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Blaschke products for finite Riemann surfaces*

by

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1. Introduction. The classical Blaschke products are used in factorization theorems for certain classes of analytic functions on the open unit disk. Since the open unit disk is the universal covering space for any finite open Riemann surface, Blaschke products can be defined there too, if one admits the so-called "multiple-valued" functions. Coifman and Weiss [2] removed this unpleasantness for the case of planar multiply-connected domains by making use of an analogue of the complex Poisson kernel. Here we treat the general case using the H^p-space theory [1], define a Blaschke product in terms of the boundary behavior and the divisibility property and give various necessary and sufficient conditions for its existence (Theorem 2.5). In general, we have to make use of the compactness of the group of reals modulo integers to obtain certain convergent subsequences in proving the convergence of a Blaschke product. In the case of planar multiply-connected domains, though, we point out a canonical choice of the parameters involved which avoids this complication (Remark 2.6). Finally, we characterize the set of points on the boundary across which a Blaschke product can be continued analytically and prove the uniqueness of a Blaschke product for a given sequence of points in the Riemann surface.

2. Inner functions and Blaschke products. Let the finite open Riemann surface R have p handles and its boundary X be the union of q non-intersecting analytic curves $\Gamma_1, \ldots, \Gamma_q$. Set $\sigma = 2p+q-1$ and let $\{\gamma_1, \ldots, \gamma_\sigma\}$ be a homology basis for the closed paths in R. Let $A(X)$ be the algebra of continuous functions on X that can be extended analytically to R. A slight modification of Wermer's proof ([3], Lemma 1) shows that there exists a basis $\{Z_1, \ldots, Z_\sigma\}$ of the invertible elements in A modulo the exponentials in A such that

$$\frac{1}{2\pi}\int_{\gamma_j} {}^*d(\log|Z_k|) = \delta_{j,k} \qquad \text{for } 1 \leqslant j, k \leqslant \sigma.$$

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Also these $Z_1, \ldots, Z_\sigma$ can be chosen to be analytic across X. We choose a fixed point z_0 in R and consider the harmonic measure m on X with respect to z_0. For $1 \leqslant p \leqslant \infty$, let $H^p(dm)$ denote the closure of A in $L^p(dm)$, where the closure is taken in the norm topology for $1 \leqslant p < \infty$ and in the weak-star topology for $p = \infty$. It is well known that $H^\infty(dm)$ can be identified with the space of all bounded analytic functions on R in a natural way. Following [1], we call a function f in $H^\infty(dm)$ an *inner function* if there exist real numbers $a_1, \ldots, a_\sigma$ such that $|f| = |Z_1|^{a_1} \ldots |Z_\sigma|^{a_\sigma}$ a.e. dm on X. A function f of the form $f = cZ_1^{m_1} \ldots Z_\sigma^{m_\sigma}$, where c is a complex constant of absolute value 1 and $m_1, \ldots, m_\sigma$ are integers, is called a *trivial inner function*. We shall say that an inner function f satisfying a certain property is *essentially unique* if any other such function differs from f by a factor of a trivial inner function. If f is an inner function, we denote by $\hat{f}$ the corresponding bounded analytic function on R.

Definition 2.1. Let $(a_n)_n$ be a sequence of points in R with repetitions allowed. A *Blaschke product* for R with respect to $(a_n)_n$ is an inner function B such that $\hat{B}$ has zeros at $(a_n)_n$ and if g is in $H^\infty(dm)$ and $\hat{g}$ has zeros at least at $(a_n)_n$, then g/B is also in $H^\infty(dm)$.

We first show rather easily that if a is any point in R, then there exists an essentially unique Blaschke factor B for R with respect to a such that B is in $A(X)$. Without loss of generality we can assume that the point a does not lie on any of the closed paths $\gamma_1, \ldots, \gamma_\sigma$. Let $G(\cdot, a)$ be the Green's function for R with singularity at a, and let

$$a_j = \frac{1}{2\pi} \int_{\gamma_j} {}^*dG, \qquad j = 1, \ldots, \sigma.$$

Consider the continuously differentiable function

$$u = \sum_{j=1}^{\sigma} a_j \log |Z_j|$$

on X. If U is the harmonic extension of u to R, define $F = U - G$. Then there exists a function B in $A(X)$ such that $|B| = \exp F$. Since $G = 0$ on X, this implies $|B| = |Z_1|^{a_1} \ldots |Z_\sigma|^{a_\sigma}$ on X. Moreover, $\hat{B}$ has a zero only where $F \to -\infty$, that is, at a, and this is a simple zero. It is easily seen that B is a required Blaschke factor. If B_1 is any other such factor, then B/B_1 is invertible in A. Since $\{Z_1, \ldots, Z_\sigma\}$ is a basis of the invertible elements of A modulo the exponentials in A, it follows that B/B_1 is a trivial inner function and we are done. The following technical lemma is the crucial first step in forming infinite products of the Blaschke factors.

LEMMA 2.2. *Let $(f_n)_n$ be a sequence of inner functions in $A(X)$ such that if*

$$|f_n| = |Z_1|^{a_{1,n}} \ldots |Z_\sigma|^{a_{\sigma,n}} \quad \text{on } X,$$

then $\sum a_{j,n} < \infty$ for $j = 1, \ldots, \sigma$. Also assume that $f_n(z_0) = b_n \neq 0$ and that $\Pi |b_n|$ converges. Let

$$f_n' = \frac{\bar{b}_n}{|b_n|} \cdot f_n.$$

Then the partial products of the infinite product $\Pi f_n'$ converge to a function in $H^2(dm)$.

Proof. Let $B_l = \prod_{n=1}^{l} f_n'$. We will show that $(B_l)_l$ is a Cauchy sequence in $H^2(dm)$.

For a σ-tuple $(\beta_1, \ldots, \beta_\sigma)$, define

$$c_\beta = c_{(\beta_1, \ldots, \beta_\sigma)} = \sup_{x \in X} |Z_1(x)|^{\beta_1} \ldots |Z_\sigma(x)|^{\beta_\sigma}.$$

Also, define

$$s_{j,l} = \sum_{n=1}^{l} a_{j,n}.$$

Then, by hypothesis, $s_{j,l}$ converge as l tends to infinity, for each $j = 1, 2, \ldots, \sigma$.

Clearly, $|B_l| = |Z_1|^{s_{1,l}} \ldots |Z_\sigma|^{s_{\sigma,l}}$ a.e. dm on X. We abbreviate this by writing $|Z|^{s_l}$. Since each Z_j is continuous and non-zero on X, there exist δ and M such that $0 < \delta < 1 < M$ and $\delta \leqslant |Z_j(x)| \leqslant M$ for every $j = 1, \ldots, \sigma$ and x in X. If $-Q \leqslant s_{j,l} \leqslant Q$ for every $j = 1, \ldots, \sigma$ and $l = 1, 2, \ldots$, where Q is a positive number, it follows that

$$c = \min(M^{-\sigma Q}, \delta^{\sigma Q}) \leqslant |B_l| \leqslant \max(M^{\sigma Q}, \delta^{-\sigma Q}) = C.$$

Let now, $m > l$. We have

$$\begin{aligned}
\int_X |B_l - B_m|^2 dm &\leqslant c_{2s_l} \int_X |B_l - B_m|^2 \cdot |Z|^{-2s_l} dm \\
&= c_{2s_l} \Big[\int_X |B_l|^2 \cdot |Z|^{-2s_l} dm + \int_X |B_m|^2 \cdot |Z|^{-2s_m} dm \\
&\qquad - 2\,\mathrm{Re} \int_X B_m / B_l \, dm + \int_X |B_m|^2 (|Z|^{-2s_l} - |Z|^{-2s_m}) \, dm \Big] \\
&= c_{2s_l} [1 + 1 - 2\,\mathrm{Re} (\hat{B_m / B_l})(z_0)] + c_{2s_l} \int_X (|Z|^{2(s_m - s_l)} - 1) \, dm \\
&= 2 c_{2s_l} [1 - \mathrm{Re} (B_m / \hat{B}_l)(z_0)] + c_{2s_l} \int_X (|Z|^{2(s_m - s_l)} - 1) \, dm.
\end{aligned}$$

Now,

$$\operatorname{Re}(B_m\hat{/}B_l)(z_0) = \operatorname{Re}[\hat{f}'_{l+1}(z_0)\dots\hat{f}'_m(z_0)] = |b_{l+1}|\dots|b_m| = \prod_{n=l+1}^{m}|b_n|.$$

Since $\prod|b_n|$ converges, $1-\operatorname{Re}(B_m\hat{/}B_l)(z_0)$ tend to zero as m and l tend to infinity.

Also $c^2\leqslant c_{2s_l}\leqslant C^2$ and $\int_X(|Z|^{2(s_m-s_l)}-1)\,dm$ tends to zero as m and l tend to infinity, by the bounded convergence theorem. Thus $(B_l)_l$ is a Cauchy sequence in $L^2(dm)$ and all B_l are in $H^2(dm)$. Hence B_l converges to a function in $H^2(dm)$.

Definition 2.3. A *boundary strip* S for a finite open Riemann surface R with boundary $X=\Gamma_1\cup\dots\cup\Gamma_q$ is an open subset S of R such that S is the disjoint union of q open subsets $S_1,\dots,S_q$ of R, each S_j conformally equivalent to an annulus $\{z\,|\,0<s_j<|z|<1\}$ under a map Φ_j, Γ_j being one of the boundaries of S_j and the continuous extension of Φ_j mapping Γ_j onto $|z|=1$. We call Φ_j's *boundary uniformizers* for the boundary strip S. It follows from classical function theory that such boundary uniformizers exist.

If $(a_n)_n$ is a sequence of points in the open unit disk, then the usual Blaschke product for $(a_n)_n$ exists if and only if $\sum_{n=1}^{\infty}1-|a_n|<\infty$. Since

$$G(a_n, z) = -\log\left|\frac{a_n-z}{1-\bar{a}_n z}\right|,$$

this condition is equivalent to $\sum_{n=1}^{\infty}G(a_n,0)<\infty$. It is important to note that this is also a necessary and sufficient condition for the existence of a bounded analytic function on the open unit disk having zeros at least at $(a_n)_n$. Consider now the annulus $K=\{z\,|\,0<r_0<|z|<1\}$, and a sequence of points $(a_n)_n$ in it satisfying $r_0^{1/2}<|a_n|$. A generalization of Jensen's theorem shows that if f is a bounded analytic function on K having zeros at a_n, then $\sum_{n=1}^{\infty}1-|a_n|<\infty$. More importantly, we have the following easily proved lemma:

LEMMA 2.4. *Let f be a real-valued continuous function on the closed annulus $\bar{K}=\{z\,|\,0<r_0\leqslant|z|\leqslant 1\}$ such that $f(z)=0$ for $|z|=1$.*

(i) *If f has a bounded radial derivative in K and if $(a_n)_n$ is any sequence of points in K satisfying $\sum 1-|a_n|<\infty$, then $\sum|f(a_n)|<\infty$.*

(ii) *If f is non-negative and harmonic in K such that $f(z)\neq 0$ for $|z|=r_0$ and if $(a_n)_n$ is any sequence in K satisfying $\sum f(a_n)<\infty$, then $\sum 1-|a_n|<\infty$.*

THEOREM 2.5. *Let $(a_n)_n$ be a sequence of points in $R-\{z_0\}$. Then the following are equivalent:*

(1) *There exists a bounded analytic function f on R having zeros at least at $(a_n)_n$.*

(2) *There exists a Blaschke product B for R with respect to $(a_n)_n$.*

(3) $\sum_{n=1}^{\infty}G(a_n,z_0)<\infty$.

(4) *If $S=S_1\cup\dots\cup S_q$ is a boundary strip for R, and $\Phi_1,\dots,\Phi$ are boundary uniformizers, then*

$$\sum_{n=1}^{\infty}1-|\Phi_j(a_{j,n})|<\infty,$$

where $(a_{j,n})_n=(a_n)_n\cap S_j$ for each $j=1,\dots,q$.

Proof. (2) ⇒ (1) by taking $f=\hat{B}$; (1) ⇒ (4) by considering the bounded analytic functions $f\circ\Phi_j^{-1}$ on the annuli $\{z\,|\,s_j<|z|<1\}$ and noting that their zeros are $\Phi_j(a_{j,n})$. To show (4) ⇒ (3), notice that $G\circ\Phi_j^{-1}$ is a continuous real-valued function on $\{z\,|\,s_j\leqslant|z|\leqslant 1\}$, it vanishes on $|z|=1$ and has a bounded radial derivative in $\{z\,|\,s_j<|z|<1\}$. Hence, by Lemma 2.4 (i), for each $j=1,\dots,q$,

$$\sum_{n=1}^{\infty}|G\circ\Phi_j^{-1}(\Phi_j(a_{j,n}))| = \sum_{n=1}^{\infty}G(a_{j,n},z_0)<\infty.$$

It remains to show that (3) ⇒ (2). For each n, let h_n be a Blaschke factor for R with respect to the point a_n, and let $|h_n|=|Z_1|^{a_{1,n}}\dots|Z_\sigma|^{a_{\sigma,n}}$ on X. Then for z in $R-\{a_n\}$,

$$|h_n(z)| = \exp[a_{1,n}\log|Z_1(z)|+\dots+a_{\sigma,n}\log|Z_\sigma(z)|-G(z,a)].$$

Let now $\tau_{j,l}=\sum_{j=1}^{l}a_{j,n}$. Let $N_{1,l},\dots,N_{\sigma,l}$ be integers such that $0\leqslant\tau_{j,l}-N_{j,l}\leqslant 1$ for $j=1,\dots,\sigma$ and $l=1,2,\dots$ and $s_{j,l}=\tau_{j,l}-N_{j,l}$. By compactness argument, there exist convergent subsequences (s_{j,l_n}) for each $j=1,\dots,\sigma$. Define

$$f_n = \prod_{l=l_{n-1}+1}^{l_n}h_l\cdot Z^{-(N_{l_n}-N_{l_{n-1}})},$$

where Z^{N_k} stands for $Z_1^{N_{1,k}}\dots Z_\sigma^{N_{\sigma,k}}$. Then

$$|f_n| = |Z|^{\sum_{l=l_{n-1}+1}^{l_n}a_l-(N_l-N_{l-1})} = |Z|^{\delta_n},$$

say, on X. Observe that $\sum_{n=1}^{m} \delta_{j,n} = \tau_{j,l_m} - N_{j,l_m} = s_{j,l_m}$, and hence $\sum_{n=1}^{\infty} \delta_{j,n}$ converges, for each $j = 1, \ldots, \sigma$. Also, as above,

$$|\hat{f}_n(z_0)| = \exp\Big[\log|Z_1(z_0)|\,\delta_{1,n} + \ldots + \log|Z_\sigma(z_0)|\,\delta_{\sigma,n} - \sum_{l=l_{n-1}+1}^{n} G(z_0, a_l)\Big].$$

Since $\sum G(z_0, a_l) < \infty$ by (3), it follows, if $b_n = \hat{f}_n(z_0)$, that $b_n \neq 0$ and that $\prod |b_n|$ converges. Let

$$f_n' = \frac{\bar{b}_n}{|b_n|} f_n \quad \text{and} \quad B_k = \prod_{n=1}^{k} f_n'.$$

Then, by Lemma 2.2, the sequence $(B_k)_k$ converges to a function B in $H^2(dm)$. Since a subsequence of $(B_k)_k$ converges pointwise a.e. dm to B and since each B_k is an inner function, B is also an inner function. It is also clear that $\hat{B}_k$ converge uniformly on compact subsets of R to $\hat{B}$. It is to be noticed that $\hat{B}$ has a zero of order p_n at a_n if and only if a_n is repeated p_n times in $(a_n)_n$. To see that if g is any function in $H^\infty(dm)$ having zeros at least at $(a_n)_n$, then g/B is in $H^\infty(dm)$, define $\hat{g}_k = \hat{g}/\hat{B}_k$ and notice that $c < |B_k| < C$ on X for some constants c and C. The maximum modules principle then gives the required result.

Remark 2.6. The construction of a Blaschke product with respect to $(a_n)_n$ in Theorem 2.5 depended on choosing a Blaschke factor h_n with respect a_n for each n, where $|h_n| = |Z_1|^{a_{1,n}} \ldots |Z_\sigma|^{a_{\sigma,n}}$ on X for some real numbers $a_{1,n}, \ldots, a_{\sigma,n}$. The difficulty lay in choosing these $a_{j,n}$'s so that $\sum a_{j,n} < \infty$ for each $j = 1, \ldots, \sigma$. This we overcame by using the compactness of the group reals modulo integers to get convergent subsequences. If R is a planar Riemann surface, then we do not have to go through the above complicated procedure, for a canonical choice of $a_{j,n}$'s is possible. Let the boundary X of R have q components $\Gamma_1, \ldots, \Gamma_q$ as usual, so that $\sigma = q-1$. Let ω_j be the harmonic function on $R \cup X$ such that $\omega_j = 1$ on Γ_j and $\omega_j = 0$ on $X - \Gamma_j$ (sometimes called the *harmonic measure* for Γ_j). Then

$$\omega_j(z) = -\frac{1}{2\pi} \int_{\xi\epsilon\gamma_j} {}^*dG(\xi, z).$$

If $S = S_1 \cup \ldots \cup S_q$ is a boundary strip for R, make the following choices for $a_{j,n}$. If $a_n \epsilon S_k$, $1 \leqslant k \leqslant q-1$, then $a_{j,n} = -\omega_j(a_n)$ for $1 \leqslant j \leqslant q-1$, $k \neq j$ and $a_{k,n} = -\omega_k(a_n)+1$. If $a_n \epsilon S_q$, let $a_{j,n} = -\omega_j(a_n)$ for $1 \leqslant j \leqslant q-1$. Then there exists a Blaschke factor h_n with respect to a_n such that $|h_n| = |Z_1|^{a_{1,n}} \ldots |Z_\sigma|^{a_{\sigma,n}}$ on X, and since $\omega_1 + \ldots + \omega_q = 1$, by Lemma 2.4 (i), $\Sigma a_{j,n} < \infty$ for each $j = 1, \ldots, q-1 = \sigma$.

3. Analytic continuation and uniqueness. We first make some remarks about the analytic continuation of an arbitrary inner function f across a point x of the boundary X. We claim that $\hat{f}$ can be continued analytically across x if and only if $\hat{f}$ is bounded away from zero in a neighborhood of x. Let U be a neighborhood of x in $R \cup X$ such that for every y in $R \cap U$, $|\hat{f}(y)| \geqslant \delta > 0$. Since f is an inner function, $|f| = |Z_1|^{a_1} \ldots |Z_\sigma|^{a_\sigma}$ a.e. dm on X for some real numbers $a_1, \ldots, a_\sigma$. The harmonic function

$$u = \log|\hat{f}| - \sum_{j=1}^{\sigma} a_j \log|Z_j|$$

has non-tangential limits zero on $X \cap U$ and $\hat{f}, Z_1, \ldots, Z_\sigma$ are bounded away from zero in $R \cap U$. Hence u can be extended harmonically across x. Since $Z_1, \ldots, Z_\sigma$ are analytic across x, it follows that $\hat{f}$ can be continued analytically across x. Conversely, if $\hat{f}$ is analytic across x, it must be bounded away from zero in a neighborhood of x if f is to be bounded away from zero a.e. dm on X.

PROPOSITION 3.1. *Let $(a_n)_n$ be a sequence of points in R and let B be a Blaschke product for R with respect to $(a_n)_n$ as constructed in Theorem 2.5. Let E denote the subset of X consisting of the accumulation points of $(a_n)_n$. Then $\hat{B}$ can be continued analytically across a point x in X if and only if x is not in E. Moreover, the Blaschke product B for R with respect to $(a_n)_n$ is essentially unique.*

Proof. If x is in E, then $\hat{B}$ is not bounded away from zero in any neighborhood of x, and hence $\hat{B}$ cannot be continued analytically across x. Conversely, let x be not in E, and let V be a neighborhood of x in $R \cup X$ not containing any a_n. Since B is a Blaschke product for $(a_n)_n$, by Theorem 2.5, $\sum_{n=1}^{\infty} G(z_0, a_n) < \infty$. Define

$$u_k(z) = \sum_{n=1}^{k} G(z, a_n) \quad \text{for } z \text{ in } V \text{ and } k = 1, 2, \ldots$$

Then it can be shown that $(u_k)_k$ is uniformly bounded in a smaller neighborhood U of x in $R \cup X$. Since, with the same notation as in (3) $\Rightarrow$ (2) of Theorem 2.5, for each z in $R \cap U$,

$$|\hat{B}(z)| = \lim_{k\to\infty} |\hat{B}_k(z)|,$$

where

$$|\hat{B}_k(z)| = \exp\Big[\sum_{j=1}^{\sigma}\Big(\log|Z_j(z)| \sum_{n=1}^{k} \delta_{j,n}\Big) - \sum_{n=1}^{l_k} G(z, a_n)\Big],$$

it follows that $(|\hat{B}_k|)_k$ is uniformly bounded away from zero in U. Hence $\hat{B}$ is bounded away from zero in U and can be continued analytically across x.

If B_1 is any other Blaschke product for R with respect to $(a_n)_n$, then B/B_1 and B_1/B are both in $H^\infty(dm)$. Then B/B_1 is an inner function that is bounded away from zero on R and hence it can be continued analytically across all of X. Thus B/B_1 is an invertible element in $A(X)$ and must be a trivial inner function.

Having shown the existence and uniqueness of Blaschke products for finite Riemann surfaces, we would only remark that "singular functions" and "outer functions" can also be defined in this case and the factorization of bounded analytic functions into Blaschke products, singular functions and outer functions accomplished in the usual manner.

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Über die Limitierbarkeit unbeschränkter Doppelfolgen

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1. Einleitung. S. Mazur und W. Orlicz haben im Jahr 1933 ([5], S. 33) folgenden interessanten Satz für Einfachfolgen mitgeteilt:

Limitiert eine permanente (zweidimensionale) Matrix eine beschränkte divergente Folge, so auch eine unbeschränkte. (Einen Beweis findet man z.B. bei Darevsky [1], S. 98). Wie Zeller 1951 ([8], S. 482]) und Mazur-Orlicz 1954 ([6], S. 151) zeigten, gilt der obige Satz auch für konvergenztreue Matrizen.

Es fragt sich, inwieweit dieses Ergebnis auch in der *Limitierungstheorie der Doppelfolgen* Gültigkeit hat, wenn man der Untersuchung *vierdimensionale regulär-konvergenztreue Matrizen* zugrunde legt (1). Dabei wird unter einer regulär-konvergenztreuen Matrix eine Matrix verstanden, die jede regulär-konvergente Doppelfolge — das sind (im Pringsheimschen Sinn) konvergente Doppelfolgen mit konvergenten Zeilen und Spalten — wieder in eine regulär-konvergente Doppelfolge transformiert. Für diese Klasse von Matrizen gilt der dem obigen Ergebnis für Einfachfolgen entsprechende Satz:

Transformiert eine regulär-konvergenztreue Matrix eine beschränkte, nicht regulär-konvergente Doppelfolge in eine regulär-konvergente Doppelfolge, so auch eine unbeschränkte.

2. Definitionen und Hilfssätze. Im folgenden werden Doppelfolgen komplexer Zahlen $x_{\mu\nu}\,(\mu,\nu = 0,1,\ldots)$ mit $x = (x_{\mu\nu})$ bezeichnet und zur Abkürzung

$$x_{\mu.} = \lim_{\nu\to\infty} x_{\mu\nu}, \qquad x_{.\nu} = \lim_{\mu\to\infty} x_{\mu\nu}, \qquad x_{..} = \lim_{\mu,\nu\to\infty} x_{\mu\nu} \qquad (\mu,\nu = 0,1,\ldots)$$

sowie

$$\|x\| = \sup_{0\leqslant\mu,\nu<\infty} |x_{\mu\nu}|$$

(1) Von einer wörtlichen Übertragung des obigen Satzes, d. h. von einer Übertragung auf *vierdimensionale konvergenztreue Matrizen* kann abgesehen werden, da eine (im Pringsheimschen Sinn) konvergente Doppelfolge unbeschränkt sein kann. Ohne Beweis sei bemerkt, daß eine konvergenztreue Matrix nicht notwendig regulär-konvergenztreu ist, und umgekehrt.