The estimated extension theorem, homogeneous collections and skeletons, and their applications to the topological classification of linear metric spaces and convex sets

by

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Introduction. The results of the present paper generalize and strengthen the Anderson–Kadec Theorem on the homeomorphism of all separable infinite-dimensional Fréchet spaces (= locally convex complete linear metric spaces) and the Keller–Klee Theorem on the homeomorphism of all infinite-dimensional convex compact sets in Fréchet spaces. We obtain some criteria for the existence of homeomorphisms of pairs consisting either of a Fréchet space and its dense sigma-compact linear subspace, or of an infinite-dimensional convex compact set in a Fréchet space and a dense sigma-compact subset of that set. A typical result is the following.

(A) *Let X_∞ be a dense linear subspace of an infinite-dimensional Fréchet space X. Let X_∞ be a countable union of compact convex sets. Then the pair (X, X_∞) is homeomorphic either to the pair (l^2, l^2_F) or to the pair (l^2, l^2_σ), where l^2 denotes the Hilbert space of square-summable sequences, and*

$$l^2_F = \{x = (x(i)) \in l^2 \colon x(i) = 0 \text{ for all but finitely many } i\},$$

$$l^2_\sigma = \Big\{x = (x(i)) \in l^2 \colon \sum_{i=1}^{\infty} i^2 |x(i)|^2 < +\infty\Big\}.$$

From (A) we easily derive the following facts:

(B) *All $\aleph_0$-dimensional locally convex linear metric spaces are homeomorphic.*

(C) *All locally convex linear metric core spaces are homeomorphic.*

By an $\aleph_0$-dimensional linear space we mean a linear space which has exactly $\aleph_0$ linearly independent elements. By a linear metric core space we mean a space which is a countable union of infinite-dimensional compact convex sets.

Clearly (B) generalizes results due to Klee and Long [23] (for normed linear spaces), Bessaga [5], and a recent result of Raymond Wong [28] stating that the space l^2_F is homeomorphic to the space $\sum R$.

Here $\sum R$ is the subspace of $R^{\mathcal{N}}$ (= the infinite countable product of lines) consisting of real sequences whose all but finitely many coordinates are zero.

Another result generalizes the Keller–Klee Theorem as follows:

(D) *Let W_1 and W_2 be infinite-dimensional compact convex sets in Fréchet spaces and let W_1 and W_2 be centrally symmetric with respect to the origins of the spaces. Write*

$$\operatorname{rint} W_i = \{x \in W_i\colon\ tx \in W_i \text{ for some } t > 1\} \qquad (i = 1, 2).$$

Then the pair $(W_1, \operatorname{rint} W_1)$ *is homeomorphic to the pair* $(W_2, \operatorname{rint} W_2)$.

Combining (D) with a result due to R. D. Anderson [3], we get

(E) *Let B_w denote the unit ball of l^2 equipped with the weak topology and let $S = \{x \in l^2\colon \sum_{i=1}^{\infty} |x(i)|^2 = 1\}$ denote the unit sphere. Then the pair (B_w, S) is homeomorphic to the pair* $\big([-1;1]^{\mathcal{N}}, (-1;1)^{\mathcal{N}}\big)$.

Observe that the last result can be regarded as a strengthening of Anderson's Theorem on the homeomorphism of l^2 and $R^{\mathcal{N}}$, because the countable product $(-1;1)^{\mathcal{N}}$ of open intervals is obviously homeomorphic to $R^{\mathcal{N}}$, the weak and the norm topologies on S coincide, and S is homeomorphic to l^2.

The idea of the proof of the results (A) and (D) is the following: By the theorems of Anderson–Kadec and Keller–Klee, mentioned above, there exist homeomorphisms between the first elements of the considered pairs, i.e. between X and l^2 and between W_1 and W_2. These homeomorphisms are then corrected in order to carry the second elements of the pairs, accordingly. The correction is based on the fact that the second elements of the pairs are countable unions of sets belonging to a class $\mathcal{K}$ for which a version of the "Estimated Extension Theorem" is valid. Roughly speaking, the Estimated Extension Theorem states that every homeomorphism between sets belonging to $\mathcal{K}$ which is sufficiently close to the identity can be extended to a homeomorphism of the whole space onto itself which is close to the identity. An axiomatic treatment of this method of correction leads to the concept of a homogeneous collection $\mathcal{K}$ and a $\mathcal{K}$-skeleton. A typical example of a homogeneous collection in l^2 is the family of all compact subsets of l^2; a typical skeleton with respect to this collection is the sequence (W_n) where

$$W_n = \{x \in l^2\colon \sum_{i=1}^{\infty} i^2|x(i)|^2 \leqslant n\} \qquad (n = 1, 2, \ldots).$$

The present paper consists of seven sections. Section 1 is of a preliminary nature. Section 2 contains some (known) facts on Anderson's Z-sets in the Hilbert cube. Section 3 is devoted to the proof of the Estimated Extension Theorem (Theorem 3.1) for Z-sets in the Hilbert cube and for compact sets in Fréchet spaces. This result, which is a generalization of the Anderson [2] and Klee [22] extension theorems, seems to be of some independent interest. In Section 4 the concepts of homogeneous collections and skeletons are introduced and the basic result on "correcting of homeomorphisms" (Proposition 4.3) is proved. Applying this result, we show that Z-sets in the Hilbert cube and compact subsets of an infinite-dimensional Fréchet space X (resp. finite-dimensional compact subsets of X) are homogeneous collections. Also examples of skeletons with respect to these collections are presented. Section 5 contains an application of the homogeneous collections and skeletons technique to the topological classification of linear metric spaces. Here the results (A), (B) and (C) are proved. Section 6 is devoted to other applications of the apparatus developed in Section 4. In particular, we prove there the results (D) and (E) and complete the discussion (started in Section 5) concerning topological types of Cartesian products and countable weak products of certain sigma-compact linear metric spaces and convex sets. Section 7 is an addendum. It contains some remarks on the existence of homeomorphisms between linear metric spaces preserving linear gradations of the spaces, i.e. carrying a given increasing sequence of linear subspaces of one of the spaces onto a given sequence of linear subspaces of the other space.

Acknowledgement. The authors would like to express their gratitude to R. D. Anderson for a valuable discussion (during his visit at the Mathematical Institute of the Polish Academy of Sciences, in May 1968). The discussion was the starting point for the present paper. We are also indebted to A. Szankowski, who read the manuscript and made some useful remarks.

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After this paper had been submitted for publication, the authors learned that R. D. Anderson had introduced the concept of *fdcap*, which leads in fact to $\mathcal{EC}$-skeletons. He had independently proved our Theorem 5.1 (cf. [33], remark to Problem 29 and [34]).

A version of the Estimated Extension Theorem has been announced by W. Barit [29]. For further results in this direction, see Toruńczyk [32].

H. Toruńczyk [27] has considerably improved our Proposition 4.2 by showing that if a set L is a countable union of members of a homogeneous collection $\mathcal{K}$ and contains a $\mathcal{K}$-skeletoid K_∞, then L is homeomorphic to K_∞ by means of a homeomorphism of the whole space onto itself. From this result one can easily deduce Proposition 6.5, stated in our paper without any proof.

Applying Proposition 6.5 mentioned above, the authors [30] have obtained an alternative proof of the Kadec–Anderson Theorem. For further applications of Proposition 6.5 see [31] and [3].

1. Preliminaries

Metric spaces. All topological spaces considered in the present paper are metric spaces. We shall often use the term "space" instead of "metric space". The symbol $d(\cdot,\cdot)$ will denote the metric for a space X. For any point x in X and a non-empty subset A of X we put $d(x, A) = \inf_{a\in A} d(x, a)$. If X is a normed linear space (cf. [11], p. 24, for the definition), then the metric of X is defined by $d(x, y) = \|x-y\|$ for $x, y \in X$, where $\|\cdot\|$ denotes the norm of X. The symbol $\underset{m\in M}{P} X_m$ denotes the Cartesian product of topological spaces X_m $(m \in M)$. In the case where $M = \mathcal{N}$ is the set of positive integers [resp. M is the set $\{1, 2, \ldots, n\}$ for some positive integer n] and each X_m is a copy of a space X, we shall write $X^{\mathcal{N}}$ [resp. X^n] instead of $\underset{m\in\mathcal{N}}{P} X_m$ [resp. $\underset{1\leqslant m\leqslant n}{P} X_m$].

By a *pointed space* we mean a pair (X, x) consisting of a space X and a point x of X which is called the base point. If X is either a linear space or a convex set in a linear space which contains the point zero of the space, then (unless otherwise stated) we take zero as the base point. Let $\big((X_i, x_i)\big)_{i=1}^{\infty}$ be a sequence of pointed spaces. The set

$$\sum_{i\in\mathcal{N}}(X_i, x_i) = \big\{x = \big(x(i)\big)_{i=1}^{\infty} \in \underset{i\in\mathcal{N}}{P} X_i\colon\ x(i) = x_i \text{ for all but finitely many } i\big\}$$

equipped with the topology inheredited from $\underset{i\in\mathcal{N}}{P} X_i$ is called the *weak product* of the pointed spaces (X_i, x_i) $(i = 1, 2, \ldots)$. In the sequel, if there is no doubt what points are the base points, we shall write shortly $\sum X_i$, and $\sum X$ in the case where all X_i are copies of the same space X and all x_i are equal to a given point $x \in X$.

We shall consider the following special spaces:

I—the closed interval $[-1; 1]$,

R—the real line,

l^2 — the *sequence Hilbert space* of real sequences $x = \big(x(i)\big)$ such that

$$\|x\| = \Big(\sum_{i=1}^{\infty} |x(i)|^2\Big)^{1/2} < +\infty,$$

$Q = I^{\mathcal{N}}$ — the *Hilbert cube*; the metric for Q is defined by

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i}|x(i)-y(i)| \quad \text{for } x = \big(x(i)\big) \text{ and } y = \big(y(i)\big) \in Q,$$

$\pi_n\colon Q\to R$ are coordinate functions, i. e. $\pi_n(x) = x(n)$, and $v_n \in Q$ are unit vectors, i. e. $v_n(m) = 0$ for $m \neq n$ and $v_n(n) = 1$. Clearly, for each $x \in Q$, we have $x = \sum_{i=1}^{\infty}\pi_i(x)v_i$, the sum of the series being understood coordinatewise,

$Q_{\text{odd}} = \big\{x = \big(x(i)\big) \in Q\colon\ x(i) = 0 \text{ for even } i\big\}$,

$P = \big\{x = \big(x(i)\big) \in Q\colon\ -1 < x(i) < 1 \text{ for } i = 1, 2, \ldots\big\}$ — the *pseudo-interior* of Q,

$B = Q\setminus P$ — the *pseudoboundary of* Q.

Maps. By a *map* $f\colon X\to Y$ we mean a continuous function from a space X into a space Y. If A is a non-empty subset of a space X, then a map $f\colon A\to X$ is called an *embedding* if it is a homeomorphism between A and $f(A)$. For a map $f\colon A\to X$ (A being a subset of X) we define

$$d(f) = \sup_{x\in A} d\big(f(x), x\big).$$

If $\mathcal{K}$ is a family of subsets of a space X, then a map [resp. an embedding] $f\colon A\to X$ is said to be a $\mathcal{K}$-*map* [resp. a $\mathcal{K}$-*embedding*] if both A and $f(A)$ belong to $\mathcal{K}$.

Let A and A_1 be subsets of spaces X and X_1, respectively. We shall say that the pair (X, A) is homeomorphic to the pair (X_1, A_1) if there is a homeomorphism H of X onto X_1 such that $H(A) = A_1$; H is called a *pair homeomorphism*. If spaces X and X_1 [resp. pairs (X, A) and (X_1, A_1)] are homeomorphic, then we write $X\sim X_1$ [resp. $(X, A)\sim(X_1, A_1)$]. A homeomorphism of a space X onto itself is called an *autohomeomorphism*. The set of all autohomeomorphism of a space X will be denoted by $\operatorname{Auth} X$. If A is a subset of a space X, then the set of those autohomeomorphism of X which carry A onto itself will be denoted by $\operatorname{Auth}(X, A)$.

By a *zero function* of a non-empty (closed) subset A of a space X we mean a map $f\colon X\to R$ such that $0\leqslant f\leqslant 1$ and $f^{-1}(0) = A$. An *Urysohn function* of a pair (A, B) consisting of non-empty disjoint (closed) subsets of a space X is any zero function f of A such that $f^{-1}(1) = B$.

Semicovers and limited homeomorphisms. By a *semicover* in a space X we shall mean any non-empty collection $\mathcal{U}$ of open subsets of X. We let

$$\bigcup\mathcal{U} = \bigcup_{U\in\mathcal{U}} U; \qquad \operatorname{mesh}\mathcal{U} = \sup_{U\in\mathcal{U}}\ \sup_{x\in U;\, y\in U} d(x, y).$$

A *cover* of a subset A of X is any semicover $\mathcal{U}$ in X such that $\bigcup\mathcal{U}\supset A$.

Given $A \subset X$ and a semicover $\mathfrak{U}$. A map $f\colon A \to X$ is said to be *limited by* $\mathfrak{U}$ if the conditions $x \neq f(x)$ and $x \in A$ imply that there exists an U in $\mathfrak{U}$ such that both x and $f(x)$ belong to U. In other words, f is limited by $\mathfrak{U}$ if and only if

$$\operatorname{graph} f \subset \Delta_A \cup \bigcup_{U \in \mathfrak{U}} U \times U,$$

where the graph $f = \{(x, y) \in X \times X\colon y = f(x)\}$ and $\Delta_A = \{(x, y) \in X \times X\colon x = y \text{ and } x \in A\}$.

By $\operatorname{Auth}_{\mathfrak{U}} X$ [resp. $\operatorname{Auth}_{\mathfrak{U}}(X, A)$] we shall denote the set of all autohomeomorphism of a space X [belonging to $\operatorname{Auth}(X, A)$] which are limited by a semicover $\mathfrak{U}$. If $\mathfrak{U} = \{U\}$ is a one-member semicover, then we shall write simply $\operatorname{Auth}_U X$ and $\operatorname{Auth}_U(X, A)$ instead of $\operatorname{Auth}_{\{U\}} X$ and $\operatorname{Auth}_{\{U\}}(X, A)$.

In the case of compact spaces the limitation by covers is related to metric estimations:

1.1. PROPOSITION. *Let A be a compact subset of a space X and let $\mathfrak{U}$ be a cover of A. Then there exists a finite cover $\mathfrak{V}$ of A such that if a map $f\colon A \to X$ is limited by $\mathfrak{V}$, then it is limited by $\mathfrak{U}$. Moreover, there is a $c > 0$ such that every map $f\colon A \to X$, with $d(f) < c$, is limited by $\mathfrak{U}$.*

Proof. Since A is compact, there is a finite subcover $\mathfrak{V}$ of $\mathfrak{U}$ which is a cover of A. It is easily verified that $\mathfrak{V}$ has the desired property. To prove the second assertion define a metric $d^*(\cdot, \cdot)$ on the space $X \times X$ by

$$d^*\big((x, y), (x', y')\big) = d(x, x') + d(y, y'),$$

where $d(\cdot, \cdot)$ denotes the metric of X. Since the diagonal Δ_A is a compact subset of $X \times X$ (because A is compact), we have

$$c = \inf_{a \in A} d^*\big((a, a), X \times X \setminus \bigcup_{U \in \mathfrak{U}} U \times U\big) > 0.$$

The number c has the required property.

The concept of limited homeomorphisms has been introduced by Anderson and Bing [4]. They also proved the following important fact (cf. [4], Theorem 4.2).

1.2. ANDERSON–BING CRITERION. *If $(G_i)_{i=0}^{\infty}$ is a sequence of autohomeomorphisms of a complete metric space X and if $(\mathfrak{V}_i)_{i=0}^{\infty}$ is a sequence of covers of X such that $G_{i+1} G_i^{-1} \in \operatorname{Auth}_{\mathfrak{V}_i} X$, $\operatorname{mesh} \mathfrak{V}_i < 2^{-i}$ and $\operatorname{mesh} G_i^{-1}(\mathfrak{V}_i) < 2^{-i}$ for $i = 0, 1, 2, \ldots$, then the sequence $(G_i)_{i=0}^{\infty}$ pointwise converges to an autohomeomorphism of X.*

(If $G \in \operatorname{Auth} X$ and $\mathfrak{V}$ is a semicover, then the semicover $G(\mathfrak{V})$ is defined by $G(\mathfrak{V}) = \{G(V)\}_{V \in \mathfrak{V}}$.)

Linear metric spaces. A (real) *linear metric space* is a linear space (over reals) with a translation invariant metric $d(\cdot, \cdot)$ such that the function $(t, x) \to tx$ from $R \times X$ onto X is continuous in this metric. According to a result of Eidelheit and Mazur [13] (cf. also [10]), we may assume without loss of generality that the translation invariant metric is *monotone*, i.e., for each x in X, the function $t \to d(tx, 0)$ is monotone for $t > 0$. A linear metric space X is *locally convex* if there is a base for the topology of X consisting of convex sets. A locally convex complete linear metric space is called a *Fréchet space*. Observe that if X is a linear metric [locally convex] space, then $X^{\mathcal{N}}$, X^n for $n = 1, 2, \ldots$ and $\sum X$ are (with natural structures induced by X) linear metric [locally convex] spaces. If X is a Fréchet space, then $X^{\mathcal{N}}$ and X^n for $n = 1, 2, \ldots$ have the same property. The space $\sum X$ is never a Fréchet space. *Normed linear spaces* (cf. [11], p. 24 for the definition) are the most important examples of locally convex linear metric spaces. A complete normed linear space is called a *Banach space*. Banach spaces form a proper subclass of Fréchet spaces. The sequence Hilbert space l^2 defined above is a Banach space.

The basic result on topological classification of Fréchet spaces (cf. [1], [4], [18] and [19]) is the following

1.3. ANDERSON–KADEC THEOREM. *Every infinite-dimensional separable Fréchet space is homeomorphic to l^2.*

If A and B are subsets of a linear space X, then for each $x \in X$ and $t \in R$, we write

$$A + B = \{a + b\colon a \in A,\ b \in B\}, \quad A + x = A + \{x\}, \quad tA = \{ta\colon a \in A\}.$$

2. Anderson Z-sets

The Z-sets in the Hilbert cube have been introduced and studied by R. D. Anderson [2]. The following definition is due to Toruńczyk [26].

2.1. DEFINITION. A compact subset A of the Hilbert cube Q is called a *Z-set* if for every $\varepsilon > 0$, every positive integer n and every map $f\colon I^n \to Q$ there exists a map $g\colon I^n \to Q$ such that $g(I^n) \subset Q \setminus A$ and $d\big(f(x), g(x)\big) < \varepsilon$ for $x \in I^n$.

The class of all Z-sets in Q will be denoted by $\mathfrak{Z}$; by $\mathfrak{Z}^*$ we shall denote the class of all sets in Q which are compact subsets of the pseudo-interior P.

It is not difficult to deduce that the notion of a Z-set is a topological notion and does not depend on the choice of the particular metric d on Q. Also the proof of the next proposition is a matter of routine.

2.2. PROPOSITION. (a) *If* $Z_i \in \mathfrak{Z}$ *for* $i = 1, 2, \ldots, n$ $(n = 1, 2, \ldots)$, *the* $\bigcup_{i=1}^{n} Z_i \in \mathfrak{Z}$; *every closed subset of a Z-set is a Z-set;*

(b) *every compact subset of the pseudoboundary* $Q \setminus P$ *belongs to* $\mathfrak{Z}$;

(c) *every non-empty compact subset* A *of* Q *which is flat in infinitely many directions (i.e.* $\pi_i(A) = \{0\}$ *for infinitely many* i) *is a Z-set;*

(d) *let* $-1 \leqslant a_i < b_i \leqslant 1$ *for* $i = 1, 2, \ldots, m$ *and for some* $m \geqslant 1$. *Let*

$$Q_1 = \{x \in Q\colon\ a_i \leqslant \pi_i(x) \leqslant b_i \ (i = 1, 2, \ldots, m)\}.$$

Then for every Z-set A *in* Q *the set* $A \cap Q_1$ *is a Z-set with respect to the "Hilbert cube"* Q_1.

Proof. We leave to the reader the simple checking of (a), (b), and (c), and we shall prove here only (d).

Let $\varepsilon > 0$ and let $f\colon I^n \to Q_1$ be a map. Define $f_1\colon I^n \to Q_1$ by

$$f_1(x) = f(x) - c\sum_{i=1}^{n}\big(\pi_i f(x) + (b_i + a_i)/2\big)v_i \quad \text{for} \quad x \in I^n,$$

where $c \in (0;1)$ is chosen so small that $d\big(f(x), f_1(x)\big) < \varepsilon/2$ for $x \in I^n$. Since A is a Z-set with respect to Q, there exists a map $g\colon I^n \to Q$ such that $g(I^n) \subset Q \setminus A$ and $d\big(f_1(x), g(x)\big) < c_1$ for $x \in I^n$, where $c_1 \in (0; \varepsilon/2)$ is chosen so small that $a_i \leqslant \pi_i g(x) \leqslant b_i$ for $x \in I^n$ and for $i = 1, 2, \ldots, m$. (A c_1 satisfying the above inequalities exists because $a_1 + c(b_i - a_i)/2 \leqslant \pi_i f_1(x) \leqslant b_i - c(b_i - a_i)/2$ for $x \in I^n$ and for $i = 1, 2, \ldots, m$). Hence

$$g(I^n) \subset Q_1 \setminus A$$

and

$$d\big(f(x), g(x)\big) \leqslant d\big(f(x), f_1(x)\big) + d\big(f_1(x), g(x)\big) < \varepsilon \quad \text{for} \quad x \in I^n.$$

This completes the proof.

In the sequel we shall need the following significant result

2.3. ANDERSON EXTENSION THEOREM. *Every* $\mathfrak{Z}$-*embedding* $f\colon A \to Q$ *admits an extension* $F \in \operatorname{Auth} Q$. *Moreover if the set* $A \cap (Q \setminus P)$ *is compact,* $f(x) = x$ *for* $x \in A \cap (Q \setminus P)$, *and* $f(A \cap P) \subset P$, *then* f *admits an extension* $F \in \operatorname{Auth}(Q, P)$.

Proof. The first assertion is stated in [2], Corollary 10.3. To prove the second part, observe that by our assumption concerning f and by [2], Theorem 8.5, there exists an $H \in \operatorname{Auth} Q$ such that $H^{-1}(A \cup P) = P$. Then, by [2], Theorem 3.5, there exists a $G \in \operatorname{Auth}(Q, P)$ which is an extension of the $\mathfrak{Z}^*$-embedding $H^{-1}fh$, where h denotes the restriction of H to $H^{-1}(A)$. We put $F = HGH^{-1}$.

The next result follows from Anderson's argument in [2], Section 7 but is not explicitly stated there. For $k = 1, 2, \ldots$ we put

$$Q_{\mathrm{odd}}(k) = \{x \in Q\colon\ \pi_i(x) = 0 \ \text{for all even} \ i > k\}.$$

2.4. PROPOSITION. *If* A *is a non-empty Z-set and* $\varepsilon > 0$, *then there exists an* $H \in \operatorname{Auth}(Q, P)$ *such that* $d(H) < \varepsilon$ and $H(A) \subset Q_{\mathrm{odd}}(k)$ *for some index* k.

Proof. By a result of Anderson [2], Theorem 8.1, there exists an $F \in \operatorname{Auth}(Q, P)$ such that $F(A)$ is flat in infinitely many directions, say, $F(A) \subset \{x \in Q\colon\ \pi_i(x) = 0$ for all $i \not\equiv 3 \pmod 4\}$. Pick a $k > 1$ such that

$$\sum_{i \geqslant k} 2^{-i} < c/3, \tag{2.1}$$

where $c \in (0; \varepsilon/2)$ is chosen so small that

$$\text{if } d(x, y) < c, \text{ then } d\big(F^{-1}(x), F^{-1}(y)\big) < \varepsilon/2 \ \text{for} \ x, y \in Q. \tag{2.2}$$

Let $M = \bigcup_{i=1}^{\infty}\{j_i\}$ where $j_i = i$ for $i = 1, 2, \ldots, k$ and $j_i = 4i+1$ for $i > k$. We define two auxiliary maps on Q

$$T(x) = \sum_{n=1}^{\infty} \pi_{j_n}(x)\cdot v_n; \qquad G(x) = \sum_{i \notin M} \pi_i(x)\cdot v_i + \sum_{n=1}^{\infty} \pi_n F^{-1}T(x)\cdot v_{j_n}.$$

Finally, we let $H = GF$.

We shall verify that H has the required properties. First we observe that the map G is the Cartesian product of two autohomeomorphisms such that the first acts on $\underset{i \notin M}{P} I_i$, the second acts on $\underset{i \in M}{P} I_i$ and each preserves the corresponding pseudointerior (by I_i $(i = 1, 2, \ldots)$ we denote copies of the interval I). Hence $G \in \operatorname{Auth}(Q, P)$. Thus $H \in \operatorname{Auth}(Q, P)$. Further, if $i > k$ and i is an even integer, then $i \notin M$, whence $\pi_i G(x) = \pi_i(x)$ and $\pi_i H(x) = \pi_i F(x) = 0$ for every $x \in A$, i.e. $H(A) \subset Q_{\mathrm{odd}}(k)$.

By (2.1) and the definition of the sequence (j_n), we have $d(T) \leqslant c/3 < \varepsilon/2$. Hence, using (2.1) and (2.2), we get

$$\begin{aligned} d(H) = d(GF) &= \sup_{x \in Q} d\big(G(x), F^{-1}(x)\big) \\ &\leqslant \sup_{x \in Q} d\Big(\sum_{i \leqslant k} \pi_i G(x), \sum_{i \leqslant k} \pi_i F^{-1}(x)\Big) + 2\cdot\sum_{i > k} 2^{-i} \\ &\leqslant \sup_{x \in Q} d\big(F^{-1}(T(x)), F^{-1}(x)\big) + c < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This completes the proof.

The concept of Z-sets can naturally be extended to topological spaces homeomorphic to Q. Namely, if G is a homeomorphism from a space W onto Q, then a subset $A \subset W$ is called a Z-set in W if $G(A)$ is a Z-set in Q. Equivalently, A is a Z-set in W if A satisfies Definition 2.1, with Q replaced by W and d replaced by any metric on W. A large class of topological spaces homeomorphic to Q is described by the following result:

2.5. KELLER–KLEE THEOREM. *Every infinite-dimensional compact convex subset of the Hilbert space l^2 is homeomorphic to the Hilbert cube. Every infinite-dimensional compact convex subset of an arbitrary Fréchet space is affinely homeomorphic to a convex subset of l^2.*

The first statement of this theorem was obtained by Keller [20] in 1930, the second one has been observed by Klee [22].

We recall that convex sets W and W_1 are said to be affinely homeomorphic if there exists a homeomorphism $H\colon W \xrightarrow[\text{onto}]{} W_1$ which is an affine map, i.e., $H(cx+(1-c)y) = cH(x)+(1-c)H(y)$ for $x, y \in W$ and $0 \leqslant c \leqslant 1$.

Let W be a convex set. By the radial interior of W and the radial boundary of W we mean the sets

$$\operatorname{rint} W = \{w \in W\colon \text{ if } x \in W \text{ then } w+\varepsilon(w-x) \in W \text{ for some } \varepsilon > 0\},$$

and $\operatorname{rbd} W = W \setminus \operatorname{rint} W$, respectively.

It is easy to check that

(1) if $0 \in \operatorname{rint} W$ then $\operatorname{rint} W = \{cw\colon w \in W \text{ and } 0 \leqslant c < 1\}$.

This holds in particular when W is symmetric with respect to 0.

Now we are ready for the proof of the main result of the present section.

2.6. PROPOSITION. *Let K be an infinite-dimensional compact convex subset of a Fréchet space and let $0 \in \operatorname{rint} K$. Then, for each $a \in (0;1)$, the set aK is a Z-set in K and the pair (K, aK) is homeomorphic to the pair (Q, Q_{odd}).*

Proof. Given $\varepsilon > 0$. We shall check the condition appearing in Definition 2.1. By the second statement of the Keller–Klee Theorem 2.5, we may assume that $K \subset l^2$. Let $f\colon I^n \to K$ be a map. Replacing f, if necessary, by its suitable simplicial approximation, we may assume that

$$f(I^n) \subset (1-\varepsilon/2)\cdot(Y \cap K), \tag{2.3}$$

where Y is a finite-dimensional subspace of l^2. Since Y is finite-dimensional and K is compact, there exists a versor w which is orthogonal to Y and such that

$$(x|w) < \varepsilon/2 \quad \text{for all} \quad x \in K. \tag{2.4}$$

Here $(x|w)$ denotes the scalar product of x and w. Define on the set $(1-\varepsilon/2)\cdot(Y \cap K)$ a real-valued function φ by

$$\varphi(y) = \sup\{t\colon y+tw \in K\}.$$

It is easy to verify (using the fact that Y is finite-dimensional and K is compact and convex) that the function φ is continuous. Finally we define a map $g\colon I^n \to l^2$ by

$$g(x) = f(x)+\varphi f(x)\cdot w \quad \text{for} \quad x \in I^n.$$

By (2.3), the composed function φf and therefore g are well-defined. By the definition of φ, the values of g are on the boundary of the set $K \cap Y_1$ with respect to the space $Y_1 = \operatorname{span}(Y \cup \{w\})$. Hence $g(I^n) \subset K \setminus aK$. Since $f(I^n)$ is orthogonal to w, condition (2.4) implies that $\sup_{x\in I^n}\|f(x)-g(x)\| < \varepsilon$, i.e., according to Definition 2.1, aK is a Z-set with respect to K.

To obtain the second statement of the proposition, observe first that, by the Keller–Klee Theorem, there exist homeomorphisms $G\colon K \xrightarrow[\text{onto}]{} Q$ and $G_1\colon aK \xrightarrow[\text{onto}]{} Q_{\text{odd}}$. Since aK is a Z-set with respect to K and G is a homeomorphism, $G(aK)$ is a Z-set in Q. Thus the map $f\colon G(aK) \to Q$ defined by $f(x) = G_1 G^{-1}(x)$ for $x \in G(aK)$ is a $\mathfrak{Z}$-embedding and $f(G(aK)) = Q_{\text{odd}}$. By the Anderson Extension Theorem 2.3, there exists an $F \in \operatorname{Auth} Q$ which is an extension of f. Clearly, FG is the required homeomorphism of pairs. This completes the proof.

2.7. COROLLARY. $\mathfrak{Z}^* \subset \mathfrak{Z}$, *i.e., every compact subset of the pseudo-interior P is a Z-set in Q.*

Proof. If A is a non-empty compact subset of P, then

$$a_i = \sup_{x\in A} |\pi_i(x)| < 1 \quad \text{for} \quad i = 1, 2, \ldots$$

Thus there exist $f_i \in \operatorname{Auth} I$ such that $f_i([-a_i; a_i]) \subset [-1/2; 1/2]$ for all i. Define $F \in \operatorname{Auth} Q$ by $F(x) = (f_i\pi_i(x))_{i=1}^{\infty}$ for $x \in Q$. Then $F(A) \subset \frac{1}{2}Q$. Hence, by Proposition 2.6, $F(A)$ is a Z-set. This completes the proof.

3. Estimated extensions of homeomorphisms

Recall that a $\mathfrak{Z}$-embedding ($\mathfrak{Z}^*$-embedding) is any homeomorphism $f\colon K \to Q$ such that both the domain K and the range $f(K)$ are Z-sets in Q (such that K and $f(K)$ are compact subsets of the pseudointerior P of Q).

This section is devoted to the proof of the following crucial result.

3.1. ESTIMATED EXTENSION THEOREM. *Let $K \subset Q$ be a Z-set (a compact subset of P). Then for cover any $\mathfrak{V}$ of the set K, there exists a cover $\mathfrak{U}$*

of K such that every $\mathfrak{Z}$-embedding (every $\mathfrak{Z}^*$-embedding) $f\colon K \to Q$ which is limited by $\mathfrak{U}$ admits an extension $F \in \mathrm{Auth}_{\mathfrak{U}} Q$ (admits an extension $F \in \mathrm{Auth}_{\mathfrak{U}}(Q, P)$).

The first step of the proof is the following:

INTERPOLATION LEMMA. *If $f\colon K \to Q$ is a $\mathfrak{Z}$-embedding ($\mathfrak{Z}^*$-embedding), A_0, A_1 are disjoint compact subsets of K such that $f(A_1) \cap A_0 = \emptyset$, and $\mathrm{fix} f = \{x \in K\colon f(x) = x\}$, then there exist $\mathfrak{Z}$-embeddings ($\mathfrak{Z}^*$-embeddings) f_0 and f_1 such that*

$$(3.1) \qquad d(f_i) \leqslant 2d(f) \quad \text{for} \quad i = 1, 2,$$

$$(3.2) \qquad f_1(x) = x \text{ for } x \in A_0 \cup \mathrm{fix} f \quad \text{and} \quad f_1(x) = f(x) \text{ for } x \in A_1,$$

$$(3.3) \qquad f_0 f_1 = f.$$

Proof. By Proposition 2.4, the general case can be reduced to that of $K \cup f(K)$ being flat in infinitely many directions, and, say $K \cup f(K) \subset Q_{\mathrm{odd}}(n)$, where $n \geqslant 1$ is chosen so that $\sum_{i>n} 2^{-i} \leqslant \frac{1}{2} d(f)$. Let $\varphi\colon K \to R$ be an Urysohn function of the pair (A_0, A_1), and let

$$\lambda(x) = \tfrac{1}{8}\varphi(x)\big(1 - \varphi(x)\big)\cdot d\big(x, f(x)\big),$$

$$f_1(x) = x + \varphi(x)\big(f(x) - x\big) + \lambda(x)\Big(v_{4n} + \sum_{j=1}^{\infty} \pi_j f(x) \cdot v_{4(n+j)}\Big).$$

Condition (3.2) follows directly from the last formulas and from the fact that φ is an Urysohn function of (A_0, A_1) and that $\lambda(x) = 0$ for $x \in A_0 \cup A_1 \cup \mathrm{fix} f$.

We shall show that f_1 is an embedding. Let us check that $x \neq x'$ implies $f_1(x) \neq f_1(x')$. If x, x' are both in $A_0 \cup \mathrm{fix} f$ or x, x' are both in $A_1 \cup \mathrm{fix} f$, this is obvious. If $x \in A_0$ and $x' \in A_1$, this follows from the assumption $f(A_1) \cap A_0 = \emptyset$. Finally, for $x, x' \in K \setminus (A_0 \cup A_1 \cup \mathrm{fix} f)$, this is due to the contribution of the term $\lambda(x) \cdot \sum_{j=1}^{\infty} \pi_j f(x) \cdot v_{4(n+j)}$, which is orthogonal to the remaining terms appearing in the formula for f_1 (because $K \cup f(K) \subset Q_{\mathrm{odd}}(n)$).

Now we define f_0 to satisfy (3.3), i.e. by letting $f_0 = f f_1^{-1}$.

Since there exist infinitely many integers non-divisible by 4, we conclude that the set $f_1(K)$ is flat in infinitely many directions, and therefore by Proposition 2.2, $f_1(K)$ is a Z-set (also one easily checks that $f_1(K) \subset P$, provided that $K \cup f(K) \subset P$). Hence the maps $f_1\colon K \to Q$ and $f_0\colon f_1(K) \to Q$ are $\mathfrak{Z}$-embeddings (are $\mathfrak{Z}^*$-embeddings).

Finally, using the formulas defining λ, f_1 and f_0, we easily check that $d(f_i) \leqslant 1 \cdot d(f)$. This is better than estimations (3.1) with constant 2. However, in the general situation (when not assuming $K \cup f(K) \subset Q_{\mathrm{odd}}(n)$), we have to pass through homeomorphisms h and h^{-1} of Proposition 2.4 and therefore the estimations with the constant 1 fail to be true, but any constant greater than 1 will do. This completes the proof of the lemma.

Suppose that $\varepsilon > 0$, $\delta > 0$, U is an open subset of Q and $\mathfrak{K}$ denotes one of the collections: $\mathfrak{Z}$, $\mathfrak{Z}^*$. We shall write

$$U(\varepsilon) = \{x \in Q\colon d(x, Q \setminus U) > \varepsilon\},$$

and

$$\Gamma_{\mathfrak{K}}(U, \varepsilon, \delta) = \{f\colon f \text{ is a } \mathfrak{K}\text{-embedding}, f(x) = x \text{ for } x \notin U(\varepsilon), \text{ and } d(f) \leqslant \delta\},$$

The symbol $\Gamma(U, \varepsilon, \delta)$ will denote either of the sets $\Gamma_{\mathfrak{Z}}(U, \varepsilon, \delta)$ and $\Gamma_{\mathfrak{Z}^*}(U, \varepsilon, \delta)$; that is, the expression "$f \in \Gamma(U, \varepsilon, \delta)$" will stand for "$f \in \Gamma_{\mathfrak{Z}}(U, \varepsilon, \delta)$ (resp. $f \in \Gamma_{\mathfrak{Z}^*}(U, \varepsilon, \delta)$)".

3.2. PROPOSITION. *For every open set $U \subset Q$ and $\varepsilon > 0$, there is a $\delta = \delta(U, \varepsilon) > 0$ such that every map $f \in \Gamma(U, \varepsilon, \delta)$ has an extension $F \in \mathrm{Auth}_U Q$ (resp. $F \in \mathrm{Auth}_U(Q, P)$).*

Proof. 1° Begin with the case where U is a *cubical neighbourhood*, i.e., a set of type

$$\{x \in Q\colon a_i < \pi_i(x) < b_i \text{ for } i = 1, 2, \ldots, m\},$$

where m is a positive integer, and for each $i \leqslant m$, the intersection of open intervals $(a_i; b_i) \cap (-1; 1)$ is non-empty.

Let $\delta = \varepsilon$ and let $f\colon K \to Q$, $f \in \Gamma(U, \varepsilon, \delta)$. The boundary ∂U is obviously a finite union of endslices (faces) of the small Hilbert cube $\mathrm{cl}\, U$, and therefore, by Proposition 2.2, the set $\mathrm{cl}\, U \cap K \cup f(\mathrm{cl}\, U \cap K) \cup \partial U$ is a Z-set with respect to $\mathrm{cl}\, U$. Hence the map $g\colon K \cap U \cup \partial U \to \mathrm{cl}\, U$ given by: $g(x) = f(x)$ for $x \in K \cap U$ and $g(x) = x$ for $x \in \partial U$ satisfies the assumption of the Anderson Extension Theorem, and therefore g has an extension $G \in \mathrm{Auth}(\mathrm{cl}\, U)$ (resp. $G \in \mathrm{Auth}(\mathrm{cl}\, U, U \cap P)$). Extending G as the identity map beyond ∂U, we obtain the required F.

2° Suppose that the proposition has been proved for every open set V_0 which is a union of $n-1$ cubical neighbourhoods, and suppose that $U = V_0 \cup V_1$, where V_1 is a cubical neighbourhood. Let $f\colon K \to Q$, $f \in \Gamma(U, \varepsilon, \delta)$, with

$$(3.4) \qquad \delta = \min\big(\varepsilon/4, \tfrac{1}{2}\delta(V_0, \varepsilon/4)\big).$$

Write

$$(3.5) \qquad A_0 = K \cap \mathrm{cl}\, V_0(\varepsilon), \quad A_1 = K \cap \mathrm{cl}\, V_1 \cap \big(Q \setminus V_0(\varepsilon/2)\big).$$

Since $d(f) \leqslant \delta \leqslant \varepsilon/4$, we have $f(A_0) \subset f\big(\mathrm{cl}\, V_0(\varepsilon)\big) \subset V_0(\varepsilon/2)$. Thus $f(A_0) \cap A_1 \subset f(A_0) \cap \big(Q \setminus V_0(\varepsilon/2)\big) = \emptyset$, and we are in the position of Interpolation Lemma. Let $f = f_0 f_1$ be the decomposition of Interpolation Lemma.

By (3.1)–(3.5), we have $f_i \in \Gamma(V_i, \varepsilon/4, 2\delta)$ for $i = 0, 1$, i.e., according to (3.4), we get

$$f_0 \in \Gamma\big(V_0, \varepsilon/4, \delta(V_0, \varepsilon/4)\big), \quad f_1 \in \Gamma(V_1, \varepsilon/4, \varepsilon/4). \tag{3.6}$$

Now, using 1° (in the case $i = 1$), and the inductive hypothesis (in the case $i = 0$), we infer that the map f_i admits an extension $F_i \in \mathrm{Auth}_{V_i} Q$ (resp. $F_i \in \mathrm{Auth}_{V_i}(Q, P)$), for $i = 0, 1$. Now $F = F_0 F_1$ is the required extension of f.

3° Let U be an arbitrary open set. For every $x \in \mathrm{cl}\, U(\varepsilon/2)$ let V_x be a cubical neighbourhood of x with $V_x \subset U$. Sellect a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}$ for the set $\mathrm{cl}\, U(\varepsilon/2)$, and let $V = \bigcup_{i=1}^{n} V_{x_i}$, a finite union of cubical neighbourhoods. We obviously have

$$\Gamma(V, \varepsilon/2, \delta) \supset \Gamma(U, \varepsilon, \delta).$$

Hence, it is enough to take $\delta(U, \varepsilon) = \delta(V, \varepsilon/2)$. This completes the proof of the proposition.

Recall that for any semicover $\mathfrak{U}$ the symbol $\bigcup \mathfrak{U}$ denotes the union of all the members of $\mathfrak{U}$. We have

3.3. Proposition. *For every finite semicover $\mathfrak{U}$ and $\varepsilon > 0$, there is a $\delta = \delta(\mathfrak{U}, \varepsilon) > 0$ such that every $f \in \Gamma(\bigcup \mathfrak{U}, \varepsilon, \delta)$ has an extension $F \in \mathrm{Auth}_{\mathfrak{U}} Q$ (resp. $F \in \mathrm{Auth}_{\mathfrak{U}}(Q, P)$).*

Proof. By the multiplicity of the semicover $\mathfrak{U}$ (briefly: mult $\mathfrak{U}$) we mean the maximal number of members in $\mathfrak{U}$ which have a non-empty intersection. Our proof will be inductive with respect to mult $\mathfrak{U}$.

1° Suppose that $\mathrm{mult}\, \mathfrak{U} = 1$, $\mathfrak{U} = \{U_1, \ldots, U_n\}$. Applying Proposition 3.2 to each U_i separately and taking as F the product of the homeomorphisms obtained, we get the assertion.

2° Suppose that Proposition 3.3 has been proved for all semicovers of multiplicity less than k and let $\mathfrak{U} = \{U_1, \ldots, U_n\}$ be of multiplicity k. Let

$V =$ the set of all points in Q which are covered by k members of $\mathfrak{U}$.

Then V is the union of $\binom{n}{k}$ pairwise disjoint sets

$$V_\alpha = U_{i_1} \cap \ldots \cap U_{i_k}$$

(some of them may be empty) corresponding to all k-element subsets $\alpha = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$. Let, for each $i \leqslant n$,

$$W_i = U_i(\varepsilon/2) \cap \big(V \setminus \mathrm{cl}\, V(\varepsilon/2)\big).$$

It is easily seen that

$$\mathrm{mult}\{W_1, \ldots, W_n\} < k$$

Let

$$\mathfrak{W} = \{W_1, \ldots, W_n\}, \quad \mathfrak{V} = \{V_\alpha\}_{\alpha \subset \{1, \ldots, n\}} \tag{3.7}$$

and let

$$\delta = \min\big(\varepsilon/8, \tfrac{1}{2}\delta(\mathfrak{W}, \varepsilon/8), \tfrac{1}{2}\delta(\mathfrak{V}, \varepsilon/8)\big). \tag{3.8}$$

Suppose that $f\colon K \to Q$, $f \in \Gamma(\mathfrak{U}, \varepsilon, \delta)$ with the above δ. Let us define

$$A_0 = \mathrm{cl}\, V(\varepsilon/2) \cap K, \quad A_1 = K \setminus V(\varepsilon/4). \tag{3.9}$$

Since $\delta \leqslant \varepsilon/8$, we easily check that $f(A_1) \cap A_0 = \emptyset$, and therefore there exists a decomposition $f = f_0 f_1$ of the Interpolation Lemma.

By (3.1), $d(f_0) \leqslant 2\delta$, whence, according to (3.8) we obtain

$$d(f_0) \leqslant \delta(\mathfrak{V}, \varepsilon/8). \tag{3.10}$$

We have $A_1 = K \setminus V(\varepsilon/4) \supset K \setminus V(\varepsilon/8)$; hence, by (3.2), $f_0(x) = x$ for $x \in K \setminus V(\varepsilon/8)$. This together with (3.10) gives

$$f_0 \in \Gamma\big(V, \varepsilon/8, \delta(\mathfrak{V}, \varepsilon/8)\big). \tag{3.11}$$

Using the same argument as in the proof of (3.10), we conclude that

$$d(f_1) \leqslant \delta(\mathfrak{W}, \varepsilon/8). \tag{3.12}$$

Examining the expressions for W_i ($i = 1, \ldots, n$), and δ, we easily check that

$$W(\varepsilon/\delta) \cap K \subset A_1 \cup \mathrm{fix} f, \quad \text{where} \quad W = \bigcup \mathfrak{W}$$

and therefore, by (3.2), $f_1(x) = x$ for $x \in W(\varepsilon/8)$. This together with (3.12) gives

$$f_1 \in \Gamma\big(\bigcup \mathfrak{W}, \varepsilon/8, \delta(\bigcup \mathfrak{W}, \varepsilon/8)\big). \tag{3.13}$$

By (3.7), $\mathrm{mult}\, \mathfrak{W} < k$ and $\mathrm{mult}\, \mathfrak{V} = 1$. Hence, statement 1° together with (3.11) and the inductive hypothesis together with (3.13) give the existence of homeomorphisms $F_0 \in \mathrm{Auth}_{\mathfrak{V}} Q$ and $F_1 \in \mathrm{Auth}_{\mathfrak{W}} Q$ (resp. $F_0 \in \mathrm{Auth}_{\mathfrak{V}}(Q, P)$ and $F_1 \in \mathrm{Auth}_{\mathfrak{W}}(Q, P)$) which are extensions of f_0 and f_1, respectively. Now $F = F_0 F_1$ is an extension of the embedding f, and $F \in \mathrm{Auth}\, Q$ (resp. $F \in \mathrm{Auth}(Q, P)$). It remains to check that F is limited by $\mathfrak{U}$.

Suppose that $x \in Q$. Since F_1 is limited by $\mathfrak{W}$ and, obviously $\mathfrak{W}$ is a refinement of $\mathfrak{V}$, we infer that F_1 is limited by $\mathfrak{U}$, i.e. there is an index $j \leqslant n$ such that both x and $F_1(x)$ are in U_j. Since F_0 is limited by $\mathfrak{V}$, there exists a subset $\alpha \subset \{1, \ldots, n\}$, say $\alpha = \{i_1, \ldots, i_k\}$ such that both $F_1(x)$ and $F_0 F_1(x) = F(x)$ are in V_α. But from $\mathrm{mult}\, U = k$ we conclude that $j \in \alpha$, whence both x and $F(x)$ are in U_j (because $U_j \supset U_{i_1} \cap \ldots \cap U_{i_k} = V_\alpha$). Hence F is limited by $\mathfrak{U}$. This completes the proof of the proposition.

Proof of Theorem 3.1. Assume that $K \in \mathfrak{Z}$ (resp. $K \in \mathfrak{Z}^*$) and $\mathcal{V}$ is a cover of K. Since K is compact, we may assume without loss of generality that cover the $\mathcal{V}$ is finite. Let $\varepsilon = d(K, Q \setminus \bigcup \mathcal{V})$ (i.e. $\varepsilon = \inf\{d(x, y): x \in K, y \in Q \setminus \bigcup \mathcal{V}\}$), $\delta = \delta(\mathcal{V}, \varepsilon)$. Then, by the definition of the set $\Gamma(\ldots)$, every $\mathfrak{Z}$-embedding (every $\mathfrak{Z}^*$-embedding) $f: K \to Q$ such that $d(f) \leqslant \delta$ belongs to $\Gamma(\bigcup \mathcal{V}, \varepsilon, \delta)$, and therefore, by Proposition 3.3, f has an extension in $\mathrm{Auth}_{\mathcal{V}} Q$ (in $\mathrm{Auth}_{\mathcal{V}}(Q, P)$). Letting $\mathcal{U}$ be any cover of K with mesh $\mathcal{U} < \delta$, we complete the proof.

The following result of H. Toruńczyk [27] generalizes the $\mathfrak{Z}^*$-statement of the Estimated Extension Theorem:

3.4. Theorem. *Given a closed subset $K \subset Q$. Then for every cover $\mathcal{V}$ of the set K, there is a cover $\mathcal{U}$ of K such that every embedding $f: K \to Q$ limited by $\mathcal{U}$ and such that $f(K \cap P) \subset P$, $f(K \cap (Q \setminus P)) \subset Q \setminus P$, admits an extension $F \in \mathrm{Auth}_{\mathcal{V}}(Q, P)$.*

4. Homogeneous collections and skeletons

4.1. Definitions. A non empty family $\mathcal{K}$ of closed subsets of a topological space X is called a *homogeneous collection* if it satisfies the following conditions:

(4.1) $$K \in \mathcal{K} \text{ and } H \in \mathrm{Auth}\, X \text{ imply } H(K) \in \mathcal{K},$$

and

(4.2) if $\mathcal{V}$ is a cover of a set $K \in \mathcal{K}$, then there is a cover $\mathcal{U}$ of K such that each $\mathcal{K}$-embedding $h: K \to X$ which is limited by $\mathcal{U}$ admits an extension $H \in \mathrm{Auth}_{\mathcal{V}} X$.

Let $\mathcal{K}$ be a homogeneous collection in a space X. An increasing sequence (K_n) of sets of $\mathcal{K}$ is called a $\mathcal{K}$-*skeleton* if it satisfies the following condition:

(4.3) if A is a set in $\mathcal{K}$ and $\mathcal{U}$ is its cover, then for each index m there exists an index $n = n(m, \mathcal{U}, A)$ and a $\mathcal{K}$-embedding $h: A \to X$ which is limited by $\mathcal{U}$ and such that $h(A) \subset K_n$; $h(x) = x$ for $x \in A \cap K_m$.

We shall use the notation $K_\infty = \bigcup_{n=1}^{\infty} K_n$; the set K_∞ will be called a $\mathcal{K}$-skeletoid.

The next proposition shows that the concept of a $\mathcal{K}$-skeleton is invariant under the autohomeomorphisms of the space.

4.2. Proposition. *If (K_n) is a $\mathcal{K}$-skeleton in a space X and if $H \in \mathrm{Auth}\, X$, then $(H(K_n))$ is also a $\mathcal{K}$-skeleton.*

Proof. By (4.1) the sets $H(K_n)$ belong to $\mathcal{K}$ for all n. Since $K_1 \subset K_2 \subset \ldots$, we infer that $H(K_1) \subset H(K_2) \subset \ldots$

To prove that the sequence $(H(K_n))$ satisfies (4.3) pick an $L \in \mathcal{K}$ and a cover $\mathcal{V}$ of L. Put $A = H^{-1}(L)$ and $\mathcal{U} = H^{-1}(\mathcal{V})$. Since (K_n) is a $\mathcal{K}$-skeleton, for each index m there exist a $\mathcal{K}$-embedding $h: A \to X$ and an index $n = n(m, \mathcal{U}, A)$ with the property described in (4.3). We put $g(x) = H(h(H^{-1}(x)))$ for $x \in L$ and $n(m, \mathcal{V}, L) = n(m, \mathcal{U}, A)$. Then clearly $g: L \to X$ is a $\mathcal{K}$-embedding, $g(L) \subset H(K_n)$ and $g(x) = x$ for $x \in L \cap H(K_m)$. This completes the proof.

Our next result is one of the main technical tools in the present paper.

4.3. Proposition. *Let $\mathcal{K}$ be a homogeneous collection in a complete metric space X and let (K_n) and (L_n) be $\mathcal{K}$-skeletons. Then there exists a $G \in \mathrm{Auth}\, X$ such that $G(K_\infty) = L_\infty$.*

Proof. We shall construct an increasing sequence of indices $(n(i))_{i=0}^{\infty}$, a sequence of covers $(\mathcal{V}_i)_{i=0}^{\infty}$ and a sequence of autohomeomorphisms $(G_i)_{i=0}^{\infty}$ such that

(4.4) $$G_{i+1} G_i^{-1} \in \mathrm{Auth}_{\mathcal{V}_i} X,$$

(4.5) $$\mathrm{mesh}\, \mathcal{V}_i < 2^{-i}; \quad \mathrm{mesh}\, G_i^{-1}(\mathcal{V}_i) < 2^{-i},$$

(4.6) $$G_{2k-1}(\widetilde{K}_{2k-1}) \subset \widetilde{L}_{2k-1} \subset G_{2k}(\widetilde{K}_{2k}),$$

(4.7) $$\text{if } G_{2k}(x) \in \widetilde{L}_{2k-1}, \text{ then } G_{2k+1}(x) = G_{2k}(x),$$

and

(4.8) $$\text{if } x \in \widetilde{K}_{2k-2}, \text{ then } G_{2k}(x) = G_{2k-1}(x),$$

where $\widetilde{K}_k = K_{n(k)}$ and $\widetilde{L}_k = L_{n(k)}$ for each index k.

Assume that we have done this. Then we put

$$G = \lim_i G_i.$$

Conditions (4.4) and (4.5) together with the Anderson–Bing Criterion 1.2 imply that $G \in \mathrm{Auth}\, X$. We shall show that $G(K_\infty) = L_\infty$. If $y \in L_\infty$, then $y \in \widetilde{L}_{2k-1}$ for some $k > 0$. Thus, by (4.6), there is an x in $\widetilde{K}_{2k}$ such that $y = G_{2k}(x)$. Hence, by (4.7), $y = G_{2k}(x) = G_{2k+1}(x)$. Since $x \in \widetilde{K}_{2k}$, condition (4.8) implies that $G_{2k+1}(x) = G_{2k+2}(x)$. Since $\widetilde{L}_{2r-1} \supset \widetilde{L}_{2k-1}$ for $r > k$, a simple induction leads to the conclusion that $y = G_i(x)$ for all $i \geqslant 2k$. Thus $y = G(x) \in G(\widetilde{K}_{2k})$. This proves the inclusion $G(K_\infty) \supset L_\infty$.

Conversely, if $x \in K_\infty$, then $x \in \widetilde{K}_{2k-2} \subset \widetilde{K}_{2k}$ for some $k > 0$. Thus, by (4.6), $y = G_{2k-1}(x) \in \widetilde{L}_{2k-1}$. By (4.8) and (4.7), $G_{2k-1}(x) = G_{2k}(x) = G_{2k+1}(x)$.

Again by (4.8), $y = G_{2k+1}(x) = G_{2k+2}(x)$. Since $\widetilde{K}_{2k-2} \subset \widetilde{K}_{2r-2}$ for $k < r$, a simple induction leads to the conclusion that $y = G_i(x)$ for all $i \geqslant 2k-1$. Therefore $G(x) = y \in \widetilde{L}_{2k-1}$. This shows the inclusion $G(K_\infty) \subset L_\infty$. Thus $G(K_\infty) = L_\infty$.

We define the sequences $\big(n(i)\big)_{i=0}^{\infty}$, $(G_i)_{i=0}^{\infty}$ and $(\mathcal{V}_i)_{i=0}^{\infty}$ inductively. We put $n(0) = 1$, $G_0 =$ the identity, and $\mathcal{V}_0 = X$. Suppose that for some even integer $2j \geqslant 0$ and for $0 \leqslant i \leqslant 2j$ the indices $n(i)$, autohomeomorphisms G_i and covers $\mathcal{V}_i$ have been defined to satisfy conditions (4.4)–(4.8). By (4.1), we have $G_{2j}(\widetilde{K}_{2j}) \in \mathcal{K}$. Thus, by (4.2), there exists a cover $\mathcal{U}$ of the set $A = G_{2j}(\widetilde{K}_{2j})$ such that every $\mathcal{K}$-embedding of A which is limited by $\mathcal{U}$ can be extended to an element of $\mathrm{Auth}_{\mathcal{V}_{2j}} X$. Now applying (4.3) to the $\mathcal{K}$-skeleton (L_n) we choose, for the triple $\big(n(2j), \mathcal{U}, A\big)$, an index $n(2j+1) = n\big(n(2j), \mathcal{U}, A\big)$ and a $\mathcal{K}$-embedding $h\colon A \to X$ limited by $\mathcal{U}$ in such a way that

$$h(A) \subset \widetilde{L}_{2j+1} \quad \text{and} \quad h(y) = y \quad \text{for} \quad y \in A \cap \widetilde{L}_{2j}. \tag{4.9}$$

By the definition of the cover $\mathcal{U}$, there exists in $\mathrm{Auth}_{\mathcal{V}_{2j}} X$ an extension of h, say H. We put $G_{2j+1} = HG_{2j}$ and we define the cover $\mathcal{V}_{2j+1}$ so small that it satisfies together with G_{2j+1} condition (4.5) for $i = 2j+1$.

Since (K_n) is a $\mathcal{K}$-skeleton, Proposition 4.2 implies that the sequence $\big(G_{2j+1}(K_n)\big)$ is also a $\mathcal{K}$-skeleton. Pick, according to (4.2), the cover $\widetilde{\mathcal{U}}$ of $\widetilde{L}_{2j+1}$ so that each $\mathcal{K}$-embedding of $\widetilde{L}_{2j+1}$ limited by $\widetilde{\mathcal{U}}$ can be extended to an element of $\mathrm{Auth}_{\mathcal{V}_{2j+1}} X$. Applying (4.3) to the $\mathcal{K}$-skeleton $\big(G_{2j+1}(K_n)\big)$ and to the triple $\big(n(2j+1), \widetilde{\mathcal{U}}, \widetilde{L}_{2j+1}\big)$ we find an index $n(2j+2) = n\big(n(2j+1), \widetilde{\mathcal{U}}, \widetilde{L}_{2j+1}\big)$ and a $\mathcal{K}$-embedding $\tilde{h}\colon \widetilde{L}_{2j+1} \to X$ limited by $\widetilde{\mathcal{U}}$ such that

$$\tilde{h}(\widetilde{L}_{2j+1}) \subset G_{2j+1}(\widetilde{K}_{2j+2}) \quad \text{and} \quad \tilde{h}(y) = y \quad \text{for} \quad y \in \widetilde{L}_{2j+1} \cap G_{2j+1}(\widetilde{K}_{2j+1}). \tag{4.10}$$

By the definition of the cover $\widetilde{\mathcal{U}}$, there exists in $\mathrm{Auth}_{\mathcal{V}_{2j+1}} X$ an extension of $\tilde{h}$, say $\widetilde{H}$. We put $G_{2j+2} = \widetilde{H}^{-1} G_{2j+1}$ and we define the cover $\mathcal{V}_{2j+2}$ so small that it satisfies together with G_{2j+2} condition (4.5) for $i = 2j+2$.

To complete the induction we have to show that the "extended" sequences $\big((n(i)\big)_{i \leqslant 2j+2}$, $(G_i)_{i \leqslant 2j+2}$ and $(\mathcal{V}_i)_{i \leqslant 2j+2}$ satisfy conditions (4.4)–(4.8). Clearly $G_{2j+1}G_{2j}^{-1} = H \in \mathrm{Auth}_{\mathcal{V}_{2j}} X$, and $G_{2j+2}G_{2j+1}^{-1} = \widetilde{H}^{-1} \in \mathrm{Auth}_{\mathcal{V}_{2j+1}} X$, because $\widetilde{H} \in \mathrm{Auth}_{\mathcal{V}_{2j+1}} X$. This fact, together with the inductive hypothesis, proves condition (4.4). Condition (4.5) for $i = 2j+1$ and $i = 2j+2$ follows immediately from the construction of the covers $\mathcal{V}_{2j+1}$ and $\mathcal{V}_{2j+2}$. Next we check (4.6) for $k = j+1$. By (4.9), we get

$$G_{2j+1}(\widetilde{K}_{2j}) = H\big(G_{2j}(\widetilde{K}_{2j})\big) = h(A) \subset \widetilde{L}_{2j+1}.$$

On the other hand, by (4.10), we obtain

$$\widetilde{H}(\widetilde{L}_{2j+1}) = \tilde{h}(L_{2j+1}) \subset G_{2j+1}(\widetilde{K}_{2j+2}).$$

Thus

$$\widetilde{L}_{2j+1} \subset \widetilde{H}^{-1} G_{2j+1}(\widetilde{K}_{2j+2}) = G_{2j+2}(\widetilde{K}_{2j+2}).$$

This proves (4.6) for $k = j+1$.

If $x \in G_{2j}^{-1}(\widetilde{L}_{2j-1})$ and $j > 0$, then $y = G_{2j}(x) \in \widetilde{L}_{2j-1}$. By the right-hand side inclusion (4.6) for $k = j$ (which is assumed to hold by the inductive hypothesis), we infer that $y \in G_{2j}(\widetilde{K}_{2j})$. Thus $y \in A \cap \widetilde{L}_{2j-1} \subset A \cap \widetilde{L}_{2j}$. Hence, by (4.9),

$$G_{2j}(x) = y = h(y) = HG_{2j}(x) = G_{2j+1}(x).$$

This proves (4.7) for $j = k$.

Finally if $x \in \widetilde{K}_{2j}$, then by the left-hand side inclusion (4.6) for $k = j+1$ (which has been checked above), we infer that $y = G_{2j+1}(x) \in \widetilde{L}_{2j+1}$. Therefore $y \in G_{2j}(\widetilde{K}_{2j+1}) \cap \widetilde{L}_{2j+1}$, because $\widetilde{K}_{2j} \subset \widetilde{K}_{2j+1}$. Thus, by (4.10), $G_{2j+1}(x) = y = \tilde{h}(y) = \widetilde{H}(y)$. Hence

$$G_{2j+2}(x) = \widetilde{H}^{-1} G_{2j+1}(x) = y = G_{2j+1}(x).$$

This proves (4.8) for $k = j+1$ and completes the induction.

Remark. It follows from the analysis of the above proof that, for every cover $\mathcal{V}$ of X, the autohomeomorphism G appearing in the statement of Proposition 4.2 may be chosen in $\mathrm{Auth}_{\mathcal{V}} X$.

In some special situations, to establish that a sequence (K_n) is a $\mathcal{K}$-skeleton, it is enough to construct a $\mathcal{K}$-map rather than a $\mathcal{K}$-embedding satisfying the conditions of (4.3). More precisely, we have

4.4. Proposition. *Suppose that $\mathcal{K}$ is a homogeneous collection in a metric space X and $\mathcal{K}$ is hereditary with respect to closed subsets, that is $B = \mathrm{cl}\, B \subset A \in \mathcal{K}$ implies $B \in \mathcal{K}$. Then every increasing sequence (K_n) of sets of $\mathcal{K}$ such that*

(4.11) *each pair (K_{n+1}, K_n) is homeomorphic to the pair (Q, Q_{odd})*

and

(4.12) *if $A \in \mathcal{K}$ and $\mathcal{V}$ is a cover of the set A, then for each index m there exists an index $n = n(m, \mathcal{V}, A)$ and a $\mathcal{K}$-map $f\colon A \to X$ which is limited by $\mathcal{V}$ and such that $f(A) \subset K_{n-1}$, $f(x) = x$ for $x \in A \cap K_m$,*

is a $\mathcal{K}$-skeleton.

Proof. Suppose that (K_n) satisfies the above conditions and that we are given a set $A \in \mathcal{K}$, a cover $\mathcal{U}$ and an index m. By (4.12) $f(A)$ is a closed subset of K_{n-1} and, by (4.11) the set K_{n-1} is compact. Hence $f(A)$ is compact. The set A is closed, and if we take a cover $\mathcal{V}$ of A with mesh $\mathfrak{W} < \varepsilon$, then, by (4.12), each point of A is within an ε-distance from the compact set $f(A)$, and therefore the set A is compact. Thus we infer that there exists an $\varepsilon > 0$ such that every map $h\colon A \to X$, with $d(h) < 2\varepsilon$ is limited by the cover $\mathfrak{U}$. Pick another cover $\mathfrak{V}$ for A such that mesh $\mathfrak{V} < \varepsilon$, and let f and n be those of (4.12). Then $d(f) < \varepsilon$. Let g be a homeomorphism of the pair (K_n, K_{n-1}) onto (Q, Q_{odd}). Let $\lambda\colon X \to R$ be a zero function of the set $A \cap K_m$ such that $\lambda(x) < \eta$ for all $x \in X$, where $0 < \eta < 1$ is chosen in such a way that $x, y \in Q$ and $d(x, y) < 2\eta$ imply $d\big(g^{-1}(x), g^{-1}(y)\big) < \varepsilon$. We define $f'\colon A \to Q$ by the formula

$$f'(x) = gf(x) + \lambda(x)\cdot\Big(v_2 + \sum_{i=1}^{\infty} \pi_i g(x)\cdot v_{2i+2}\Big).$$

Finally we let $h = g^{-1}f'$.

We shall verify that $h\colon A \to X$ satisfies conditions (4.3). First, since $\eta < 1$, we easily conclude that $f'(x) \in Q$ for all $x \in A$, and therefore $h(x)$ is well-defined and $h(x) \in K$. Also the continuity of the map h is obvious. Further, for $x \in A \cap K_m$ we have $\lambda(x) = 0$, whence $f'(x) = gf(x)$ and $h(x) = f(x) = x$.

Assume that $x, y \in A$, $x \neq y$. Then the natural projections onto the orthogonal complement of Q_{odd} of the points $f'(x)$ and $f'(y)$ are:

$$\lambda(x)\cdot\Big(v_2 + \sum_{i=1}^{\infty} \pi_i g(x)\cdot v_{2i+2}\Big) \quad \text{and} \quad \lambda(y)\cdot\Big(v_2 + \sum_{i=1}^{\infty} \pi_i g(y)\cdot v_{2i+2}\Big),$$

respectively, and they do not coincide unless $\lambda(x) = \lambda(y) = 0$. This means that $f'(x) \neq f'(y)$ if at least one of the numbers $\lambda(x)$, $\lambda(y)$ is not zero. But $\lambda(x) = \lambda(y) = 0$ means that $x, y \in A \cap K_m$ and $f'(x) = g(x) \neq g(y) = f'(y)$, because g is a homeomorphism. Hence f' is one-to-one and therefore h is one-to-one. From the compactness of the domain A of the map h it follows that h is a homeomorphism.

Since $h(A) \subset K_n$ and the collection $\mathfrak{K}$ is hereditary, we get $h(A) \in \mathfrak{K}$; this means that h is a $\mathfrak{K}$-embedding. This completes the proof.

In the last two propositions in this section we shall present examples of homogeneous collections and skeletons. In order to state them we need the following notation. If X is a metric space, then $\mathfrak{C}_X$ denotes the family of all compact subsets of X. The subfamily of $\mathfrak{C}_X$ consisting of those subsets of X which are finite-dimensional will be denoted by $\mathcal{E}\mathfrak{C}_X$.

4.5. PROPOSITION. *If X is an infinite-dimensional separable Fréchet space (= locally convex complete linear metric space), then $\mathfrak{C}_X$ and $\mathcal{E}\mathfrak{C}_X$ are homogeneous collections.*

Proof. Clearly, if a subcollection of a homogeneous collection satisfies (4.1), then it is itself a homogeneous collection. Thus it is enough to show that $\mathfrak{C}_X$ is a homogeneous collection. By the Anderson–Kadec Theorem 1.3, the space X is homeomorphic to the pseudointerior of the Hilbert cube. Hence condition (4.2) for $\mathfrak{C}_X$ is an obvious consequence of the Estimated Extension Theorem 3.1.

Remark. The statement of the proposition above holds true without the assumption of separability of X. This follows from Toruńczyk ([32], Theorem).

Our next proposition shows that increasing sequences of compact convex subsets of an infinite-dimensional Fréchet space X are, under some geometric conditions, either $\mathfrak{C}_X$- or $\mathcal{E}\mathfrak{C}_X$-skeletons.

4.6. PROPOSITION. *Let X be an infinite-dimensional separable Fréchet space and let (K_n) be a sequence of compact subsets of X which are convex, symmetric with respect to zero and such that*

(4.13) $\quad K_n \subset 2^{-1}K_{n+1}$ *for all* $n \geq 1$

and

(4.14) $\quad$ *the set* $\bigcup_{n=1}^{\infty} K_n$ *is dense in* X,

then

(a) *the condition* $\dim K_n < +\infty$ *for all* $n \geq 1$ *implies that* (K_n) *is an* $\mathcal{E}\mathfrak{C}_X$*-skeleton,*

(b) *the condition* $\dim K_m = +\infty$ *for some* m *implies that* (K_n) *is a* $\mathfrak{C}_X$*-skeleton.*

The proof of this proposition is based on the following

4.7. LEMMA. *Under the assumption of Proposition 4.6, for each triple $(m, \mathfrak{V}, A)$ consisting of a compact set A, a cover $\mathfrak{V}$ of A and an index m, there exist an index $p = p(m, \mathfrak{V}, A)$ and a map $r\colon A \to K_p$ which is limited by $\mathfrak{U}$ and is identity on $A \cap K_m$.*

Proof. The compactness of A and the local convexity of X imply that there exist a finite subset $\{a_1, a_2, \ldots, a_N\}$ of A and a convex symmetric neighbourhood of zero, say W, such that for each $a \in A$ there is an $U \in \mathfrak{U}$ such that $3W + a \subset U$ and $\bigcup_{i=1}^{N} W_i \supset A$, where $W_i = W + a_i$ for $i = 1, 2, \ldots, N$. Condition (4.14) and again the compactness of A imply that there is an index $p = p(m, \mathfrak{U}, A) \geq m$ such that $W_i \cap K_p \neq \emptyset$ for $i = 1, 2, \ldots, N$. Then the set $K_p \cap \operatorname{cl} W_i$ is an absolute retract, as a closed convex set in a Fréchet space (cf. [17], pp. 60, 83–84). Let r_i denote the retraction of X onto $K_p \cap \operatorname{cl} W_i$. Furthermore, let $\{\lambda_i\}_{i=1}^{N}$ be a partition of unity of the set $\bigcup_{i=1}^{N} W_i$ such that each λ_i vanishes outside W_i.

Let us put

$$r(a) = \sum_{i=1}^{N} \lambda_i(a)\cdot r_i(a) \quad \text{for} \quad a \in A.$$

Clearly, $r(A) \subset K_p$ (because K_p is a convex set). If $a \in K_p \cap A$ and $\lambda_i(a) \neq 0$, then $a \in W_i$ and $r_i(a) = a$. Thus

$$r(a) = \sum_{i \in T_a} \lambda_i(a) \cdot a = a, \quad \text{where} \quad T_a = \{i\colon \lambda_i(a) \neq 0\}.$$

In particular, $r(a) = a$ for $a \in K_m \cap A$, because $K_m \subset K_p$. Finally, if $\lambda_i(a) \neq 0$, then $a - a_i \in W$ and $r_i(a) - a_i \in \operatorname{cl} W \subset 2W$. Thus

$$a - r(a) = \sum_{i \in T_a} \lambda_i(a)\big(a - r_i(a)\big) = \sum_{i \in T_a} \lambda_i(a)(a - a_i) + \sum_{i \in T_a} \lambda_i(a)\big(a_i - r_i(a)\big) \in 3W.$$

This shows that r is limited by $\mathcal{U}$ and completes the proof of the lemma.

Remark. The argument used above gives, in fact, the following: If K is an infinite-dimensional compact convex subset of a Fréchet space and $K_1 \subset K_2 \subset K_3 \ldots$ are compact convex subsets of K such that $\operatorname{cl} \bigcup_{i=1} K_i = K$, then, for each triple $(m, \mathcal{U}, A)$ consisting of a compact set $A \subset K$, a cover $\mathcal{U}$ of A and an index m there exists an index $p = p(m, \mathcal{U}, A)$ and a map $r\colon A \to K_p$ which is limited by $\mathcal{U}$ and is identity on $A \cap K_m$.

Proof of Proposition 4.6. Case (a). We have to check condition (4.3). Suppose that $A \in \mathcal{EC}_X$, $\mathcal{U}$ is a cover of A and m is an index. We pick an index $n = n(m, A, \mathcal{U})$ so that $\dim K_n - \dim K_p > k$, where $p = p(m, \mathcal{U}, A)$ is that of Lemma 4.7, and k is so large that the Euclidean $(k-1)$-space contains a subset homeomorphic to A. Since X is infinite-dimensional, conditions (4.13) and (4.14) imply that the dimensions of the sets K_n are monotonely diverging to $+\infty$, and therefore the index n with the above properties exists. Next, let us pick in $K_n \setminus K_p$ linearly independent vectors $e_1, e_2, \ldots, e_k$ so that $K_p \cap \operatorname{span}\{e_1, e_2, \ldots, e_k\} = \{0\}$. Let g be an $\mathcal{EC}$-embedding of A into the $\operatorname{span}\{e_1, e_2, \ldots, e_{k-1}\}$ and let φ be a zero function of K_m. Let us put

$$h(x) = r(x) + \varepsilon\big(\varphi(x) \cdot g(x) + \varphi(x) \cdot e_k\big) \quad \text{for} \quad x \in A, \tag{4.15}$$

where the map $r\colon A \to K_p$ is that of Lemma 4.7 (limited by $\mathcal{U}$ and the identity on $A \cap K_m$), and $\varepsilon > 0$ is chosen so small that $h(x)$ is still limited by $\mathcal{U}$ and

$$\varepsilon\big(\varphi(x) \cdot g(x) + \varphi(x) \cdot e_k\big) \in K_n \cap \operatorname{span}\{e_1, \ldots, e_k\} \quad \text{for} \quad x \in A.$$

(This is possible because of the compactness of A.) Thus, by (4.13), $h(x) \in 2K_p \subset K_n$. Owing to the contribution of the second term on the right-hand side of (4.15) h is one-to-one. Indeed, if $h(x_1) - h(x_2) = 0$ for some x_1, x_2 in A, then $r(x_1) - r(x_2) = 0$; $\varphi(x_1) g(x_1) - \varphi(x_2) g(x_2) = 0$ and $\varphi(x_1) - \varphi(x_2) = 0$ because the vectors $r(x_1) - r(x_2)$, e_k and $\varphi(x_1) g(x_1) - \varphi(x_2) g(x_2)$ belong to linearly independent and finite-dimensional subspaces. Thus either $\varphi(x_1) = \varphi(x_2) \neq 0$ or $\varphi(x_1) = \varphi(x_2) = 0$. In the first case $g(x_1) = g(x_2)$ and $x_1 = x_2$ because g is one-to-one. In the second case both x_1 and x_2 belong to K_m because φ is a zero function of K_m. Hence x_1 and x_2 are in $K_m \cap A$, and $0 = r(x_1) - r(x_2) = x_1 - x_2$. Thus h is one-to-one and therefore it is an $\mathcal{EC}$-embedding (because of the compactness of A). Clearly, if $x \in K_m \subset K_p$, then $\varphi(x) = 0$. Thus $h(x) = r(x)$. Since r is the identity on $A \cap K_m$, the $\mathcal{EC}$-embedding h has the same property. This completes the proof of the proposition in case (a).

(b) By Definition 4.1, we may assume without loss of generality that $\dim K_1 = +\infty$. Thus, by Proposition 2.6, each pair (K_{n+1}, K_n) is homeomorphic to the pair (Q, Q_{odd}). Therefore, by Lemma 4.7 and Proposition 4.4, (K_n) is a $\mathcal{C}$-skeleton. This completes the proof.

5. Topological classification of sigma-compact linear metric spaces

The main results of this section are consequences of Proposition 4.3 applied to the skeletons of Proposition 4.6.

We recall that a linear metric space X is $\aleph_0$-*dimensional* if it has exactly $\aleph_0$-linearly independent vectors, that is, there is an infinite sequence (e_n) of linearly independent vectors in X such that $\operatorname{span}\{(e_n)\} = X$.

5.1. THEOREM. *Let X_0 and Y_0 be $\aleph_0$-dimensional dense linear subspaces of Fréchet spaces X and Y, respectively. Then the pairs (X, X_0) and (Y, Y_0) are homeomorphic.*

Proof. First observe that the spaces X and Y are separable because their dense subspaces X_0 and Y_0 have the same property. Next note that X_0 is an $\mathcal{EC}_X$-skeletoid and Y_0 is an $\mathcal{EC}_Y$-skeletoid. Indeed, let $X_0 = \operatorname{span}\{x_i\}_{i=1}^{\infty}$, where (x_i) is an appropriate linearly independent sequence of vectors in X_0. Let $X_n = \operatorname{span}\{x_i\}_{i=1}^{n}$. Then $X_1 \subset X_2 \subset \ldots$ and $X_0 = \bigcup_{n=1}^{\infty} X_n$. Since each X_n is exactly n-dimensional, there exists a compact, symmetric with respect to zero and convex subsets L_n of X_n such that $\bigcup_{m=1}^{\infty} mL_n = X_n$. Therefore each compact subset of X_n is contained in $m \cdot L_n$ for some m. Thus one can define inductively an increasing sequence of indices $m(n)$ so that for $K_n = m(n) \cdot L_n$, the following conditions are satisfied:

$$K_n \subset 2^{-1} K_{n+1}, \quad \bigcup_{n=1}^{\infty} K_n = X_0, \quad \dim K_n = n.$$

Since $\operatorname{cl} X_0 = X$, by Proposition 4.6 (a), we infer that (K_n) is an $\mathcal{EC}_X$-skeleton. The same argument shows that Y_0 is an $\mathcal{EC}_Y$-skeletoid.

Now, by the Anderson–Kadec Theorem 1.3, there exists a homeomorphism, say H from X onto Y. Clearly H takes the collection $\mathcal{E}\mathcal{C}_X$ onto $\mathcal{E}\mathcal{C}_Y$. Therefore $(H(K_n))$ is an $\mathcal{E}\mathcal{C}_Y$-skeleton. By Proposition 4.3, there exists a $G \in \operatorname{Auth} Y$ such that $G\big(\bigcup_{n=1}^{\infty} H(K_n)\big) = Y_0$. Thus GH is the required homeomorphism of the pair (X, X_0) onto the pair (Y, Y_0).

5.2. Corollary. *All $\aleph_0$-dimensional locally convex linear metric spaces are homeomorphic.*

It seems convenient to restate Theorem 5.1 in the following "more effective" form

5.3. Corollary. *Each pair (X, X_0), consisting of a Fréchet space X and its dense $\aleph_0$-dimensional linear subspace X_0, is homeomorphic to the pair (l^2, l^2_F) as well as to the pair $(R^{\mathcal{N}}, \sum R)$, where l^2_F and $\sum R$, denote the subspaces of l^2 and of $R^{\mathcal{N}}$, respectively, consisting of those sequences which have all but finitely many coordinates equal to zero.*

Now we pass to another class of sigma-compact linear metric spaces.

A linear metric spaces E is called a *core space* if there is a sequence (W_n) of compact convex subsets of E such that $\bigcup_{n=1}^{\infty} W_n = E$ and $\dim W_j = +\infty$ for some index j. If E is a core space and is a linear subspace of a linear metric space X, then E is called a core linear subspace of X.

We observe here that every core space can be decomposed into its convex subsets in a certain regular way; namely, we have

5.4. Lemma. *If E is a core space, then there is a sequence (K_n) of compact convex and symmetric with respect to zero subsets of E such that $\bigcup_{n=1}^{\infty} K_n = E$, $K_n \subset 2^{-1} K_{n+1}$ for all $n \geqslant 1$, $\dim K_j = +\infty$ for some j.*

Proof. Let $E = \bigcup_{n=1}^{\infty} W_n$, where W_n are compact convex sets and $\dim W_j = +\infty$ for some index j. We put, for $n = 1, 2, \ldots$,

$$K_n = 2^n\big(W_1 + W_2 + \ldots + W_n + (-W_1) + (-W_2) + \ldots + (-W_n)\big).$$

We leave to the reader the verification that the sequence (K_n) has the required properties.

Our next result is a full analogue of Theorem 5.1.

5.5. Theorem. *If X_σ and Y_σ are dense core linear subspaces of Fréchet spaces X and Y, respectively, then the pairs (X, X_σ) and (Y, Y_σ) are homeomorphic.*

Proof. Observe first that the spaces X and Y are separable because their dense subspaces X_σ and Y_σ are separable. Next, note that X_σ is a $\mathcal{C}_X$-skeletoid and Y_σ is a $\mathcal{C}_Y$-skeletoid. This is an obvious consequence of Lemma 5.4 and Proposition 4.6. The remaining part of the proof is the same as that of Theorem 5.1.

5.6. Corollary. *All locally convex core spaces are homeomorphic.*

5.7. Corollary. *Every pair (X, X_σ) consisting of a Fréchet space X and its dense core linear subspace X_σ is homeomorphic to the pair (l^2, l^2_σ) as well as to the pair $(R^{\mathcal{N}}, (R^{\mathcal{N}})_\sigma)$, where*

$$l^2_\sigma = \Big\{x \in l^2 \colon \sum_{n=1}^{\infty} |x(n)|^2 n^2 < \infty\Big\} \quad \text{and} \quad (R^{\mathcal{N}})_\sigma = \{x \in R^{\mathcal{N}} \colon \sup_n |x(n)| < \infty\}.$$

Recall that, for every topological vector space X, the symbol $\sum X$ denotes the space consisting of all X-valued sequences $x = (x(i))$ having all but finitely many coordinates equal to zero. The topology in $\sum X$ is that inherited from $X^{\mathcal{N}}$. The symbol "$\sim$" means "is homeomorphic to". Under this notation we have

5.8. Corollary. *Suppose that X_0 is an $\aleph_0$-dimensional locally convex linear metric space and X_σ is a locally convex core space, and let $I = [-1; 1]$, $I^+ = (0; 1]$, $J = (-1; 1)$ be the closed, the half-open an the open intervals respectively. Then the following formulas hold*

$$X_\sigma \times X_\sigma \sim X_\sigma, \quad X_0 \times X_\sigma \sim X_\sigma, \quad \sum X_\sigma \sim X_\sigma; \tag{5.1}$$

$$X_0 \times X_0 \sim X_0, \quad \sum X_0 \sim X_0; \tag{5.2}$$

$$X_\sigma \times I \sim X_\sigma \times I^+ \sim X_\sigma \times J \sim X_\sigma; \tag{5.3}$$

$$X_0 \times I \sim X_0 \times I^+ \sim X_0 \times J \sim X_0; . \tag{5.4}$$

Proof. Identities (5.2) are obvious consequences of Corollary 5.2 because the spaces $X_0 \times X_0$ and $\sum X_0$ have the natural structure of an $\aleph_0$-dimensional space compactible with their topologies.

Now we shall establish the identity $X_0 \times X_\sigma \sim X_\sigma$. By the definition of a core space, we have $X_\sigma = \bigcup_{n=1}^{\infty} W_n$, where W_n are convex compact sets and $\dim W_j = \infty$ for at least one index j. Similarly (cf. the proof of Theorem 5.1), one can find finite-dimensional convex compact sets K_m such that $X_0 = \bigcup_{m=1}^{\infty} K_m$. Clearly, $X_0 \times X_\sigma$ is a locally convex linear metric space, and

$$X_0 \times X_\sigma = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_m \times W_n,$$

each of the sets $K_m \times W_n$ is compact and convex, and $\dim(K_1 \times W_j) \geqslant \dim W_j = \infty$.

Hence $X_0 \times X_\sigma$ is a locally convex core space. Thus by Corollary 5.6, $X_0 \times X_\sigma \sim X_\sigma$. The proof of the other two identities in (5.1) is similar.

Proof of (5.3). Assume that X_σ is a dense linear set in a Fréchet space X, and that (K_n) is a $\mathfrak{C}$-skeleton in X, with $\bigcup_{n=1}^{\infty} K_n = X_\sigma$. By [21], III. (1.3), and by Anderson–Kadec Theorem 1.3, we have $X \times I \sim X \times I^+ \sim X \times J \sim X$. Let $A_n = K_n \times I$, $B_n = K_n \times I^+$, $C_n = K_n \times J$. It is easy to check that the sequences (A_n), (B_n), (C_n) are $\mathfrak{C}$-skeletons in the spaces $X \times I$, $X \times I^+$, $X \times J$, respectively, and $A_\infty = X_\sigma \times I$, $B_\infty = X_\sigma \times I^+$, $C_\infty = X_\sigma \times J$. Hence, applying Proposition 4.3, we establish formulas (5.3).

The proof of (5.4) is similar.

5.9. Corollary. *If E is a locally convex linear metric space which is either $\aleph_0$-dimensional or a core space, then every closed convex body W in the space E is homeomorphic to E.*

Proof. By [7], W is homeomorphic to the product of k intervals $(0 \leqslant k < \infty)$ and a closed subspace Y of E of codimension k. Using formulas (5.3) and (5.4) of the last corollary we get the assertion.

There are several interesting examples of core spaces. For instance, the space of all real-valued functions defined on the interval I which satisfy the Hölder condition with a fixed exponent α $(0 < \alpha \leqslant 1)$, regarded in the supremum-norm topology, i.e. regarded as a subspace of $C(I)$, is a core space. These spaces, as well as the spaces l^2_σ and $(R^{\mathcal{N}})_\sigma$ defined before, belong to a special class of so called simply generated core spaces.

A core space E is called *simply generated* if there exists a compact convex and symmetric with respect to zero subset W of E such that $\dim W = \infty$ and $E = \bigcup_{n=1}^{\infty} nW$.

Observe that not every core space is simply generated. Indeed, if X_σ is an arbitrary core space, then the core space $\sum X_\sigma$ is not simply generated.

We shall need the following concept.

A bounded linear operator $u\colon Z \to X$ acting from a Banach space Z into a linear metric space X is said to be closed-compact if the set $u(\{z \in Z\colon \|z\| \leqslant 1\})$ is compact.

We have

5.10. Proposition. *Let E be a linear metric space. Then the following conditions are equivalent:*

(i) *E is a simply generated core space,*

(ii) *there is a Banach space Z and a closed-compact operator $u\colon Z \to E$ such that $u(Z) = E$.*

Proof. (ii)$\Rightarrow$(i). Put $W = u(\{z \in Z\colon \|z\| \leqslant 1\})$.

(i)$\Rightarrow$(ii). Let Z be the space $l^1(W)$ of all scalar-valued functions $f(\cdot)$ defined on W such that

$$\|f\| = \sup_{A \in \mathfrak{F}_W} \sum_{b \in A} |f(b)| < \infty .$$

Here $\mathfrak{F}_W$ denotes the family of all finite subsets of W. Next we define the operator $u\colon l^1(W) \to E$ by

$$u(f) = \sum_{b \in W} f(b) \cdot b .$$

To show that u is well-defined for each $f \in W$, observe that if U is a neighbourhood of zero in E, then from the compactness of W it follows that there is a $\delta > 0$ such that $\delta \cdot W \subset U$. On the other hand, by the definition of the norm $\|f\|$ there is an $A_0 \in \mathfrak{F}_W$ such that for each $A \in \mathfrak{F}_W$

$$\sum_{b \in A \setminus A_0} |f(b)| < \delta ,$$

and therefore, by the convexity and central symmetry of W,

$$\sum_{b \in A \setminus A_0} f(b) \cdot b \in \delta W \subset U .$$

This shows that the family $\big(\sum_{b \in A} f(b) \cdot b\big)_{A \in \mathfrak{F}_W}$ is a Cauchy net in $\|f\| \cdot W$. Since the set $\|f\| \cdot W$ is compact, this net is convergent to an element of the set $\|f\| W$. This element is denoted by $\sum_{b \in W} f(b) \cdot b$. Clearly $u(\{f \in l^1(W)\colon \|f\| \leqslant 1\}) = W$. The linearity of u is obvious. Finally, the continuity of u follows from the fact that it takes the unit ball of the space $l^1(W)$ onto W which is a compact, and therefore is a bounded subset of the linear metric space E (cf. [12], pp. 51, 54). This completes the proof.

There exist compact linear operators which are not closed-compact (for instance the natural embedding of the space of all scalar-valued functions, having the first derivative continuous, defined on the interval I, into the space $C(I)$ of all continuous functions on I, both spaces regarded under the supremum-norm). However, we have

5.11. Proposition. *If u is a compact linear operator from a reflexive Banach space Z into an arbitrary linear metric space, then u is closed-compact and the range of u is a simply generated core space.*

Proof. The argument is very easy in the case where the range of u is a locally convex linear metric space. Indeed, the reflexivity of Z implies that the unit ball of Z, say S, is weakly compact, [11], p. 56. Therefore the set $W = u(S)$ is weakly compact, hence weakly closed. Thus it is closed. On the other hand, the compactness of u implies that the set W is totally bounded (pre-compact). Hence W is compact.

In the case where $u(Z)$ is an arbitrary linear metric space, the argument is more sophisticated. As before, let W be the image under u of the unit ball S of the space ω. We have to show that W is closed. Let (w_n) be a sequence of elements of W. Since W is totally bounded, there exists a limit point, say w_0, of the set $\{w_n\}_{n=1}^{\infty}$. Hence there is a subsequence (w_{k_n}) such that $\sum_{n=1}^{\infty} d(w_{k_n}, w_0) < +\infty$. (Here $d(\cdot, \cdot)$ denotes a translation invariant and monotone admissible metric of the space $u(Z)$, i.e. $d(x, y) = d(x-y, 0)$ for all $x, y \in u(Z)$ and $d(tx, 0)$ is a monotone function of t for $t > 0$, for every fixed $x \in u(Z)$). Let $w_{k_n} = u(z_n)$ for some $z_n \in S$ $(n = 1, 2, \ldots)$. Since Z is reflexive, there is a subsequence (z_{m_n}) which weakly converges to an element z_0 of S. Thus, by a result of Mazur ([11], p. 40), there is a sequence (λ_k) of real numbers and an increasing sequence $(n(j))_{j=1}^{\infty}$ of indices such that

$$0 \leqslant \lambda_n \leqslant 1, \qquad \sum_{n=n(j)+1}^{n(j+1)} \lambda_n = 1, \qquad \Big\| \sum_{n=n(j)+1}^{n(j+1)} \lambda_n z_{m_n} - z_0 \Big\| < j^{-1}$$

for $j = 1, 2, \ldots$ Hence

$$\lim_j d\Big(\sum_{n=n(j)+1}^{n(j+1)} \lambda_n w'_n - u(z_0), 0 \Big) = 0, \tag{5.5}$$

where $w'_n = w_{k_{m_n}}$ $(n = 1, 2, \ldots)$.

On the other hand, for $j = 1, 2, \ldots$, we have

$$\begin{aligned} d\Big(\sum_{n=n(j)+1}^{n(j+1)} \lambda_n w'_n - w_0, 0 \Big) &= d\Big(\sum_{n=n(j)+1}^{n(j+1)} \lambda_n (w'_n - w_0), 0 \Big) \\ &\leqslant \sum_{n=n(j)+1}^{n(j+1)} d\big(\lambda_n (w'_n - w_0), 0\big) \leqslant \sum_{n=n(j)+1}^{n(j+1)} d(w'_n, w_0) \\ &\leqslant \sum_{n=j}^{\infty} d(w_{k_n}, w_0). \end{aligned}$$

Since $\sum_{n=1}^{\infty} d(w_{k_n}, w_0) < +\infty$, we infer that

$$\lim_j d\Big(\sum_{n=n(j)+1}^{n(j+1)} \lambda_n w'_n - w_0, 0 \Big) = 0. \tag{5.6}$$

Clearly, (5.5) and (5.6) imply that $w_0 = u(z_0)$. Hence $w_0 \in W$. This establishes the closedness of W and completes the proof of the proposition.

6. Other applications

In the first place we shall discuss applications related to homogeneous collections and skeletons in the Hilbert cube Q.

In Section 5 we defined the class $\mathfrak{Z}$ of Z-sets in Q. The notion of Z-sets can be naturally transferred to any topological space W which is homeomorphic to Q. The class of all Z-sets in W will be denoted by $\mathfrak{Z}_W$. By $\mathfrak{E}\mathfrak{Z}$ and $\mathfrak{E}\mathfrak{Z}_W$ we shall denote the collections of finite-dimensional Z-sets in Q and in W, respectively.

6.1. Proposition. *The classes $\mathfrak{Z}$ and $\mathfrak{E}\mathfrak{Z}$ are homogeneous collections.*

Proof. The statement about $\mathfrak{Z}$ follows directly from the Estimated Extension Theorem 3.1. Since $\mathfrak{E}\mathfrak{Z}$ is a subclass of $\mathfrak{Z}$ invariant under autohomeomorphisms of Q, we obtain that $\mathfrak{E}\mathfrak{Z}$ is also a homogeneous collection.

Now let us assume that:

(6.1) *W is an infinite-dimensional compact convex subset of a Fréchet space with a non-empty* rint W.

The notions of a radial interior (rint) and the radial boundary (rbd) of a convex set have been defined in Section 2. We recall here that if $0 \in \operatorname{rint} W$ (in particular if O is the centre of symmetry for W), then

$$\operatorname{rint} W = \{tx\colon\ x \in W \text{ and } 0 \leqslant t < 1\}.$$

In particular, we have $\operatorname{rint} Q = \{x \in Q\colon \sup |\pi_n(x)| < 1\}$; that is, the radial interior of Q is smaller than the pseudointerior and the radial boundary is bigger than the pseudoboundary of Q.

6.2. Proposition. *If $A_1 \subset A_2 \subset A_3 \subset \ldots$ are members of $\mathfrak{Z}_W$ such that each pair (A_{n+1}, A_n) is homeomorphic to (Q, Q_{odd}) and $W = \operatorname{cl} \bigcup_{n=1}^{\infty} A_n$, then (A_n) is a $\mathfrak{Z}_W$-skeleton. In particular, if $0 \in \operatorname{rint} W$, then the sequence $\big((1-1/n)\cdot W\big)$ is a $\mathfrak{Z}_W$-skeleton.*

Proof. By Lemma 4.7, the sequence (A_n) has property (4.12). Hence, by Proposition 4.4, it is a $\mathfrak{Z}_W$-skeleton. The second statement follows from the first one and from Proposition 2.6.

The next result gives an improvement of Keller–Klee Theorem 2.5.

6.3. Corollary. *Under condition (6.1) there is a homeomorphism h of W onto Q such that $h(\operatorname{rbd} W) = \operatorname{rbd} Q$.*

Proof. Without loss of generality we may assume that $0 \in \operatorname{rint} W$. Then it is enough to apply Proposition 4.3 to the skeletons $\big((1-1/n)\cdot W\big)$ and $\big((1-1/n)\cdot Q\big)$.

6.4. Corollary. *The radial boundary of Q is homeomorphic to the Hilbert space l^2.*

Proof. The set $C = \{x = (x(n)) \in l^2\colon \sum_{n=1}^{\infty} n^2|x(n)|^2 \leqslant 1\}$ is compact convex and symmetric with respect to 0. Its radial boundary $\operatorname{rbd} C = \{x\colon \sum_{n=1}^{\infty} n^2|x(n)|^2 = 1\}$ is an ellipsoid and is homeomorphic to the unit sphere S of l^2 under the map: $(x(n)) \to (nx(n))$. By Klee [21], III. (3.1), we have $S \sim l^2$. Hence $\operatorname{rbd} C \sim l^2$, and by Corollary 6.3, we get the assertion.

From the last result it follows that $\operatorname{rbd} C$ is homeomorphic with P, the pseudointerior (not pseudoboundary!) of the Hilbert cube. In fact, a stronger statement holds.

6.5. Proposition. *Under condition* (6.1) *the pair* $(W, \operatorname{rbd} W)$ *is homeomorphic to* (Q, P).

In view of Corollary 6.3 it is enough to establish this fact for $W = Q$, and this has been done by Anderson [3], of Toruńczyk [27].

6.6. Corollary. *Every locally convex core space is homeomorphic with* $Q\backslash P$, *the pseudoboundary of the Hilbert cube.*

Proof. By Proposition 6.5, $Q\backslash P \sim \operatorname{rint} Q$. The space $\operatorname{rint} Q$ is evidently homeomorphic to the core space $Y = \{x \in R^{\mathcal{N}}\colon \sup_i |x(i)| < \infty\}$. By Corollary 5.6, every locally convex core space is homeomorphic to Y.

The next proposition presents examples of $\mathfrak{E}\mathfrak{Z}$-skeletons.

6.7. Proposition. *The sequences* (K_n), (L_n), (M_n) *are* $\mathfrak{E}\mathfrak{Z}$-*skeletons in* Q, *where*

$$K_n = \{x \in Q\colon \pi_i(x) = -1 \text{ for } i > n\}, \quad L_n = \{x \in Q\colon \pi_i(x) = 0 \text{ for } i > n\},$$
$$M_n = (1-1/n)\cdot L_n \quad \text{for} \quad n = 1, 2, \ldots$$

Proof. The argument is similar to that of Proposition 4.6 and is the same for all the three skeletons. Let (T_i) be any of these skeletons. Given a set $A \in \mathfrak{E}\mathfrak{Z}$, its cover $\mathfrak{U}$ and an index m. By Remark to Lemma 4.7, there exist an index p and a map $r\colon A \to Q$ which is limited by $\mathfrak{U}$ and such that

$$r(A) \subset T_p, \quad r(x) = x \quad \text{for} \quad x \in A \cap T_m.$$

Since $A \in \mathfrak{E}\mathfrak{Z}$, there is a homeomorphism $g\colon A \xrightarrow[\text{into}]{} I^{k-1}$ for some $k < \infty$. Let $g(x) = (g_1(x), g_2(x), \ldots, g_{k-1}(x))$ and let $\lambda\colon A \to R$ be a zero function of the set $A \cap K_m$. We define

$$h(x) = r(x) + \varepsilon\lambda(x)\cdot\Big(v_{p+k} + \sum_{i=1}^{k-1} g_i(x)\cdot v_{p+i}\Big)$$

(recall that v_j is the jth unit vector in Q).

For every $0 < \varepsilon < 1$, the above h is a well defined homeomorphic embedding of A into Q, and clearly, $h(x) = r(x) = x$ for $x \in A \cap T_m$, and $h(A) \subset T_{p+k}$. For $\varepsilon = 0$, $h(x) = r(x)$, if $\varepsilon > 0$ is sufficiently small, then the map h preserves the property of being limited by $\mathfrak{U}$. This completes the proof.

The symbol $\sum X$ denoting the weak product of spaces with the base point 0 has been introduced in Preliminaries; here we shall also consider the weak product of intervals $[-1; 1]$ with the base point -1:

$$\sum{}^{*} I = \{x \in Q\colon x(i) = -1 \text{ for all but finitely many values of } i\}.$$

6.8. Corollary. *Under the notation of Corollary* 5.8, *we have* $\sum I \sim \sum^{*} I \sim \sum J \sim \sum R \sim X_0$, *where* $J = (-1; 1)$ *is the open interval, and* $\sum Q \sim Q \times X_0 \sim Q \times X_\sigma \sim X_\sigma$. *The spaces* X_0, X_σ, Q, l^2, *and* $X_0 \times l^2$ *represent different topological types.*

Proof. Let $(K_n), (L_n), (M_n)$ be the skeletons of Proposition 6.7. Then $K_\infty = \sum^{*} I$, $L_\infty = \sum I$, $M_\infty = \sum R$. Hence, applying Proposition 4.3, we get $\sum^{*} I \sim \sum I \sim \sum J$. The homeomorphism $\sum J \sim \sum R$ is obvious, and $\sum R \sim X_0$ has been established in Corollary 5.6.

To establish the second series of homeomorphisms, we put for $n = 1, 2, \ldots$

$$A_n = \{x = (x(i)) \in Q^{\mathcal{N}}\colon x(i) = 0 \text{ for } i > n\},$$
$$D_n = (1-1/n)\cdot Q, \quad C_n = Q \times D_n, \quad B_n = Q \times M_n,$$

where (M_n) is the skeleton of Proposition 6.7. By Proposition 6.2 the sequence (D_n) is a $\mathfrak{Z}$-skeleton in Q. Clearly, (A_n) is a $\mathfrak{Z}$-skeleton in $Q^{\mathcal{N}}$, and (C_n) and (B_n) are $\mathfrak{Z}$-skeletons in $Q \times Q$. Since the spaces $Q, Q \times Q$ and $Q^{\mathcal{N}}$ are mutually homeomorphic, the desired conclusion follows from Proposition 4.3 and from the relations: $A_\infty \sim \sum Q$, $B_\infty \sim Q \times X_0$, $C_\infty \sim Q \times X_\sigma$, $D_\infty \sim X_\sigma$.

It remains to prove the last statement of the proposition. We observe that among the spaces X_0, X_σ, Q, l^2, $X_0 \times l^2$ sigma-compact are only X_0, Q, X_σ; the Hilbert cube Q is the only compact one. The spaces X_0 and X_σ are differentiated by the property that every compact set in X_0 is a countable union of members of $\mathfrak{E}\mathfrak{C}$, while the space X_σ contains subsets homeomorphic to Q, which are not representable as countable unions of sets from $\mathfrak{E}\mathfrak{C}$. The space l^2 as a complete metric space is an absolute G_σ, and $l^2 \times X_0$ naturally embedded in $l^2 \times l^2$ is not of type G_δ (if X is a Banach space, then the only dense linear G_δ set in X is the whole space X, see [24]). This completes the proof.

In the tables below we summarize the results of the Cartesian multiplication of the spaces considered and of the action of the functors $\sum$ and $(\cdot)^{\mathcal{N}}$.

	X_0	X_σ	X	X_τ	Q	I	R	$[0,1)$
X_0	X_0	X_σ	X_τ	X_τ	X_σ	X_0	X_0	X_0
X_σ	X_σ	X_σ	X_τ	X_τ	X_σ	X_σ	X_σ	X_σ
X	X_τ	X_τ	X	X_τ	X	X	X	X
X_τ	X_τ	X_τ	X_τ	X_τ	X_τ	X_τ	X_τ	X_τ
Q	X_σ	X_σ	X	X_τ	Q	Q	$Q\times R$	Q^+

	X_0	X_σ	X	X_τ	Q	$I,[0;1]$	$[0;1),[-1,1),R$
$\sum$	X_0	X_σ	X_τ	X_τ	X_σ	X_0	X_0
$(\cdot)^{\mathcal{N}}$	X_T	X_T	X	X_T	Q	Q	X

We have denoted by X an arbitrary infinite-dimensional separable Fréchet space; $X_\tau = X_0\times X$, $X_T = X_0^{\mathcal{N}}$, $Q^+ = Q\times[0,1)$. It can be shown that different symbols appearing in the tables represent topologically distinct spaces, except the open question:

PROBLEM 1. Is X_T homeomorphic to X_τ?

The entries $\sum X = X_\tau$ and $\sum X_\tau = X_\tau$ do not follow from Corollaries 5.8 and 6.8, but can be derived from some generalized skeletons based on Toruńczyk's Estimated Extension Theorem, Theorem 3.4.

The following problem is related to Corollary 6.8:

PROBLEM 2. Let M be a pointed metric space, i.e. a compact metric space with a base point. Is it true that if M is a finite dimensional absolute retract then $\sum M \sim \sum I$?

Now we shall discuss applications related to homogeneous collections consisting of finite sets.

A space X is said to be *locally homogeneous* at the point x_0 if the point x_0 admits a basis of open neighbourhoods U each of which has the following property:

($*$) *for every $y\in U$ there is an $f\in \mathrm{Auth}_U X$ such that $f(x_0)=y$.*

The set of all the points of local homogeneity of the space X will be denoted by $X^\wedge$.

From the above definition the following is clear:

6.9. PROPOSITION. *If X is a complete metric space such that $X^\wedge$ is non-empty, then the class $\mathcal{F}_X$ of all finite subsets of $X^\wedge$ is a homogeneous collection in X. If $(x_i)_{i=1}^\infty$ is any sequence of elements of $X^\wedge$ such that the set $\{x_i\}_{i=1}^\infty$ is dense in X, then the sequence of sets $(\{x_i\}_{i=1}^n)$ is an $\mathcal{F}_X$-skeleton in X.*

A space X is said to be *locally homogeneous* if $X^\wedge = X$. Since local homogeneity is a local property, it passes from a model to any manifold built on this model. Hence, *every Euclidean manifold, more generally: every Fréchet-space-manifold (without boundary), is locally homogeneous.* We also easily check that *if M is a Fréchet-space-manifold with a boundary and, say, the boundary of M is C, then $M^\wedge \supset M\setminus C$ (in the finite-dimensional case: $M^\wedge = M\setminus C$).*

M. K. Fort [14] has proved that if $Y_1 = \underset{i\in N}{P} M_i$ is an infinite countable product of connected compact Euclidean manifolds with a boundary, then Y_1 is homogeneous, i.e. for every two points $y_0, y_1\in Y_1$ there exists a homeomorphism $h_0\in \mathrm{Auth}\, Y_1$ such that $h_0(y_0)=y_1$. Analysing Fort's proof, we easily check that the map h_0 can be connected with the identity h_1 by an isotopy

$$(h_t)_{0\leqslant t\leqslant 1}. \tag{6.2}$$

This fact enables us to prove

6.10. PROPOSITION. *If M_n, $n=1,2,\ldots$ are compact connected Euclidean manifolds with a boundary, then the space $Y = \underset{i\in N}{P} M_i$ is locally homogeneous. In particular the Hilbert cube Q is locally homogeneous.*

Proof. Any point $x_0\in Y$ has a basis of closed neighbourhoods each of which is homeomorphic to a product $Y_1\times D_n$, where Y_1 is of a form $\underset{i>m}{P} M_i$ and D_n is the closed unit ball in the Euclidean n-space

$$(n = \dim M_1 + \ldots + \dim M_m),$$

and obviously m and n vary with the neighbourhoods of the basis. (This fact is evident for the points x_0 having, for all i, the ith coordinate not on the boundary of M_i, and, by the homogeneity of Y it is true for an arbitrary point $x_0\in Y$.). Now it remains to show that, for given points $y_0, y_1\in Y_1$ and $z_0\in \mathrm{int}\, D_n$, there is a homeomorphism $f\in \mathrm{Auth}(Y_1\times D_n)$ such that $f(y_0,0)=(y_1,z_0)$ and $f(x,z)=(x,z)$ for all $(x,z)\in Y_1\times S_n$, where 0 is the centre of D_n and $S_n = D_n\setminus \mathrm{int}\, D_n$. The required map f can be defined by means of isotopy (6.2) as follows:

$$f(x,z) = \big(h_{|z|}(x), g(z)\big),$$

where $|x| = d(x, 0)$ and $g \in \operatorname{Auth} D_n$, with $g(0) = z_0$, $g(z) = z$ for $z \in S_n$. This completes the proof. We observe here that an alternative proof of the statement concerning the Hilbert cube follows directly from the Estimated Extension Theorem 3.1 and also from Proposition 3.2.

Combining the facts stated above with Proposition 4.3, we get

6.10. COROLLARY. *If X is one of the spaces:*

(a) *a separable complete metrizable Fréchet-space-manifold without a boundary,*

(b) *a Hilbert-cube-manifold,*

(c) *an infinite countable product of connected compact Euclidean manifolds with a boundary,*

and K, L are countable dense subsets of X, then the pairs (X, K) and (X, L) are homeomorphic.

If X is a complete metrizable Fréchet-space-manifold with a boundary and the boundary of X is C, and K, L are countable dense subsets of $X \setminus C$, then the pairs (X, K) and (X, L) are homeomorphic.

This result, in the case where X is a Euclidean space, has been obtained by Fréchet [15]; in case (b) it has been obtained by Fort [14].

7. Homeomorphisms of linearly gradated spaces

Let X be a linear metric space and let (X_n) be a sequence of closed linear subspaces of X. The sequence (X_n) is called a *linear gradation* of X, if $\bigcup_{n=1}^{\infty} X_n$ is dense in X; (X_n) is called an *l-d gradation* (= linear-dimensional gradation), if $\bigcup_{n=1}^{\infty} X_n$ is dense in X and, moreover, $\dim X_n = n$ for $n = 1, 2, \ldots$

Let (X_n) and (Y_n) be linear gradations of linear metric spaces X and Y, respectively. A homeomorphism $H\colon X \to Y$ is said to be *gradation preserving*, if $H(X_n) = Y_n$ for $n = 1, 2, \ldots$

A pair $(X, (X_n))$ consisting of a linear metric space X and its linear gradation (X_n) is called a linearly gradated space. Two linearly gradated spaces are said to be homeomorphic, if there exists a gradation preserving homeomorphism between them.

The linearly gradated linear metric spaces appear naturally in several problems of functional analysis (for instance in approximation theory, perturbation theory of linear operators, in basic constructions of cohomology and cohomotopy functors on the Lerey–Schauder category, cf. Gęba and Granas [16]). Therefore the question of homeomorphism of given linearly gradated spaces seems to be worth considering.

There are natural gradations of l_F^2 and $\sum R$ which are defined as follows. Let us put, for $n = 1, 2, \ldots$,

$$l_n^2 = \{x \in l^2\colon x(i) = 0 \text{ for } i > n\}, \qquad R^n = \{x \in R^{\mathcal{N}}\colon x(i) = 0 \text{ for } i > n\}.$$

Clearly the sequences (l_n^2) and (R^n) are l-d gradations of the spaces l_F^2 and $\sum R$, respectively.

We have ample information on *l-d* gradations of $\aleph_0$-dimensional linear metric spaces. Restating in this languague some results of Klee and Long [23], and Bessaga [5] we get

7.1. PROPOSITION. *Let $(X_0, (X_n))$ be a l-d gradation of an $\aleph_0$-dimensional locally convex linear metric spaces $X_0 = \bigcup_{n=1}^{\infty} X_n$ and let*

(∗) *no linear subspace of X_0 be linearly homeomorphic to $\sum R$.*

Then the linearly gradated space $(X_0, (X_n))$ is homeomorphic to $(l_F^2, (l_n^2))$.

Observe that condition (∗) is always satisfied in the case where X_0 is a normed linear space. Next observe that condition (∗) is also necessary for the linearly gradated spaces $(X_0, (X_n))$ and $(l_F^2, (l_n^2))$ to be homeomorphic. Namely, we have

7.2. PROPOSITION. *Let $(X_0, (X_n))$ be an l-d gradation of an $\aleph_0$-dimensional locally convex linear metric space $X = \bigcup_{n=1}^{\infty} X_n$. Suppose that the linearly gradated spaces $(X_0, (X_n))$ and $(l_F^2, (l_n^2))$ are homeomorphic. Then no linear subspace of X_0 is linearly homeomorphic to $\sum R$.*

The proof of this proposition is based upon an unpublished result due to R. D. Anderson, which we include here with his permission.

7.3. LEMMA. *Let $h\colon \sum R \to l^2$ be a function such that for each n the restriction of h to R^n is a homeomorphism and $h(R^n) \subset l_{m(n)}^2$ for some index $m(n)$. Then h is not continuous.*

Proof. Let, for $n = 1, 2, \ldots$

$$L_n = \Big\{x \in \sum R\colon x(i) = 0 \text{ for all } i \neq n\Big\}.$$

Clearly $L_n \subset R^n$. Since L_n is not compact, the assumption on h implies that the set $h(L_n)$ is unbounded. Hence, for each n, there is an $x_n \in L_n$ such that $\|h(x_n)\| > n$. Thus the sequence $(h(x_n))$ does not converge to any element of l^2, while, by the definition of L_n, the sequence (x_n) converges to the zero vector of the space $\sum R$. This completes the proof.

Proof of Proposition 7.2. Let $u\colon \sum R \to X_0$ be a one-to-one continuous linear operator. Suppose that there exists a gradation preserving homeomorphism, say $H\colon X_0 \to l_F^2$. Then the map $h = Hu$ satisfies the assumption of Lemma 7.3. Hence h is discontinuous, which is a contradiction.

COROLLARY 7.4. *The linearly gradated spaces* $(l_F^2, (l_n^2))$ *and* $(\sum R, (R^n))$ *are not homeomorphic.*

The following problems seem to be open.

PROBLEM 3. Let (X_n) be an l-d gradation of a linear metric (locally convex) space $X_0 = \bigcup_{n=1}^{\infty} X_n$. Is then true that the homeomorphism between linearly gradated spaces $(X_0, (X_n))$ and $(\sum R, (R^n))$ implies the linear homeomorphism of the spaces X_0 and $\sum R$?

PROBLEM 4. Give a complete topological classification of l-d gradated spaces $(X_0, (X_n))$ such that $X_0 = \bigcup_{n=1} X_n$ and X_0 is a locally convex linear metric space.

Now we pass to the discussion of homeomorphisms of l-d gradated Fréchet spaces. The analysis of the proof of the Kadec theorem on homeomorphism of Banach spaces with bases [18], [19] gives the following:

PROPOSITION 7.5. *Let X be a Banach space with a basis (e_j). Let E_n denote the n-dimensional space spanned by vectors $e_1, e_2, \ldots, e_n$ $(n = 1, 2, \ldots)$. Then the linearly gradated spaces $(X, (E_n))$ is homeomorphic to the gradated space $(l^2, (l_n^2))$.*

We recall that a sequence (e_n) of elements of a linear metric space X is a basis for X, if for each x in X there is a unique sequence of scalar (t_n) such that $x = \sum_{n=1}^{\infty} t_n e_n$.

There are several problems related to Proposition 7.5.

PROBLEM 5. Let (X_n) be an l-d gradation of a Banach space X. Are the linearly gradated spaces $(X, (X_n))$ and $(l^2, (l_n^2))$ homeomorphic?

In particular, we do not know whether the linearly gradated space $(C(I), (\mathfrak{I}_n))$ is homeomorphic to $(l^2, (l_n^2))$ where $C(I)$ denotes the space of all real-valued continuous functions on the interval $I = [-1, 1]$ and $\mathfrak{I}_n$ denotes the n-dimensional subspace of $C(I)$ consisting of all polynomials of degree $\leqslant n-1$ $(n = 1, 2, \ldots)$.

It is also not clear how to generalize Proposition 7.5 to the case of Fréchet spaces with bases.

PROBLEM 6. Let X be a Fréchet space with a basis (e_n). Suppose that

(⁎⁎) no linear subspace of X is linearly homeomorphic to $R^{\mathcal{N}}$.

Does it imply that the linearly gradated space $(X, (E_n))$ is homeomorphic to $(l^2, (l_n^2))$, where E_n denotes the space spanned by the vectors $e_1, e_2, \ldots, e_n$?

Observe that condition (⁎⁎) is necessary. Indeed, if a Fréchet space X with a basis (e_n) does not satisfy (⁎⁎), then there is a subsequence $(m(n))_{n=1}^{\infty}$ such that the space E consisting of all finite linear combinations of the elements of the sequence $(e_{m(n)})_{n=1}^{\infty}$ is linearly homeomorphic to the space $\sum R$ (cf. [8], Lemma 4). Hence, by Proposition 7.2, the linearly gradated space $\left(\bigcup_{n=1}^{\infty} E_n, (E_n)\right)$ is not homeomorphic to the space $(l_F^2, (l_n^2))$. Consequently the linearly gradated space $(X, (E_n))$ is not homeomorphic to the space $(l^2, (l_n^2))$.

It follows from a construction given in [9], Lemma 1, cf. also [6], 7.2, that the answer to Problem 6 is affirmative for nuclear Köthe sequence spaces and, equivalently, for nuclear Fréchet spaces with bases, cf. [25].

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Reçu par la Rédaction le 28. 2. 1969