$, which means that $f(a_1) \wedge f(a_2)' = 0$ or $f(a_1) \leqslant f(a_2)$. In the same way we may prove that $f(a_1) \leqslant f(a_2) \to a_1 \leqslant a_2$. Hence, f is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Now let π be induced by another bijection g of A onto B. Then for every $d \in H_{\mathfrak{A}}$ $\pi(d)\big(f(a)\big) = 0 \leftrightarrow d(a) = 0 \leftrightarrow \pi(d)\big(g(a)\big) = g(0)$. If g is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then $g(0) = 0$ and $\pi(d)\big(f(a)\big) = \pi(d)\big(g(a)\big)$ for all $d \in H_{\mathfrak{A}}$. Therefore $f(a) = g(a)$ and $f = g$.

Theorem 3 is proved.

Corollary 3 may be deduced from Theorem 3 along the same lines as Corollary 1 from Theorem 1. If $\mathfrak{A}$ is an $\mathfrak{M}$-representable Boolean algebra and $\mathfrak{M}$ is permissible, then the semigroup $\mathcal{E}_{\mathfrak{M}}(\mathfrak{A})$ is sufficient which may be proved by using an $\mathfrak{M}$-representability criterion from [2].

4. Proof of Theorem 4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be as in Theorem 4. Then $\mathcal{E}(\mathfrak{B})$ contains all the constants from $\mathcal{C}(B)$. Exactly as in the proof of Theorem 1 we may define a mapping f of A onto B and prove formula (1). If $\mathfrak{B}$ is degenerate, then f and π are degenerate (here π denotes a given homomorphism of $\mathcal{E}(\mathfrak{A})$ onto $\mathcal{E}(\mathfrak{B})$). Suppose $\mathfrak{B}$ is not degenerate. The argument in Case 1.2 carries through in our situation since $\mathcal{E}(\mathfrak{A})$ is, clearly, a sufficient subsemigroup of itself. Hence, π is a disconnecting homomorphism of $\mathcal{E}(\mathfrak{A})$ into $\mathcal{C}(B)$. In § 1 we have seen that if $\Phi = \mathcal{E}(\mathfrak{A})$, then π is a homomorphism onto $\mathcal{C}(B)$. Therefore $\mathcal{E}(\mathfrak{B}) = \mathcal{C}(B)$. By a lemma of Gluskin [5], $\mathfrak{B}$ is a quasi-ordered set and ϱ is trivial. If π is neither degenerate nor disconnecting, then the argument in Case 1.3 shows that π is an isomorphism induced by f. By Gluskin's result [5], f is either an isomorphism or an anti-isomorphism. The unicity of f may be shown in the same way as in Case 1.3.

Theorem 4 is proved.

5. Proof of Theorem 5. Let $\mathfrak{A}$, $\mathfrak{B}$ be as in Theorem 5. Clearly, if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic or anti-isomorphic, or $\mathfrak{B}$ is degenerate, then there exists a homomorphism of $\mathcal{E}(\mathfrak{A})$ onto $\mathcal{E}(\mathfrak{B})$.

Now let π be a non-degenerate homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Clearly, $\mathcal{E}(\mathfrak{A})$ contains all the constants from $\mathcal{C}(\mathfrak{A})$, $\mathcal{E}(\mathfrak{B})$ all the constants from $\mathcal{C}(\mathfrak{B})$. $\mathcal{E}(\mathfrak{A})$ is a sufficient subsemigroup of $\mathcal{E}(\mathfrak{A}^{\cdot})$ and the ordered set $\mathfrak{A}^{\cdot}$ is connected. The argument in the proof of Theorem 1 is applicable here, and it shows that π is an isomorphism induced by a uniquely defined bijection f.

Let $\{a_1, a_2\}$ be a subsemilattice of $\mathfrak{A}$. Then there exists a $\varphi \in \mathcal{E}(\mathfrak{A})$ with the range $\{a_1, a_2\}$ [11]. The range of $\pi(\varphi)$ is $\{f(a_1), f(a_2)\} = B_1$. It follows that B_1 is a subalgebra of $\mathfrak{B}$. Suppose B_1 is a singular semigroup, α denoting its non-trivial automorphism and β a non-trivial permutation of $\{a_1, a_2\}$. Then $\beta \circ \varphi = \pi^{-1}\big(\alpha \circ \pi(\varphi)\big)$ and $\alpha \circ \pi(\varphi) \in \mathcal{E}(\mathfrak{B})$, whence $\beta \circ \varphi \in \mathcal{E}(\mathfrak{A})$. Hence, β is an automorphism of the subsemilattice $\{a_1, a_2\}$ — a contradiction. Therefore B_1 cannot be a singular semigroup. But every two-element algebra with one binary idempotent operation either is a singular semigroup or a semigroup having zero and identity.

Define a binary relation $\leqslant_{\mathfrak{B}}$ on B: $b_1 \leqslant_{\mathfrak{B}} b_2 \leftrightarrow b_1 = b_1 \circ b_2 = b_2 \circ b_1$. If $b_1 \leqslant_{\mathfrak{B}} b_2$ or $b_2 \leqslant_{\mathfrak{B}} b_1$, we say that b_1 and b_2 are comparable. Hence, if a_1 and a_2 are comparable (relative to the natural order of $\mathfrak{A}$), then $f(a_1)$ and $f(a_2)$ are comparable.

Let g be a mapping of $\{a_1, a_2\}$ onto a subset $\{a_3, a_4\} \subset A$, $g(a_1) = a_3$, $g(a_2) = a_4$ and h be the corresponding mapping of $\{f(a_1), f(a_2)\}$ onto $\{f(a_3), f(a_4)\}$. Now (g is an isomorphism) $\leftrightarrow g \circ \varphi \in \mathcal{E}(\mathfrak{A}) \leftrightarrow h \circ \pi(\varphi) = \pi(g \circ \varphi) \in \mathcal{E}(\mathfrak{B}) \leftrightarrow$ (h is an isomorphism).

Clearly, a_1 and a_2 are comparable in $\mathfrak{A}^{\cdot}$, we may suppose that $a_1 \leqslant_{\mathfrak{A}} a_2$. Consider the following two cases:

Case 5.1. Let $f(a_1) \leqslant_{\mathfrak{B}} f(a_2)$. Then $a_3 \leqslant_{\mathfrak{A}} a_4 \leftrightarrow$ (g is an isomorphism) $\leftrightarrow$ (h is an isomorphism) $\leftrightarrow f(a_3) \leqslant_{\mathfrak{B}} f(a_4)$. Therefore f is an isomorphism between $\mathfrak{A}^{\cdot}$ and $(B, \leqslant_{\mathfrak{B}})$.

Let $r_a(a_0) = a_0 a$ for all $a_0 \in A$ (here $a_0 a$ is the product of a_0 and a in $\mathfrak{A}$). Then $r_a \in \mathcal{E}(\mathfrak{A})$ for every $a \in A$. Suppose $f(a_3) o f(a_4) = f(a_0)$. Then $f(a_0 a) = f\big(r_a(a_0)\big) = \pi(r_a)\big(f(a_0)\big) = \pi(r_a)\big(f(a_3)\big) \, o \, \pi(r_a)\big(f(a_4)\big) = f\big(r_a(a_3)\big) \, o \, f\big(r_a(a_4)\big) = f(a_3 a) \, o \, f(a_4 a)$. Substitute a_0 for a in this formula. Then $f(a_3) o f(a_4) = f(a_0) = f(a_0 a_0) = f(a_3 a_0) o f(a_4 a_0) = f(a_0 a_3) o f(a_0 a_4) = \big(f(a_3 a_3) o f(a_4 a_3)\big) \, o \, \big(f(a_3 a_4) o f(a_4 a_4)\big) = \big(f(a_3) o f(a_3 a_4)\big) \, o \, \big(f(a_3 a_4) o f(a_4)\big) = f(a_3 a_4) o f(a_3 a_4) = f(a_3 a_4)$, whence f is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Case 5.2. Let $f(a_2) \leqslant_{\mathfrak{B}} f(a_1)$. Then $a_3 \leqslant_{\mathfrak{A}} a_4 \leftrightarrow$ (g is an isomorphism $\leftrightarrow$ (h is an isomorphism) $\leftrightarrow f(a_4) \leqslant_{\mathfrak{B}} f(a_3)$. Therefore, f is an „anti-isomorphism" between $\mathfrak{A}^{\cdot}$ and $(B, \leqslant_{\mathfrak{B}})$.

A prime ideal of $\mathfrak{B}$ is any proper subset of B which is an ideal of $\mathfrak{B}$ and whose complement is a subalgebra of $\mathfrak{B}$. Prime ideals of $\mathfrak{A}$ are defined in the same way. Clearly, the join of two prime ideals is again a prime ideal, provided it is a proper subset. Evidently, a proper subset $C \subset A$ is a prime ideal of $\mathfrak{A}$ if and only if $d_C \in \mathcal{E}(\mathfrak{A})$ where d_C maps all the elements of C onto a_1 and all the elements of C' onto a_2. Let D be a proper subset of B. Define a mapping $d_D \in \mathcal{C}(B)$: d_D maps all the elements of D onto $f(a_2)$ and all the elements of D' onto $f(a_1)$. If D is a prime ideal, then one can easily verify that $d_D \in \mathcal{E}(\mathfrak{B})$. Conversely, let $d_D \in \mathcal{E}(\mathfrak{B})$. Evidently, $\{f(a_2)\}$ is an ideal of $\{f(a_2), f(a_1)\}$, whence $D = d_D^{-1}\big(f(a_2)\big)$ is an ideal of $\mathfrak{B}$. $D' = d_D^{-1}\big(f(a_1)\big)$ and $f(a_1)$ is an idempotent, whence D' is a subalgebra of $\mathfrak{B}$. It follows that D is a prime ideal of $\mathfrak{B}$ if and only if $d_D \in \mathcal{E}(\mathfrak{B})$.

Since π is induced by f, we conclude that C is a prime ideal of $\mathfrak{A}$ if and only if $f(C')$ is a prime ideal of $\mathfrak{B}$. If C_1 and C_2 are prime ideals of $\mathfrak{A}$, then $f(C_1')$ and $f(C_2')$ are prime ideals of $\mathfrak{B}$, whence $f(C_1')\cup f(C_2') = f(C_1'\cup C_2') = f((C_1\cap C_2)')$ either is a prime ideal of $\mathfrak{B}$ or equals B. It follows that $C_1\cap C_2$ either is a prime ideal of $\mathfrak{A}$ or is empty. The meet of two ideals of a semigroup cannot be empty, whence, the meet of two prime ideals of $\mathfrak{A}$ is a prime ideal of $\mathfrak{A}$.

For every $a\in A$ the set $[a,+\infty)'$ is a prime ideal of $\mathfrak{A}$ unless empty. It is empty if and only if a is the smallest element of $\mathfrak{A}^{\cdot}$. For every $a_3, a_4\in A$ $[a_3,+\infty)'\cup[a_4,+\infty)'$ is a prime ideal of $\mathfrak{A}$ or empty. Hence $[a_3,+\infty)\cup[a_4,+\infty)$ is a subsystem of $\mathfrak{A}$. This subsystem contains a_3 and a_4. Therefore, it contains a_3a_4. It follows that either $a_3\leqslant_{\mathfrak{A}}a_3a_4$ or $a_4\leqslant_{\mathfrak{A}}a_3a_4$, which means that either $a_3\leqslant_{\mathfrak{A}}a_4$ or $a_4\leqslant_{\mathfrak{A}}a_3$. Therefore $\mathfrak{A}$ is a linear semilattice. It follows that $\mathfrak{B}$ also is a linear semilattice and f is an anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.

Theorem 5 is proved.

6. Proof of Theorem 6. The "if" part of Theorem 6 may be verified straightforwardly. Now let $\mathfrak{A}$ and $\mathfrak{B}$ be as in Theorem 6 and let π be a non-degenerate homomorphism of $\mathcal{E}(A)$ onto $\mathcal{E}(\mathfrak{B})$. Since $\mathcal{E}(\mathfrak{A})$ is a sufficient subsemigroup of $\mathcal{E}(\mathfrak{A}^{\cdot})$ (it follows from the distributivity of $\mathfrak{A}$), the same argument as in the proof of Theorem 1 shows that π is an isomorphism induced by a uniquely defined bijection f.

Consider two binary relations on B: $\leqslant_{\mathfrak{B}}$, which was introduced in the proof of Theorem 5, and $\leqslant^{\mathfrak{B}}$, defined by the following formula: $b_1\leqslant^{\mathfrak{B}}b_2\leftrightarrow b_2 = b_2tb_1 = b_1tb_2$. Let $\{a_1, a_2\}$ be a sublattice of $\mathfrak{A}$. Precisely as in the proof of Theorem 5, we may verify that $\{f(a_1), f(a_2)\}$ is a subalgebra of $\mathfrak{B}$ which cannot be a singular semigroup under either o or t (provided $a_1\neq a_2$). Now for arbitrary $a_3, a_4\in A$ define the mappings g and h as has been done in the previous proof and by the same argument prove that g is an isomorphism if and only if h is. Continuing the same argument, we prove the following four statements:

1) if $f(a_1)\leqslant_{\mathfrak{B}}f(a_2)$, then f is an isomorphism of $\mathfrak{A}^{\cdot}$ onto $(B,\leqslant_{\mathfrak{B}})$;
2) if $f(a_2)\leqslant_{\mathfrak{B}}f(a_1)$, then f is an anti-isomorphism of $\mathfrak{A}^{\cdot}$ onto $(B,\leqslant_{\mathfrak{B}})$;
3) if $f(a_1)\leqslant^{\mathfrak{B}}f(a_2)$, then f is an isomorphism of $\mathfrak{A}^{\cdot}$ onto $(B,\leqslant^{\mathfrak{B}})$;
4) if $f(a_2)\leqslant^{\mathfrak{B}}f(a_1)$, then f is an anti-isomorphism of $\mathfrak{A}^{\cdot}$ onto $(B,\leqslant^{\mathfrak{B}})$.

For every $a, a_0\in A$ define $r_a^{\wedge}(a_0) = a_0\wedge a$ and $r_a^{\vee}(a_0) = a_0\vee a$. Evidently, $r_a^{\wedge}, r_a^{\vee}\in\mathcal{E}(\mathfrak{A})$ for all $a\in A$.

If the first of the aforesaid cases holds, then, using the endomorphisms $r_a^{\wedge}$ and the operation o, we may prove that f is an isomorphism of $(A,\wedge)$ onto (B, o) (the proof is the same as given in the Case 5.1).

If the second case holds, then, using the endomorphisms $r_a^{\vee}$ and the operation o, we prove that f is an isomorphism of $(A,\vee)$ onto (B, o).

In the third case we use $r_a^{\vee}$ and t to prove that f is an isomorphism of $(A,\vee)$ onto (B, t).

In the fourth case we use $r_a^{\wedge}$ and t to prove along the lines of Case 5.1 that f is an isomorphism of $(A,\wedge)$ onto (B, t).

If both the first and the third cases hold, then f is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$; if both the second and the fourth cases hold then, clearly, f is an isomorphism of $(A,\vee,\wedge)$ onto $\mathfrak{B}$, whence f is an anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. If both the first and the fourth cases hold, then f is an isomorphism of $(A,\wedge)$ onto (B, o) and of $(A,\wedge)$ onto (B, t), whence $o = t$. $(A,\wedge)$ is a semillattice, whence $\mathfrak{B}$ is a bi-semilattice and f is an order-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. It follows that π^{-1} is an isomorphism of $\mathcal{E}(\mathfrak{B}) = \mathcal{E}((B, o))$ onto $\mathcal{E}((A,\wedge))$, whence $\mathcal{E}(\mathfrak{A}) = \mathcal{E}((A,\wedge))$. By Lemma 1, $\mathfrak{A}$ is a linear lattice. Hence, $\mathfrak{B}$ is a linear bi-semilattice. If both the cases 2 and 3 hold, then in the same way we prove that $\mathfrak{A}$ is linear, $\mathfrak{B}$ is a linear bi-semilattice and f is an order anti-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

LEMMA 1. *Let $\mathfrak{A} = (A,\wedge,\vee)$ be a lattice. $\mathfrak{A}$ is linear if and only if $\mathcal{E}(\mathfrak{A}) = \mathcal{E}((A,\wedge))$.*

Proof. If $\mathfrak{A}$ is linear, then every endomorphism of $\mathfrak{A}^{\cdot}$ is also an endomorphism of $(A,\wedge)$ and of $(A,\vee)$, which proves the "only if" part.

Now let $\mathcal{E}(\mathfrak{A}) = \mathcal{E}((A,\wedge))$ for some lattice $\mathfrak{A}$. If C is a prime ideal of $(A,\wedge)$, then $d_C\in\mathcal{E}((A,\wedge))$; hence $d_C\in\mathcal{E}(\mathfrak{A})$. Let $a_3\vee a_4 = a$ for some $a_3, a_4, a\in A$ and $a\notin C$. Then $d_C(a_3)\vee d_C(a_4) = d_C(a) = a_2$ (provided $a_1\leqslant_{\mathfrak{A}}a_2$). This is possible only if $d_C(a_3) = a_2$ or $d_C(a_4) = a_2$, i.e., only if $a_3\notin C$ or $a_4\notin C$. If a is not the smallest element of $\mathfrak{A}^{\cdot}$, then $[a,+\infty)'$ is a prime ideal, whence $a_3\in[a,+\infty)$ or $a_4\in[a,+\infty)$, i.e. $a\leqslant_{\mathfrak{A}}a_3$ or $a\leqslant_{\mathfrak{A}}a_4$. It follows that either $a_4\leqslant_{\mathfrak{A}}a_3$ or $a_3\leqslant_{\mathfrak{A}}a_4$, and if a is the smallest element, then $a_3 = a = a_4$. Therefore, $\mathfrak{A}$ is linear.

Lemma 1 is proved.

Now we must consider the case where $\{f(a_1), f(a_2)\}$ is a singular semigroup under one of the operations o and t.

Let $\{f(a_1), f(a_2)\}$ be a singular semigroup under the operation t. We shall consider only this case and omit the consideration of the completely analogous case where $\{f(a_1), f(a_2)\}$ is a singular semigroup relative to the operation o. Furthermore, we will consider only the case where $\{f(a_1), f(a_2)\}$ is a left zero semigroup under t, omitting a detailed consideration of the dual case of right zero semigroups. We must show that (B, t) is a left zero semigroup and $\mathfrak{A}$ is linearly ordered. Since $\{f(a_1), f(a_2)\}$ cannot be a singular semigroup both under o and t, we infer that either the first or the second of above-mentioned cases holds, i.e., f is an isomorphism of $(A,\wedge)$ or of $(A,\vee)$ onto (B, o). Provided $\mathfrak{A}$ is linear, we infer that $\mathfrak{B}$ is a singular chain and f is an order-isomorphism or an order-anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. To shorten our argument still more, we shall consider only the case where f is an isomorphism of $(A, \wedge)$ onto (B, o) (if f is an isomorphism of $(A, \vee)$ onto (B, o), we may consider $(A, \vee, \wedge)$ instead of $\mathfrak{A}$ and reduce this case to the previous one).

If $a_1 \leqslant_{\mathfrak{A}} a_2$ and $a_3 \leqslant_{\mathfrak{A}} a_4$, then g is an isomorphism, whence h is an isomorphism. Therefore $\{f(a_3), f(a_4)\}$ is a left zero semigroup under t. Let b_3, b_4 be two arbitrary elements of B. If b_3 and b_4 are comparable under $\leqslant_{\mathfrak{B}}$, then $a_3 = f^{-1}(b_3)$ and $a_4 = f^{-1}(b_4)$ are comparable under $\leqslant_{\mathfrak{A}}$. Hence $b_3 t b_4 = b_3$.

Now let b_3 and b_4 be incomparable under $\leqslant_{\mathfrak{B}}$, which means that a_3 and a_4 are not comparable under $\leqslant_{\mathfrak{A}}$. Define $a_0 = a_3 \wedge a_4$ and $a_5 = a_3 \vee a_4$. $\mathfrak{A}$ is isomorphic to a subdirect product of two-element lattices $\{0, 1\}$ [1]; hence there exist homomorphisms d_1 and d_2 of $\mathfrak{A}$ onto $\{0, 1\}$ such that $d_1(a_0) \neq d_1(a_3)$ and $d_2(a_5) \neq d_2(a_4)$. Define a transformation $d \in \mathcal{T}(A)$ for every $a \in A$: $d(a) = a_0 \leftrightarrow d_1(a) = 0 = d_2(a)$, $d(a) = a_3 \leftrightarrow d_1(a) \neq 0 = d_2(a)$, $d(a) = a_4 \leftrightarrow d_1(a) = 0 \neq d_2(a)$ and $d(a) = a_5 \leftrightarrow d_1(a) \neq 0 \neq d_2(a)$. One can easily verify that $d \in \mathcal{E}(\mathfrak{A})$ and $A_1 = \{a_0, a_3, a_4, a_5\}$ is the range of d. Let $b_0 = f(a_0)$ and $b_5 = f(a_5)$. Then $B_1 = \{b_0, b_3, b_4, b_5\}$ is the range of $\pi(d)$, whence B_1 is a subsystem of $\mathfrak{B}$. It follows that $b_3 t b_4 \in B_1$. Let p and q be the transformations of B_1 corresponding (under f) to the transformations $\overline{p}$ and $\overline{q}$ of A_1 defined as follows: $\overline{p} = \begin{pmatrix} a_0 a_3 a_4 a_5 \\ a_0 a_0 a_4 a_4 \end{pmatrix}$ and $\overline{q} = \begin{pmatrix} a_0 a_3 a_4 a_5 \\ a_0 a_3 a_0 a_3 \end{pmatrix}$. Clearly, $\overline{p}$ and $\overline{q}$ are endomorphisms of A_1, whence $\overline{p} \circ d, \overline{q} \circ d \in \mathcal{E}(\mathfrak{A})$ and $p \circ \pi(d)$, $q \circ \pi(d) \in \mathcal{E}(\mathfrak{B})$. It follows that p and q are endomorphisms of B_1.

Suppose $b_3 t b_4 = b_5$. Using the fact that $a_0 \leqslant_{\mathfrak{A}} a_4$, which implies $b_0 t b_4 = b_0$ we obtain $b_0 = b_0 t b_4 = p(b_3) t p(b_4) = p(b_3 t b_4) = p(b_5) = b_4$ — a contradiction. Suppose now that $b_3 t b_4 = b_4$. Repeating the previous equalities, we obtain $b_0 = p(b_3 t b_4) = p(b_4) = b_4$ — a contradiction. Now let $b_3 t b_4 = b_0$. Using the fact that $a_0 \leqslant_{\mathfrak{A}} a_3$, which implies $b_3 = b_3 t b_0$, we obtain $b_3 = b_3 t b_0 = q(b_3) t q(b_4) = q(b_3 t b_4) = q(b_0) = b_0$ — a contradiction. Thus, $b_3 t b_4 = b_3$.

It follows that (B, t) is a left zero semigroup.

Evidently, $\mathcal{E}((B, t)) = \mathcal{T}(B)$, whence $\mathcal{E}(\mathfrak{B}) = \mathcal{E}((B, o)) \cap \mathcal{E}((B, t)) = \mathcal{E}((B, o))$. It follows that $\mathcal{E}(\mathfrak{A}) = \pi^{-1}(\mathcal{E}(\mathfrak{B})) = \pi^{-1}(\mathcal{E}((B, o))) = \mathcal{E}((A, \wedge))$. By Lemma 1, $\mathfrak{A}$ is linear.

Theorem 6 is proved.

§ 4. Applications. An obvious corollary to Theorem 6 is

COROLLARY 4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be lattices, $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ being isomorphic. If $\mathfrak{A}$ is distributive then $\mathfrak{B}$ is distributive.*

If $\mathfrak{A}$ is a semilattice, then $\mathcal{E}(\mathfrak{A}) = \mathcal{E}_{\wedge}(\mathfrak{A}^{\cdot})$; if $\mathfrak{A}$ is a lattice, then $\mathcal{E}(\mathfrak{A}) = \mathcal{E}_{\wedge\vee}(\mathfrak{A}^{\cdot})$, which permits the application of Theorem 1 not only to ordered sets but also to semilattices and lattices. If $\mathfrak{A}$ is a semilattice, then $\mathcal{E}(\mathfrak{A})$ always contains a sufficient subsemigroup (e.g., $\mathcal{E}(\mathfrak{A})$ itself). If $\mathfrak{A}$ is a lattice, then $\mathcal{E}(\mathfrak{A})$ contains a sufficient subsemigroup if and only if $\mathcal{E}(\mathfrak{A})$ is its own sufficient subsemigroup. This is the case precisely when $\mathfrak{A}$ is distributive. If $\mathfrak{A}$ is a semilattice with zero (or with identity) or a distributive lattice with zero (or with identity), then the set of all endomorphisms of $\mathfrak{A}$ respecting zero (or identity) is a sufficient subsemigroup of $\mathcal{E}((\mathfrak{A}, \leqslant_{\mathfrak{A}}, 0))$, which permits the application of Theorem 2 in the case of semilattices and distributive lattices. One can consider semigroups of complete endomorphisms of complete (or not necessarily complete) semilattices and representable distributive lattices; these semigroups also characterize the underlying algebras up to isomorphism. If $\mathfrak{m}$ is a cardinal number and $\mathfrak{A}$ is an $\mathfrak{m}$-field of sets [13], then the semigroup of all $\mathfrak{m}$-endomorphisms of $\mathfrak{A}$ (i.e. those endomorphisms of $\mathfrak{A}$ which respect meets and joins of subsets having cardinality $\mathfrak{m}$ at most) characterizes $\mathfrak{A}$ up to an isomorphism.

Let $\mathfrak{A}$ be an algebraic system with the basic set A and $\mathcal{S}(A)$ denote the symmetric group of all permutations of A. There exists a natural homomorphism h: $\mathcal{A}(\mathfrak{A}) \to \mathcal{A}(\mathcal{E}(\mathfrak{A}))$ — if $\alpha \in \mathcal{A}(\mathfrak{A})$, then $h(\alpha)$ is the automorphism of $\mathcal{E}(\mathfrak{A})$ induced by α. If h is a surmorphism, we say that every automorphism of $\mathcal{E}(\mathfrak{A})$ is *inner*. If h is a bijection we say that every automorphism of $\mathcal{E}(\mathfrak{A})$ is *strictly inner*. The latter being the case, $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ is naturally isomorphic with $\mathcal{A}(\mathfrak{A})$.

In the general case $\alpha \in \mathcal{A}(\mathcal{E}(\mathfrak{A}))$ is called *inner* (*strictly inner*) if it is induced by some (uniquely defined) automorphism of $\mathfrak{A}$.

Let $\mathfrak{A}$ be an ordered set, a semilattice or a lattice. Then one can consider the group $\mathcal{AJ}(\mathfrak{A})$ of all automorphisms and anti-automorphisms of $\mathfrak{A}$ (of course, $\mathfrak{A}$ may well possess no anti-automorphisms). The homomorphism h may be extended to h_t: $\mathcal{AJ}(\mathfrak{A}) \to \mathcal{A}(\mathcal{E}(\mathfrak{A}))$. An automorphism α of $\mathcal{E}(\mathfrak{A})$ is called (*strictly*) *inner in a wider sense* if α is induced by an (uniquely defined) element of $\mathcal{AJ}(\mathfrak{A})$.

COROLLARY 5. *Let $\mathfrak{A}$ be an ordered set, Φ a sufficient semigroup of endomorphisms of $\mathfrak{A}$. Then $\mathcal{ES}(\Phi) = \mathcal{A}(\Phi)$ and every automorphism of Φ is strictly inner in a wider sense. In particular, $\mathcal{ES}(\mathcal{E}(\mathfrak{A})) = \mathcal{A}(\mathcal{E}(\mathfrak{A}))$ and $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ is naturally isomorphic with $\mathcal{AJ}(\mathfrak{A})$.*

If a set $\mathfrak{A}$ is trivially ordered and $\mathfrak{A} = (A, =)$ is the corresponding ordered set, then $\mathcal{E}(\mathfrak{A}) = \mathcal{T}(A)$. Hence we obtain

COROLLARY 6. $\mathcal{ES}(\mathcal{T}(A)) = \mathcal{A}(\mathcal{T}(A))$, *every automorphism of $\mathcal{T}(A)$ is strictly inner and $\mathcal{A}(\mathcal{T}(A))$ is naturally isomorphic with $\mathcal{S}(A)$.*

Automorphisms of $\mathcal{T}(A)$ were first found in [12].

COROLLARY 7. *Let $\mathfrak{A}$ be an ordered set with zero, and Φ a sufficient* *semigroup of endomorphisms of* $\mathfrak{A}$. *Every automorphism of* Φ *is strictly inner. In particular,* $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *is naturally isomorphic with* $\mathcal{A}(\mathfrak{A})$.

Consider a set $A_0 = A \cup \{0\}$ where $0 \notin A$. Order A_0 in the following way: $a_1 \leqslant a_2$ if and only if $a_1 = a_2$ or $a_1 = 0$. Then $\mathcal{E}(A_0) = \mathcal{T}_0(A_0)$ where $\mathcal{T}_0(A_0)$ is the set of all transformations of A_0 for which 0 is a fixed point. Let $\mathcal{F}(A)$ be the semigroup of all partial transformations of A. Then $\mathcal{F}(A)$ is naturally isomorphic with $\mathcal{T}_0(A_0)$ (if $\varphi \in \mathcal{F}(A)$ then φ_0 denotes the extension of φ: $\varphi_0(a) = 0$ for all $a \in A_0$ for which $\varphi(a)$ is not defined; the correspondence $\varphi \to \varphi_0$ is the natural isomorphism between $\mathcal{F}(A)$ and $\mathcal{T}_0(A_0)$. We obtain

COROLLARY 8. *Every automorphism of* $\mathcal{F}(A)$ *is strictly inner and* $\mathcal{A}(\mathcal{F}(A))$ *is naturally isomorphic with* $\mathcal{S}(A)$ *for every set* A.

Automorphisms of $\mathcal{F}(A)$ were first found in [4].

A distributive lattice $\mathfrak{A} = (A, \wedge, \vee)$ is called *self-dual* if $\mathfrak{A}$ possesses an anti-automorphism (i.e., if $\mathfrak{A}$ is isomorphic with $(A, \vee, \wedge)$).

A semilattice $\mathfrak{A}$ is called *self-dual* if $\mathfrak{A}^*$ possesses an anti-automorphism.

COROLLARY 9. *Let* $\mathfrak{A}$ *be a semilattice. If* $\mathfrak{A}$ *is linear and self-dual, then every automorphism of* $\mathcal{E}(\mathfrak{A})$ *is strictly inner in a wider sense and* $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *is naturally isomorphic with* $\mathcal{A}\mathcal{I}(\mathfrak{A}^*)$. *Otherwise, every automorphism of* $\mathcal{E}(\mathfrak{A})$ *is strictly inner and* $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *is isomorphic with* $\mathcal{A}(\mathfrak{A})$. *In every case* $\mathcal{E}\mathcal{S}(\mathcal{E}(\mathfrak{A})) = \mathcal{A}(\mathcal{E}(\mathfrak{A}))$.

COROLLARY 10. *Let* $\mathfrak{A}$ *be a distributive lattice. If* $\mathfrak{A}$ *is self-dual, then every automorphism of* $\mathcal{E}(\mathfrak{A})$ *is strictly inner in a wider sense and* $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *is naturally isomorphic with* $\mathcal{A}\mathcal{I}(\mathfrak{A})$. *Otherwise, every automorphism of* $\mathcal{E}(\mathfrak{A})$ *is strictly inner and* $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *is naturally isomorphic with* $\mathcal{A}(\mathfrak{A})$. *In every case* $\mathcal{E}\mathcal{S}(\mathcal{E}(\mathfrak{A})) = \mathcal{A}(\mathcal{E}(\mathfrak{A}))$.

COROLLARY 11. *Let* $\mathfrak{A}$ *be a Boolean algebra. If* Φ *is a sufficient semigroup of endomorphisms of* $\mathfrak{A}$, *then* $\mathcal{E}\mathcal{S}(\Phi) = \mathcal{A}(\Phi)$ *and every automorphism of* Φ *is strictly inner. In particular,* $\mathcal{E}\mathcal{S}(\mathcal{E}(\mathfrak{A})) = \mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *and* $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ *is naturally isomorphic with* $\mathcal{A}(\mathfrak{A})$.

If $\mathcal{I}(A)$ is the semigroup of all one-to-one partial transformations of a set A, then the isomorphic image of $\mathcal{I}(A)$ under the natural isomorphism of $\mathcal{F}(A)$ into $\mathcal{T}_0(A_0)$ is a sufficient semigroup of endomorphisms of the ordered set A_0, whence we obtain

COROLLARY 12. *Every automorphism of* $\mathcal{I}(A)$ *is strictly inner and* $\mathcal{A}(\mathcal{I}(A))$ *is isomorphic with* $\mathcal{S}(A)$.

Automorphisms of $\mathcal{I}(A)$ were first found in [7].

We may also conclude that if A and B are sets of unequal cardinality, $|B| > 1$, then there are no surmorphisms of $\mathcal{T}(A)$ onto $\mathcal{T}(B)$.

Let $\mathfrak{P}(A \times A)$ be the semigroup of all binary relations on a set A. If $\varrho \in \mathfrak{P}(A \times A)$, then $\breve{\varrho} \in \mathcal{T}(\mathfrak{P}(A))$ is the transformation naturally corresponding to ϱ: $a \in \breve{\varrho}(\mathfrak{a})$ if and only if $(a_1, a) \in \varrho$ for some $a_1 \in \mathfrak{a}$. Clearly, the correspondence $\varrho \to \breve{\varrho}$ is an isomorphism of $\mathfrak{P}(A \times A)$ into $\mathcal{T}(\mathfrak{P}(A))$. One can straightforwardly verify that the isomorphic image of $\mathfrak{P}(A \times A)$ in $\mathcal{T}(\mathfrak{P}(A))$ is a sufficient subsemigroup of $\mathcal{E}(\mathfrak{P}(B))$ (here $\mathcal{E}(\mathfrak{P}(B))$ is considered as the endomorphism semigroup of the naturally ordered set $\mathfrak{P}(A)$ of all subsets of A).

We obtain

COROLLARY 13. $\mathcal{E}\mathcal{S}(\mathfrak{P}(A \times A)) = \mathcal{A}(\mathfrak{P}(A \times A))$ *and every automorphism of* $\mathfrak{P}(A \times A)$ *is strictly inner.* $\mathcal{A}(\mathfrak{P}(A \times A))$ *is naturally isomorphic with* $\mathcal{S}(A)$. $\mathfrak{P}(A \times A)$ *may be homomorphically mapped onto* $\mathfrak{P}(B \times B)$ *if and only if* A *and* B *have the same cardinality or if* $B = \emptyset$.

Automorphisms of $\mathfrak{P}(A \times A)$ were first found in [16].

In exactly the same way one can prove that every automorphism of Φ is strictly inner and $\mathcal{A}(\Phi)$ is naturally isomorphic with $\mathcal{S}(A)$ if Φ is the semigroup of all reflexive binary relations, or the semigroup of all rectangular binary relations, or of all full binary relations, or of all dense binary relations, or of all (r, s)-relations over a set A (a binary relation $\varrho \in \mathfrak{P}(A \times A)$ is called full if its domain is A, ϱ is called dense if ϱ is full and the range of ϱ is A, ϱ is called an (r, s)-relation if the cardinality of its domain $\leqslant r$ and the cardinality of its range $\leqslant s$). To prove this one should consider either the isomorphism $\varrho \to \breve{\varrho}$ of Φ into $\mathcal{T}(\mathfrak{P}(B))$ or the isomorphism $\varrho \to \breve{\varrho}_0$ (where $\breve{\varrho}_0$ is the restriction of $\breve{\varrho}$ on $\mathfrak{P}_0(A)$) of Φ into $\mathcal{T}(\mathfrak{P}_0(A))$ and verify that the isomorphic image of Φ is a sufficient subsemigroup of the endomorphism semigroup of $\mathfrak{P}(A)$ or of $\mathfrak{P}_0(A)$ ordered by their inclusion relation. $\mathfrak{P}(A)$ is considered as an ordered set with zero or as an ordered set with identity). We omit the proofs. Automorphisms of these semigroups were first found by different authors and by different methods (cf. [6, 16, 17]). In particular, every result of [6] is a simple corollary to Theorem 2. These results may be strengthened $(\mathcal{E}\mathcal{S}(\Phi) = \mathcal{A}(\Phi))$ if one uses the fact that the isomorphic image of Φ is a semigroup of endomorphisms of $\mathfrak{P}(A)$ considered as join-semilattice. As further corollaries we may prove that surmorphisms of the aforesaid semigroups onto one another are either trivial or isomorphisms.

Using Theorem 1 (or other theorems) one can easily construct exclusive families of arbitrary cardinality (a family $\mathfrak{F}$ of semigroups is called exclusive if the only surmorphisms between the members of $\mathfrak{F}$ are trivial automorphisms).

Let $\mathfrak{A} = (A, \cdot, +, 1)$ be a Boolean ring (i.e., a ring satisfying the identity $x^2 = x$) with identity 1 and let $\mathcal{E}(\mathfrak{A})$ be the endomorphism semigroup of $\mathfrak{A}$ (all endomorphisms considered respect 1). Clearly, $\mathcal{E}(\mathfrak{A})$ coincides with the endomorphism semigroup of the Boolean algebra corresponding to $\mathfrak{A}$.

We obtain

Corollary 14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two non-degenerate Boolean rings with identities, and let $\Phi \subset \mathcal{E}(\mathfrak{A})$ and $\Psi \subset \mathcal{E}(\mathfrak{B})$ be sufficient subsemigroups. Then every homomorphism of Φ onto Ψ is an isomorphism induced by a uniquely defined isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. In particular, $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ are isomorphic if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. $\mathcal{ES}(\mathcal{E}(\mathfrak{A})) = \mathcal{A}(\mathcal{E}(\mathfrak{A}))$, every automorphism of $\mathcal{E}(\mathfrak{A})$ is strictly inner and $\mathcal{A}(\mathcal{E}(\mathfrak{A}))$ is naturally isomorphic with $\mathcal{A}(\mathfrak{A})$.*

The same corollary could be proved for Boolean rings without identity (one should previously prove an analog of Theorem 3 for distributive lattices with relative complements considered as algebras with three binary operations: join, meet and subtraction).

Let $\mathfrak{A}$ be a Boolean space (i.e., a totally disconnect compact Hausdorff topological space). It is well known [13] that continuous transformations of $\mathfrak{A}$ correspond to endomorphisms of the Boolean algebra $\overline{\mathfrak{A}}$ dual to $\mathfrak{A}$.

The semigroup $\mathcal{C}(\mathfrak{A})$ of all continuous transformations of $\mathfrak{A}$ is naturally anti-isomorphic to $\mathcal{E}(\overline{\mathfrak{A}})$, which gives

Corollary 15. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be non-degenerate Boolean spaces, and let $\Phi \subset \mathcal{C}(\mathfrak{A})$ and $\Psi \subset \mathcal{C}(\mathfrak{B})$ be sufficient subsemigroups. Every homomorphism of Φ onto Ψ is an isomorphism induced by a homeomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. In particular, $\mathcal{C}(\mathfrak{A})$ and $\mathcal{C}(\mathfrak{B})$ are isomorphic if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are homeomorphic. $\mathcal{ES}(\mathcal{C}(\mathfrak{A})) = \mathcal{A}(\mathcal{C}(\mathfrak{A}))$, every automorphism of $\mathcal{C}(\mathfrak{A})$ is strictly inner and $\mathcal{A}(\mathcal{C}(\mathfrak{A}))$ is naturally isomorphic with the group $\mathcal{H}(\mathfrak{A})$ of all homeomorphisms of $\mathfrak{A}$.*

A weaker version of this Corollary was first proved in [14].

Remark. Sufficient semigroups in the cases of Boolean rings or Boolean spaces are defined in the obvious way by using the same concept for Boolean algebras.

Let S be a topological space, and ϱ a binary relation over S. ϱ is called *closed* if $\check{\varrho}(\mathfrak{a})$ is closed for every closed $\mathfrak{a} \subset S$. ϱ is called a *multi-homeomorphism* if ϱ and its converse ϱ^{-1} are closed and ϱ is dense (i.e., the domain and the range of ϱ coincide with S). The set $\mathcal{H}_m(S)$ of all multi-homeomorphisms of S is a semigroup of binary relations. Let C be the set of all non-empty closed subsets of S, $\mathfrak{C} = (C, \subset, S)$ i.e., $\mathfrak{C}$ is the inclusion-ordered set C with the largest element S. For every $f \in \mathcal{H}_m(S)$ let $\bar{f}$ denote the restriction of $\check{f}$ to the set C. Clearly $\bar{f}_2 \circ \bar{f}_1 = \overline{f_2 \circ f_1}$ for every $f_1, f_2 \in \mathcal{H}_m(S)$. If $\bar{f}_1 = \bar{f}_2$ and S is a T_1-space, then $\bar{f}_1(\{a\}) = \bar{f}_2(\{a\})$ for every $a \in S$, which means that $f_1(a) = f_2(a)$ and $f_1 = f_2$. Therefore the correspondence $f \to \bar{f}$ is an isomorphism of $\mathcal{H}_m(S)$ into $\mathcal{E}(\mathfrak{C})$. Let $\mathfrak{a}_i$ be proper closed subsets of S. Then $f = (S \times \mathfrak{a}_1) \cup (\mathfrak{a}_2 \times S)$ is an element of $\mathcal{H}_m(S)$ and the range of $\bar{f}$ is $\{\mathfrak{a}_1, S\}$. For every $\mathfrak{a}_0 \in C$, $\mathfrak{a}_0 \cap \mathfrak{a}_2 = \emptyset$ $\bar{f}(\mathfrak{a}_0) = \mathfrak{a}_1$.

Now let $\bar{a}_1 \not\subset \bar{a}_2$ for some $\bar{a}_1, \bar{a}_2 \in C$. Then $\bar{a}_1 \backslash \bar{a}_2 \neq \emptyset$. Let $a \subset \bar{a}_1 \backslash \bar{a}_2$. Then $\bar{f}(\bar{a}_1) = S \neq a_1 = \bar{f}(\bar{a}_2)$. Hence, the isomorphic image of $\mathcal{H}_m(S)$ in $\mathcal{E}(\mathfrak{C})$ is a sufficient subsemigroup.

Corollary 16. *Let S_0 and S be T_1-spaces. $\mathcal{H}_m(S_0)$ and $\mathcal{H}_m(S)$ are isomorphic if and only if S_0 and S are homeomorphic. Every isomorphism between $\mathcal{H}_m(S_0)$ and $\mathcal{H}_m(S)$ is induced by a uniquely defined homeomorphism between S_0 and S. Every automorphism of $\mathcal{H}_m(S)$ is strictly inner and $\mathcal{A}(\mathcal{H}_m(S))$ is isomorphic with $\mathcal{H}(S)$.*

Proof. Let $\mathcal{H}_m(S_0)$ and $\mathcal{H}_m(S)$ be isomorphic. Then the isomorphic images of these semigroups are isomorphic sufficient subsemigroups of $\mathcal{E}(\mathfrak{C}_0)$ and of $\mathcal{E}(\mathfrak{C})$ respectively. By Theorem 2, the isomorphism between these sufficient subsemigroups is induced by an isomorphism between $\mathfrak{C}_0$ and $\mathfrak{C}$. By [15], every isomorphism between $\mathfrak{C}_0$ and $\mathfrak{C}$ is induced by a uniquely defined homeomorphism between S_0 and S. Hence, the isomorphism between $\mathcal{H}_m(S_0)$ and $\mathcal{H}_m(S)$ is induced by a homeomorphism between S_0 and S. If this isomorphism is induced by two homeomorphisms f and g, then $g \circ f^{-1}$ is a homeomorphism of S inducing the trivial automorphism of $\mathcal{E}(\mathfrak{C})$, whence $f = g$.

Corollary 16 is proved.

A continuous mapping of a closed subspace of S into S is called a partial continuous mapping. One can show, as another corollary of Theorem 2, that two T_1-spaces with isomorphic semigroups of partial continuous transformations are homeomorphic (this result may be deduced from a far stronger result from [10]). Analogously, one can prove the main result of [3] and of many other papers.

The following question is natural after Theorem 5 and Corollary 4: if $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ are isomorphic for two non-distributive lattices $\mathfrak{A}$ and $\mathfrak{B}$, need $\mathfrak{A}$ and $\mathfrak{B}$ be isomorphic or anti-morphic? Clearly, $\mathfrak{A}$ and $\mathfrak{B}$ must have the same order. One can easily construct two finite lattices $\mathfrak{A}$ and $\mathfrak{B}$ which are neither isomorphic, nor anti-isomorphic and for which $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ are isomorphic. The author has such an example for lattices of order 10.

Added in proof. In the abstract: А. Я. Айзенштат, Т. Б. Шварц, Об определяемости структур структурными эндоморфизмами, IX Всесоюзн. алгебр. коллокв. Резюме научных сообщений, Гомель, 1968. pp. 5-6, there is announced that such examples do exist if and only if the lattices have more than 7 elements. A weaker version of Theorem 6 (if $\mathcal{E}(\mathfrak{A})$ and $\mathcal{E}(\mathfrak{B})$ are isomorphic, $\mathfrak{A}$ and $\mathfrak{B}$ are lattices and $\mathfrak{A}$ is distributive, then $\mathfrak{A}$ and $\mathfrak{B}$ are either isomorphic or anti-isomorphic) has been announced without proof in the same summary. A corollary to Theorem 1 in the case when π is an isomorphism, Φ and Ψ are semigroups of all n-valued endomorphisms from $\mathcal{E}_\wedge(\mathfrak{a})$ and $\mathcal{E}_\wedge(\mathfrak{b})$ respectively, where $n \leqslant m$, m is a fixed integer, $m \geqslant 2$, has been announced in the abstract: Ю. М. Важенин, Об inf-эндоморфизмах упорядоченных множеств, IX Всесоюзн. алгедп. коллокв., Резюме научных соодщений, Гомель, 1968, pp. 39-40.

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On the Baire system generated by a linear lattice of functions

by

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Suppose G is a linear space of real functions defined over a point set S such that if f is in G, then $|f|$ is in G; G is a linear lattice of functions over S. Also, suppose G contains the constant functions over S. Let $B_0(G) = G$ and for each ordinal number α, $0 < \alpha < \Omega$, let $B_\alpha(G)$ denote the collection of all pointwise limits of sequences from the collection $\sum_{\gamma<\alpha} B_\gamma(G)$. Sierpiński [1] and Tucker [2] have given necessary and sufficient conditions on a function f in order that it be in $B_1(G)$. These conditions are in terms of particular sequences of functions which converge in a uniform or a monotonic sense. Since for each ordinal $\alpha > 0$, the collection $\sum_{\gamma<\alpha} B_\gamma(G)$ is a linear lattice of real functions over S and it contains the constant function over S, these results may be extended to give necessary and sufficient conditions on a function f in order that it be in $B_\alpha(G)$. In this paper we characterize the collection $B_\alpha(G)$, $\alpha > 0$, in terms of an associated collection of Baire sets (Theorem 7) and give some relationships between these collections and the collections described by Hausdorff in [3].

Notation. If K is a lattice of functions, then K_u denotes the collection of all functions which are uniform limits of sequences from K, USK the collection of all functions which are limits of nonincreasing sequences from K and LSK the collection of all functions which are limits of nondecreasing sequences from K. The Baire system of functions generated by K is denoted by $B(K)$. If f is a bounded function, $\|f\|$ denotes the l.u.b. norm of f.

THEOREM 1. *If f is a bounded function in G, then f^2 is in G_u.*