von Rellich [9] folgt, daß die Friedrichssche Erweiterung von $A-\mu_\varkappa E$ kontinuierliches Spektrum besitzt. Also ist auch $C(\tilde{A})\neq\emptyset$, wenn $\tilde{A}$ eine beliebige selbstadjungierte Erweiterung von A bedeutet. Damit ist Satz 3 vollständig bewiesen.

Literaturnachweis

[1] N. I. Achieser und J. M. Glasmann, *Theorie der linearen Operatoren im Hilbertraum*, Berlin 1965.

[2] M. Sch. Birman, *Störung quadratischer Formen und Spektren singulärer Randwertaufgaben* (russ.), DAN SSSR 125 (1959), S. 471-474.

[3] I. Brink, *Selfadjointness and spectra of Sturm-Liouville operators*, Math. Scand. 7, Nr. 1 (1959), S. 219-239.

[4] J. M. Glazman, *Direkte Methoden der qualitativen Spektralanalysis singulärer Differentialoperatoren* (russ.), Moskau 1963.

[5] R. S. Ismagilov, *Über Bedingungen für die Halbbeschränktheit und Diskretheit des Spektrums eindimensionaler Differentialoperatoren* (russ.), DAN SSSR 140 (1961), S. 33-36.

[6] L. W. Kantorowitsch und G. P. Akilow, *Funktionalanalysis in normierten Räumen*, Berlin 1964.

[7] A. M. Molčanov, *Über Bedingungen für die Diskretheit des Spektrums selbstadjungierter Differentialgleichungen zweiter Ordnung* (russ.), Tr. Mosk. matem. ob-va 2 (1953), S. 169-200.

[8] M. A. Neumark, *Lineare Differentialoperatoren*, Berlin 1963.

[9] F. Rellich, *Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung*, Math. Ann. 122 (1951), S. 343-368.

[10] S. L. Sobolew, *Einige Anwendungen der Funktionalanalysis auf Gleichungen der mathematischen Physik*, Berlin 1964.

Reçu par la Rédaction le 11. 2. 1969

STUDIA MATHEMATICA, T. XXXIV. (1970)

Determinant system for composite of generalized Fredholm operators

by

A. BURACZEWSKI (Warszawa)

1. Introduction. The main purpose of this paper is to give a general formula for the determinant system for composite of two generalized Fredholm operators provided their determinant systems are known.

Let Ω and X be two fixed linear spaces over the real or complex field $\mathfrak{F}$. The letters x, y, z will denote elements of X, the letters ω, η, ζ elements of Ω and the letters a, b, c numbers of $\mathfrak{F}$. Every mapping into $\mathfrak{F}$ will be called a *functional*. Following Sikorski [3], we assume that Ω and X are *conjugate*, i.e. there exists a bilinear functional defined on $\Omega\times X$ whose value at a point (ω, x) is denoted by ωx and which satisfies two conditions:

(a) if $\omega x=0$ for every $\omega\in\Omega$, then $x=0$;

(a′) if $\omega x=0$ for every $x\in X$, then $\omega=0$.

If $\omega x=0$, then ω, x are said to be *orthogonal*. In the following $\mathfrak{A}$ will denote the class of all bilinear functionals on $\Omega\times X$ such that:

(b) For every fixed $x\in X$ there exists a $y\in X$ such that $\omega Ax=\omega y$ for every $\omega\in\Omega$ (this unique element y will be denoted by Ax).

(b′) For every fixed $\omega\in\Omega$ there exists an $\eta\in\Omega$ such that $\omega Ax=\eta x$ for every $x\in X$ (this unique element η will be denoted by ωA).

Thus, every bilinear functional $A\in\mathfrak{A}$ can simultaneously be interpreted as the endomorphism $y=Ax$ in X and the endomorphism $\eta=\omega A$ in Ω. $\mathfrak{A}$ is a ring with the following definition of multiplication: if $A_1, A_2\in\mathfrak{A}$, then by A_1A_2 we understand the bilinear functional $\omega(A_1A_2)x=(\omega A_1)(A_2x)$. It is evident that the product A_1A_2 interpreted as an endomorphism in X (in Ω) is the composite of the endomorphisms A_2, A_1 in X (A_1, A_2 in Ω). The bilinear functional $I\in\mathfrak{A}$ such that $\omega Ix=\omega x$, will be called the *identity bilinear functional*. By definition, $Ix=x$ for each $x\in X$ and $\omega I=\omega$ for each $\omega\in\Omega$.

If x_0 and ω_0 are fixed, then the bilinear functional K defined by the formula $\omega Kx=\omega x_0\cdot\omega_0 x$ is called *one-dimensional* and is denoted by $x_0\cdot\omega_0$. Any finite sum of one-dimensional bilinear functionals is called a *finite-dimensional bilinear functional*.

2. Some definitions and properties of Fredholm functionals. For any $A\in\mathfrak{A}$ we introduce the following notation:

$$\text{(1)}\qquad \begin{aligned} &\mathscr{Y}(A) = \{\omega A\colon \omega\in\Omega\}, &&\mathscr{Z}(A) = \{\omega\colon \omega A = 0,\ \omega\in\Omega\},\\ &Y(A) = \{Ax\colon x\in X\}, &&Z(A) = \{x\colon Ax = 0,\ x\in X\}.\end{aligned}$$

The following definitions and basic properties of generalized Fredholm bilinear functions will be used, which can be found in [2]. A bilinear functional (endomorphism) $A\in\mathfrak{A}$ is said to be a *generalized Fredholm bilinear functional* if:

(c) $\dim\mathscr{Z}(A) = m'$, $\dim Z(A) = n'$;

(c_1) the equation $Ax = x_0$ has a solution x if and only if $\omega x_0 = 0$ for every $\omega\in\mathscr{Z}(A)$;

(c_2) the equation $\omega A = \omega_0$ has a solution ω if and only if $\omega_0 x = 0$ for every $x\in Z(A)$.

The integers $r = \min(m', n')$ and $d = n'-m'$ will be called the *order* and *defect* of A, respectively. A bilinear functional B is said to be a *quasi-inverse* of A if $ABA = A$ and $BAB = B$.

Instead of using the notation $D_0, D_1, \ldots, D_n, \ldots$, as in [2], for the *determinant system* of order r and defect d, we shall use this notation:

$$D_0^d,\ D_1^{d+1},\ \ldots,\ D_m^{m+d},\ \ldots \qquad \text{if } d\geqslant 0,$$

and

$$D_{-d}^0,\ D_{-d+1}^1,\ \ldots,\ D_{-d+n}^n,\ \ldots \qquad \text{if } d< 0,$$

or, more briefly, $\{D_m^n\}$ in both cases, $D_m^n\begin{pmatrix}\omega_1, \ldots, \omega_n\\ x_1, \ldots, x_m\end{pmatrix}$ being the value of $(n+m)$-linear functional D_m^n at a point $(\omega_1, \ldots, \omega_n, x_1, \ldots, x_m)\in\Omega^n\times X^m$.

(i) (cf. [2]) *Every generalized Fredholm operator A of order r and defect d has a determinant system also of order r and defect d.*

Moreover, if $\{D_m^n\}$ and $\{\overline{D}_m^n\}$ are determinant systems for A, then there exists a constant $k\neq 0$ such that $\{\overline{D}_m^n\} = \{kD_m^n\}$.

(ii) *Let $\{D_m^n\}$ be a determinant system for the generalized Fredholm bilinear functional A, $r = \min(m', n')$, $d = n'-m'$ being the order and defect of A, respectively. Let $\eta_1, \ldots, \eta_{n'}$ and $y_1, \ldots, y_{m'}$ be points such that*

$$\delta := D_{m'}^{n'}\begin{pmatrix}\eta_1, \ldots, \eta_{n'}\\ y_1, \ldots, y_{m'}\end{pmatrix}\neq 0.$$

Then the elements $\zeta_1, \ldots, \zeta_{m'}$ and $z_1, \ldots, z_{n'}$ determined by the formulae

$$\text{(2)}\qquad \zeta_i x = \frac{1}{\delta} D_{m'}^{n'}\begin{pmatrix}\eta_1, \ldots\ldots\ldots\ldots\ldots, \eta_{n'}\\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_{m'}\end{pmatrix} \quad \textit{for every } x\in X$$

and

$$\text{(3)}\qquad \omega z_j = \frac{1}{\delta} D_{m'}^{n'}\begin{pmatrix}\eta_1, \ldots, \eta_{j-1}, \omega, \eta_{j+1}, \ldots, \eta_{n'}\\ y_1, \ldots\ldots\ldots\ldots\ldots, y_{m'}\end{pmatrix} \quad \textit{for every } \omega\in\Omega$$

form complete systems of solutions of $\omega A = 0$ and $Ax = 0$, respectively. A bilinear functional B, defined by

$$\text{(4)}\qquad \omega Bx = \frac{1}{\delta} D_{m'+1}^{n'+1}\begin{pmatrix}\omega, \eta_1, \ldots, \eta_{n'}\\ x, y_1, \ldots, y_{m'}\end{pmatrix}$$

is a quasi-inverse of A.

Moreover

$$\text{(5)}\qquad AB = I - \sum_{i=1}^{m'} y_i\cdot\zeta_i \quad\textit{and}\quad BA = I - \sum_{j=1}^{n'} z_j\cdot\eta_j,$$

where $\eta_i y_k = \delta_{ik}$ and $\eta_p z_j = \delta_{pj}$ $(i, k = 1, \ldots, m';\ p, j = 1, \ldots, n')$.

(iii) *Let $\zeta_1, \ldots, \zeta_{m'}$ and $z_1, \ldots, z_{n'}$ be bases of $\mathscr{Z}(A)$ and $Z(A)$ respectively, and let $B\in\mathfrak{A}$ be any quasi-inverse of A. The sequence $\{D_m^n\}$ defined by the formulae*

$$\text{(6)}\qquad D_m^n\begin{pmatrix}\omega_1, \ldots, \omega_n\\ x_1, \ldots, x_m\end{pmatrix} = 0 \quad (n = \max(d, 0), \ldots, n'-1;\ \ m = n-d),$$

$$\text{(7)}\qquad D_{m'}^{n'}\begin{pmatrix}\omega_1, \ldots, \omega_{n'}\\ x_1, \ldots, x_{m'}\end{pmatrix} = \begin{vmatrix}\omega_1 z_1 & \ldots & \omega_1 z_{n'}\\ \cdot & \cdots & \cdot\\ \omega_{n'} z_1 & \ldots & \omega_{n'} z_{n'}\end{vmatrix}\cdot\begin{vmatrix}\zeta_1 x_1 & \ldots & \zeta_1 x_m\\ \cdot & \cdots & \cdot\\ \zeta_{m'} x_1 & \ldots & \zeta_{m'} x_{m'}\end{vmatrix},$$

and for $k = 1, 2, \ldots$

$$\text{(8)}\qquad \begin{aligned} &D_{m'+k}^{n'+k}\begin{pmatrix}\omega_1, \ldots, \omega_{n'+k}\\ x_1, \ldots, x_{m'+k}\end{pmatrix}\\ &\quad = \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\operatorname{sgn}\mathfrak{q}\begin{vmatrix}\omega_{p_1} Bx_{q_1} & \ldots & \omega_{p_1} Bx_{q_k}\\ \cdot & \cdots & \cdot\\ \omega_{p_k} Bx_{q_1} & \ldots & \omega_{p_k} Bx_{q_k}\end{vmatrix} D_{m'}^{n'}\begin{pmatrix}\omega_{p_{k+1}}, \ldots, \omega_{p_{k+n'}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+m'}}\end{pmatrix}\end{aligned}$$

is a determinant system for A, where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{k+n'})$ and $\mathfrak{q} = (q_1, \ldots, q_{k+m'})$ of the integers $1, \ldots, k+n'$ and $1, \ldots, k+m'$, respectively such that

$$\text{(9)}\qquad \begin{aligned} &p_1 < p_2 < \ldots < p_k, && p_{k+1} < \ldots < p_{k+n'},\\ &q_1 < q_2 < \ldots < q_k, && q_{k+1} < \ldots < q_{k+m'}.\end{aligned}$$

The determinant system $\{D_m^n\}$ for A defined by (6), (7) *and* (8) *does not depend on the choice of B.*

It is easy to verify the relationship between two arbitrary quasi-inverses B and C of A,

$$C = B + \sum_{i=1}^{n'} z_i \cdot \sigma_i + \sum_{j=1}^{m'} u_j \cdot \zeta_j + \sum_{i=1}^{n'} \sum_{j=1}^{m'} (\sigma_i A u_j) z_i \cdot \zeta_j, \tag{10}$$

where $\sigma_i \in \mathscr{Y}(B)$ $(i = 1, \ldots, n')$ and $u_j \in Y(B)$ $(j = 1, \ldots, m')$.

I now recall the formula for the generalized expansion of a classical determinant which will be used below:

(11)
$$\begin{vmatrix} a_{1,1} \ldots a_{1,k+n} \\ \cdots\cdots\cdots \\ a_{1,k+n} \ldots a_{k+n,k+n} \end{vmatrix} = \sum_{\mathfrak{p}} \operatorname{sqn} \mathfrak{p} \begin{vmatrix} a_{p_1,1} \ldots a_{p_1,k} \\ \cdots\cdots \\ a_{p_k,1} \ldots a_{p_k,k} \end{vmatrix} \cdot \begin{vmatrix} a_{p_{k+1},k+1} \ldots a_{p_{k+1},k+n} \\ \cdots\cdots\cdots \\ a_{p_{k+n},k+1} \ldots a_{p_{k+n},k+n} \end{vmatrix},$$

where $\sum_{\mathfrak{p}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{k+n})$ of the integers $1, \ldots, k+n$ such that $p_1 < p_2 < \ldots < p_k$, $p_{k+1} < p_{k+2} < \ldots < p_{k+n}$.

We shall use the following notation throughout the paper. A_1 and A_2 will denote fixed bilinear generalized Fredholm operators of orders $r' = \min(m', n')$, $r'' = \min(m'', n'')$ and defects $d' = n' - m'$ and $d'' = n'' - m''$, respectively. Let $\{D_m^n\}$ and $\{T_m^n\}$ be determinant systems for A_1 and A_2, respectively. Using formulae (2) and (3) we can find a basis $z_1', \ldots, z_{n'}'$ of $Z(A_1)$, a basis $\zeta_1', \ldots, \zeta_{m'}'$ of $\mathscr{Z}(A_1)$, a basis $z_1'', \ldots, z_{m''}''$ of $Z(A_2)$ and a basis $\zeta_1'', \ldots, \zeta_{m''}''$ of $\mathscr{Z}(A_2)$. We have the following

Lemma 1. *Let B_1 and B_2 be arbitrary quasi-inverses of A_1 and A_2, let $z_1', \ldots, z_{\bar{n}'}'$ and $\zeta_1'', \ldots, \zeta_{\bar{m}''}''$ be bases of $Z(A_1) \cap Y(A_2)$ and $\mathscr{Z}(A_2) \cap \mathscr{Y}(A_1)$ respectively. Then the elements,*

$$B_2 z_1', \ldots, B_2 z_{\bar{n}'}', \quad z_1'', \ldots, z_{n''}'', \tag{12}$$

$$\zeta_1', \ldots, \zeta_{m'}', \quad \zeta_1'' B_1, \ldots, \zeta_{\bar{m}''}'' B_1' \tag{12'}$$

are solutions of the equations $A_1 A_2 x = 0$ and $\omega A_1 A_2 = 0$, respectively.

It is easy to show that every solution of $A_1 A_2 x = 0$, $\omega A_1 A_2 = 0$ is a linear combination of the elements given by formulae (12), (12′), resp. Furthermore, there exist elements $\eta_1', \ldots, \eta_{n'}'$ and $y_1'', \ldots, y_{m''}''$ such that

$$B_1 A_1 = I - \sum_{i=1}^{n'} z_i' \cdot \eta_i', \qquad A_2 B_2 = I - \sum_{i=1}^{m''} y_i'' \cdot \zeta_i'',$$

where $\eta_i' z_j' = \delta_{ij}$ $(i, j = 1, \ldots, n')$, $\zeta_i'' y_j'' = \delta_{ij}$ $(i, j = 1, \ldots, m'')$. It follows immediately from these formulae that $\eta_i' A_2 B_2 z_j' = \delta_{ij}$ $(i, j = 1, \ldots, \bar{n}')$, $\zeta_i'' A_1 B_1 y_{ij}'' = \delta_{ij}$ $(i, j = 1, \ldots, \bar{m}'')$ and, since no solution of the homogeneous Fredholm equation belongs to the range of its quasi-inverse, the linear independence of the elements (12) and (12′) has been proved.

Since $d(A_1 A_2) = d(A_1) + d(A_2) = d' + d''$, [1], we have the relationship $\bar{n}' + n'' - (m' + \bar{m}'') = d' + d''$ from which one obtains $n' - \bar{n}' = m'' - \bar{m}''$. Putting $s = n' - \bar{n}'$ we easily assert that

$$r = r(A_1 A_2) = \min(m^*, n^*), \tag{13}$$

where $n^* = n' + n'' - s$ and $m^* = m' + m'' - s$.

Since $\bar{n}' = \dim\big(Z(A_1) \cap Y(A_2)\big)$, $m'' = \dim\big(\mathscr{Z}(A_2) \cap \mathscr{Y}(A_1)\big)$ and $s = n' - \bar{n}' = m'' - \bar{m}''$, we can denote by $w_1, \ldots, w_s$ and $\psi_1, \ldots, \psi_s$ linearly independent solutions of $A_1 x = 0$ and $\omega A_2 = 0$ respectively, such that $w_j \notin Y(A_2)$ and $\psi_j \notin \mathscr{Y}(A_1)$ for $j = 1, \ldots, s$. We also assume that $\psi_1, \ldots, \psi_s$ and $w_1, \ldots, w_s$ form a biorthogonal system, i.e.

$$\psi_i w_j = \delta_{ij} \quad (i, j = 1, \ldots, s). \tag{14}$$

Since $\psi_j \notin \mathscr{Y}(A_1)$ and $w_j \notin Y(A_2)$, $j = 1, \ldots, s$, we can easily prove that there exist points $\eta_1', \ldots, \eta_{\bar{n}'}' \in \Omega$, $y_1', \ldots, y_{m'}' \in X$ and $\eta_1'', \ldots, \eta_{n''}'' \in \Omega$, $y_1'', \ldots, y_{\bar{m}''}'' \in X$ such that $\eta_1', \ldots, \eta_{\bar{n}'}'$ are orthogonal to all $y_1'', \ldots, y_{\bar{m}''}''$ and

$$\delta' = D_{m'}^{n'} \begin{pmatrix} \eta_1', \ldots, \eta_{\bar{n}'}', \psi_1, \ldots, \psi_s \\ y_1', \ldots\ldots\ldots, y_{m'}' \end{pmatrix} \neq 0,$$

$$\delta'' = T_{m''}^{n''} \begin{pmatrix} \eta_1'', \ldots\ldots\ldots, \eta_{n''}'' \\ y_1'', \ldots, y_{\bar{m}''}'', w_1, \ldots, w_s \end{pmatrix} \neq 0.$$

With the above assumptions we have the following

Lemma 2. *If B_1 and B_2 are quasi-inverses of A_1 and A_2 respectively, defined by*

$$\omega B_1 x = \frac{1}{\delta'} D_{m'+1}^{n'+1} \begin{pmatrix} \omega, \eta_1', \ldots, \eta_{\bar{n}'}', \psi_1, \ldots, \psi_s \\ x, y_1', \ldots\ldots\ldots, y_{m'}' \end{pmatrix} \tag{15}$$

and

$$\omega B_2 x = \frac{1}{\delta''} T_{m''+1}^{n''+1} \begin{pmatrix} \omega, \eta_1'', \ldots\ldots\ldots, \eta_{n''}'' \\ x, y_1'', \ldots, y_{\bar{m}''}'', w_1, \ldots, w_s \end{pmatrix}, \tag{15'}$$

then $B_2 B_1$ is a quasi-inverse of $A_1 A_2$.

In the same manner as in (ii) we can obtain complete systems of linearly independent solutions $\zeta_1', \ldots, \zeta_{m'}'$ of $\omega A_1 = 0$, $z_1', \ldots, z_{n'}'$ of $A_1 x = 0$ and $\zeta_1'', \ldots, \zeta_{m''}''$ of $\omega A_2 = 0$, $z_1'', \ldots, z_{n''}''$ of $A_2 x = 0$. Therefore, by (5), we obtain

$$A_1 A_2 B_2 B_1 = I - \sum_{i=1}^{m'} y_i' \cdot \zeta_i' - \sum_{i=1}^{\bar{m}''} A_1 y_i'' \cdot \zeta_i'' B_1. \tag{16}$$

Since $\zeta_1'', \ldots, \zeta_{\bar{m}''}''$ are orthogonal to $w_1, \ldots, w_s$, we can easily verify that $\zeta_1'', \ldots, \zeta_{\bar{m}''}'' \in \mathscr{Y}(A_1)$.

Hence multiplying (16) on the right-hand side by A_1A_2, we obtain $A_1A_2B_2B_1A_1A_2 = A_1A_2$. Since the determinant system is determined up to a constant scalar, it can be easily shown, by virtue of (7), that $z'_{\bar{n}'+j} = w_j$ $(j = 1, \ldots, s)$. Hence multiplying (16) on the left-hand side by B_2B_1, then applying (5) to B_1A_1 and remembering that y''_i $(i = 1, \ldots, \bar{m}'')$ are orthogonal to all η'_j $(j = 1, \ldots, \bar{n}')$ we obtain $B_2B_1A_1A_2B_2B_1 = B_2B_1$. This completes the proof.

3. Proof of the main theorem.

THEOREM. *Let $\{D^n_m\}$ and $\{T^n_m\}$ be determinant systems for A_1 and A_2 of order $r' = \min(n', m')$, $r'' = \min(n'', m'')$ and defects d' and d'', respectively. Let C_1 and C_2 be arbitrary quasi-inverses of A_1 and A_2 and let $\psi_1, \ldots, \psi_s$ and $w_1, \ldots, w_s$ be complete systems of solutions of $\omega A_2 = 0$ and $A_1 x = 0$ respectively such that $\psi_i \notin \mathscr{Y}(A_1)$, $w_i \notin Y(A_2)$ and $\psi_i w_j = \delta_{ij}$ $(i = 1, 2, \ldots, s)$, where $s = n' - \dim\big(Z(A_1) \cap Y(A_2)\big) = m'' - \dim\big(\mathscr{Z}(A_2) \cap \mathscr{Y}(A_1)\big)$. The sequence $\{S^n_m\}$ defined by the formulae*

$$S^n_m\begin{pmatrix}\omega_1, \ldots, \omega_n\\ x_1, \ldots, x_m\end{pmatrix} = 0 \qquad (n = \max(d'+d'', 0), \ldots, n'+n''-s-1;$$
$$m = n-(d'+d''))$$

and, for $n \geqslant n'+n''-s$,

$$(17)\qquad S^n_m\begin{pmatrix}\omega_1, \ldots, \omega_n\\ x_1, \ldots, x_m\end{pmatrix}$$
$$= \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\operatorname{sgn}\mathfrak{q}\, D^{n-n''+s}_{m-m''+s}\begin{pmatrix}\omega_{p_1}C_2, \ldots, \omega_{p_{n-n''}}C_2, \psi_1, \ldots, \psi_s\\ x_{q_1}, \ldots\ldots\ldots\ldots, x_{q_{m-m''+s}}\end{pmatrix}\times$$
$$\times T^{n''}_{m''}\begin{pmatrix}\omega_{p_{n-n''+1}}, \ldots\ldots\ldots\ldots\ldots, \omega_{p_n}\\ C_1x_{q_{m-m''+s+1}}, \ldots, C_1x_{q_m}, w_1, \ldots, w_s\end{pmatrix},$$

where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_n)$ and $\mathfrak{q} = (q_1, \ldots, q_m)$ of the integers $1, \ldots, n$ and $1, \ldots, m$, respectively, such that

$$(18)\qquad \begin{gathered} p_1 < p_2 < \ldots < p_{n-n''}, \qquad p_{n-n''+1} < \ldots < p_n;\\ q_1 < q_2 < \ldots < q_{m-m''+s}, \qquad q_{m-m''+s+1} < \ldots < q_m \end{gathered}$$

is a determinant system for A_1A_2 which depends neither on C_1 and C_2 nor on the points $\psi_1, \ldots, \psi_s$ and $w_1, \ldots, w_s$.

Proof. Let B_1 and B_2 be quasi-inverses defined by (15) and (15′) for A_1 and A_2, i.e. B_1B_2 is a quasi-inverse for A_1A_2. Since the determinant system is determined up to a constant factor $k \neq 0$, we can assume that the determinant systems $\{D^n_m\}$ and $\{T^n_m\}$ for A_1 and A_2, respectively are defined in a similar way as in (iii).

By (iii) and Lemma 1 the sequence $\{S^n_m\}$ defined by

$$(19)\qquad S^n_m = 0 \quad \text{for } n = \max(d'+d'', 0), \ldots, n'+n''-s-1,$$

$$(20)\qquad S^{n^*}_{m^*}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*}\\ x_1, \ldots, x_{m^*}\end{pmatrix}$$
$$= \begin{vmatrix} \omega_1B_2z'_1, \ldots, \omega_1B_2z'_{\bar{n}'}, & \omega_1z''_1, \ldots, \omega_1z''_{n''}\\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\\ \omega_{n^*}B_2z'_1, \ldots, \omega_{n^*}B_2z'_{\bar{n}'}, & \omega_{n^*}z''_1, \ldots, \omega_{n^*}z''_{n''}\end{vmatrix} \cdot \begin{vmatrix} \zeta'_1x_1, & \ldots, \zeta'_1x_{m^*}\\ \ldots\ldots & \ldots\ldots\\ \zeta'_{m'}x_1, & \ldots, \zeta'_{m'}x_{m^*}\\ \zeta''_1B_1x_1, & \ldots, \zeta''_1B_1x_{m^*}\\ \ldots\ldots & \ldots\ldots\\ \zeta''_{\bar{m}''}B_1x_1, & \ldots, \zeta''_{\bar{m}''}B_1x_{m^*}\end{vmatrix}$$

and for $k = 1, 2, \ldots$

$$(21)\qquad S^{n^*+k}_{m^*+k}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*+k}\\ x_1, \ldots, x_{m^*+k}\end{pmatrix}$$
$$= \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\operatorname{sgn}\mathfrak{q} \begin{vmatrix}\omega_{p_1}B_2B_1x_{q_1}, \ldots, \omega_{p_1}B_2B_1x_{q_k}\\ \ldots\ldots\ldots\ldots\ldots\\ \omega_{p_k}B_2B_1x_{q_1}, \ldots, \omega_{p_k}B_2B_1x_{q_k}\end{vmatrix} S^{n^*}_{m^*}\begin{pmatrix}\omega_{p_{k+1}}, \ldots, \omega_{p_{k+n^*}}\\ x_{q_{k+1}}, \ldots, x_{q_{k+m^*}}\end{pmatrix}$$

is a determinant system for A_1A_2 of order $r = \min(n^* = n'+n''-s, m^* = m'+m''-s)$ and defect $d'+d''$, where $\sum_{\mathfrak{p},\mathfrak{q}}$ is extended over all permutations $\mathfrak{p} = (p_1, \ldots, p_{k+n^*})$ and $\mathfrak{q} = (q_1, \ldots, q_{k+m^*})$ of the integers $1, \ldots, k+n^*$ and $1, \ldots, k+m^*$ respectively such that

$$(22)\qquad \begin{gathered} p_1 < p_2 < \ldots < p_k, \qquad p_{k+1} < p_{k+2} < \ldots < p_{k+n^*};\\ q_1 < q_2 < \ldots < q_k, \qquad q_{k+1} < q_{k+2} < \ldots < q_{k+m^*}.\end{gathered}$$

Let $\bar{C}_1$ be any other quasi-inverse of A_1 defined as follows:

$$(23)\qquad \bar{C}_1 = B_1 + \sum_{i=1}^{\bar{n}'} z'_i\cdot\sigma'_i + \sum_{j=1}^{m'} u'_j\cdot\zeta'_j + \sum_{i=1}^{\bar{n}'}\sum_{j=1}^{m'}(\sigma'_iAu'_j)z'_i\cdot\zeta'_j,$$

where B_1 is given by (15), $\sigma'_i \in \mathscr{Y}(B_1)$, $i = 1, \ldots, \bar{n}'$ and $u'_j \in Y(B_1)$, $j = 1, \ldots, m'$. Thus it is easy to see that

$$(24)\qquad \psi_i\bar{C}_1 = 0 \quad (i = 1, \ldots, s).$$

Now let C_2 be any fixed quasi-inverse of A_2, i.e.

$$(25)\qquad C_2 = B_2 + \sum_{i=1}^{n''} z''_i\cdot\sigma''_i + \sum_{j=1}^{m''} u''_j\cdot\zeta''_j + \sum_{i=1}^{n''}\sum_{j=1}^{m''}(\sigma''_iA_2u''_j)z''_i\cdot\zeta''_j,$$

where B_2 is given by (15′), $\sigma''_i \in \mathscr{Y}(B)$, $u''_j \in Y(B_2)$, $i = 1, \ldots, n''$ and $j = 1, \ldots, m''$. We also assume that $\zeta''_{\bar{m}''+i} = \psi_i$, $i = 1, \ldots, s$. In general, $C_2\bar{C}_1$ is not a quasi-inverse of A_1A_2, yet we can take $\bar{C}_1$ and C_2 in place of B_1 and B_2 in (20) and (21).

Thus bearing in mind that $z_1', \ldots, z_{\bar n'}'$ are orthogonal to all $\zeta_1'', \ldots, \zeta_{\bar m''}''$ and ζ_j'' $(j = 1, \ldots, \bar m'')$ are orthogonal to all $z_1', \ldots, z_{n'}'$ we assert that formula (20) remains the same.

As far as (21) is concerned we notice that $C_2\bar C_1 = B_2B_1 + K$, where K is a finite-dimensional operator which we may write as

$$K = \sum_{i=1}^{\bar n'} B_2 z_i'\cdot \varrho_i + \sum_{i=1}^{n''} z_i''\cdot \varrho_{\bar n'+i}'' + \sum_{i=1}^{m'} s_i\cdot \zeta_i' + \sum_{i=1}^{\bar m''} s_{m'+i}\cdot \zeta_i'' B_1.$$

Thus, replacing B_2B_1 by $C_2\bar C_1$, writing the second factor as the product in which there is no need to replace B_1 by $\bar C_1$ and B_2 by C_2 (since its value remains the same) and then using (11), we conclude that the term K can be removed so that we shall come back to the same formula (21). Our main purpose is now to express the determinant system $\{S_m^n\}$, for A_1A_2 defined by (19), (20) and (21), in terms of the determinant systems $\{D_m^n\}$ and $\{T_m^n\}$ for A_1 and A_2, respectively.

Starting from (20) with $\bar C_1$ and C_2 in place of B_1 and B_2, remembering that $\psi_i z_{\bar n'+j} = \delta_{ij}$, $\zeta_{\bar m'+j}'' w_i = \delta_{ij}$ $(i, j = 1, \ldots, s)$, and by (11), we obtain

$$S_{m^*}^{n^*}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*}\\ x_1, \ldots, x_{m^*}\end{pmatrix} = \sum_{\mathfrak p,\mathfrak q}\operatorname{sgn}\mathfrak p\operatorname{sgn}\mathfrak q\begin{vmatrix}\omega_{p_1}C_2z_1', \ldots, & \omega_{p_1}C_2z_{\bar n'}'\\ \omega_{p_{\bar n'}}C_2z_1', \ldots, & \omega_{p_{\bar n'}}C_2z_{\bar n'}'\end{vmatrix}\cdot\begin{vmatrix}\zeta_1'x_{q_1}, \ldots, & \zeta_1'x_{q_{m'}}\\ \cdots & \cdots\\ \zeta_{m'}'x_{q_1}, \ldots, & \zeta_{m'}'x_{q_{m'}}\end{vmatrix}\times$$

$$\times\begin{vmatrix}\omega_{p_{\bar n'+1}}z_1'', \ldots, & p_{\bar n'+1}z_{m''}''\\ \cdots & \cdots\\ \omega_{p_{n^*}}z_1'', \ldots, & \omega_{p_{n^*}}z_{m''}''\end{vmatrix}\cdot\begin{vmatrix}\zeta_1''\bar C_1x_{q_{m'+1}}, \ldots, & \zeta_1''\bar C_1x_{q_{m^*}}\\ \cdots & \cdots\\ \zeta_{\bar m''}''\bar C_1x_{q_{m'+1}}, \ldots, & \zeta_{\bar m''}''\bar C_1x_{q_{m^*}}\end{vmatrix}$$

$$= \sum_{\mathfrak p,\mathfrak q}\operatorname{sgn}\mathfrak p\operatorname{sgn}\mathfrak q\begin{vmatrix}\omega_{p_1}C_2z_1', \ldots, \omega_{p_1}C_2z_{n'}'\\ \cdots\cdots\cdots\\ \omega_{p_{\bar n'}}C_2z_1', \ldots, \omega_{p_{\bar n'}}C_2z_{n'}'\\ \psi_1z_1', \ldots, \psi_1z_{n'}'\\ \cdots\cdots\\ \psi_sz_1', \ldots, \psi_sz_{n'}'\end{vmatrix}\times$$

$$\times\begin{vmatrix}\zeta_1'x_{q_1}, \ldots, & \zeta_1'x_{q_{m'}}\\ \cdots & \cdots\\ \zeta_{m'}'x_{q_1}, \ldots, & \zeta_{m'}'x_{q_{m'}}\end{vmatrix}\cdot\begin{vmatrix}\omega_{p_{\bar n'+1}}z_1'', \ldots, & \omega_{p_{\bar n'+1}}z_{n''}''\\ \cdots & \cdots\\ \omega_{p_{n^*}}z_1'', \ldots, & \omega_{p_{n^*}}z_{n''}''\end{vmatrix}\times$$

$$\times\begin{vmatrix}\zeta_1''\bar C_1x_{q_{m'+1}}, \ldots, \zeta_1''\bar C_1x_{q_{m^*}}, & \zeta_1''w_1, \ldots, & \zeta_1''w_s\\ \cdots & \cdots & \cdots\\ \zeta_{m''}''\bar C_1x_{q_{m'+1}}, \ldots, \zeta_{m''}''\bar C_1x_{q_{m^*}}, & \zeta_{m''}''w_1, \ldots, & \zeta_{m''}''w_s\end{vmatrix}.$$

Hence, by (7),

$$S_{m^*}^{n^*}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*}\\ x_1, \ldots, x_{m^*}\end{pmatrix} \tag{26}$$

$$= \sum_{\mathfrak p,\mathfrak q}\operatorname{sqn}\mathfrak p\cdot\operatorname{sqn}\mathfrak q\, D_{m'}^{n'}\begin{pmatrix}\omega_{p_1}C_2, \ldots, \omega_{p_{\bar n'}}C_2, \psi_1, \ldots, \psi_s\\ x_{q_1}, \ldots\ldots\ldots\ldots, x_{q_{m'}}\end{pmatrix}\times$$

$$\times T_{m''}^{n''}\begin{pmatrix}\omega_{p_{\bar n'+1}}, \ldots\ldots\ldots\ldots, \omega_{p_{n^*}}\\ \bar C_1x_{q_{m'+1}}, \ldots, \bar C_1x_{q_{m^*}}, w_1, \ldots, w_s\end{pmatrix},$$

where $\sum_{\mathfrak p,\mathfrak q}$ is extended over all permutations $\mathfrak p = (p_1, \ldots, p_{n^*})$ and $\mathfrak q = (q_1, \ldots, q_{n^*})$ respectively such that

$$\begin{aligned}&p_1 < p_2 < \ldots < p_{\bar n'}, \quad p_{\bar n'+1} < \ldots < p_{n^*};\\ &q_1 < q_2 < \ldots < q_{m'}, \quad q_{m'+1} < \ldots < q_{m^*}.\end{aligned}\tag{27}$$

This proves the theorem for $n = n^* = \bar n' + n''$, $m = m^* = m' + \bar m''$.

Starting from (21) with $\bar C_1$ and C_2 in place of B_1 and B_2 and using formula (26), we obtain for $k = 1, 2, \ldots$

$$S_{m^*+k}^{n^*+k}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*+k}\\ x_1, \ldots, x_{q_{m^*+k}}\end{pmatrix} = \sum_{\mathfrak s,\mathfrak t}\operatorname{sgn}\mathfrak s\operatorname{sgn}\mathfrak t\begin{vmatrix}\omega_{s_1}C_2\bar C_1x_{t_1}, \ldots, \omega_{s_1}C_2\bar C_1x_{t_k}\\ \cdots\cdots\cdots\cdots\\ \omega_{s_k}C_2\bar C_1x_{t_1}, \ldots, \omega_{s_k}C_2\bar C_1x_{t_k}\end{vmatrix}\times \tag{28}$$

$$\times\sum_{\mathfrak i,\mathfrak j}\operatorname{sgn}\mathfrak i\cdot\operatorname{sgn}\mathfrak j\, D_{m'}^{n'}\begin{pmatrix}\omega_{s_{k+i}}C_2, \ldots, \omega_{s_{k+i_{\bar n'}}}C_2, \psi_1, \ldots, \psi_s\\ x_{t_{k+j_1}}, \ldots\ldots\ldots\ldots, x_{t_{k+j_{m'}}}\end{pmatrix}\times$$

$$\times T_{m''}^{n''}\begin{pmatrix}\omega_{s_{k+i_{\bar n'+1}}}, \ldots\ldots\ldots\ldots, \omega_{s_{k+i_{\bar n'+n''}}}\\ \bar C_1x_{t_{k+j_{m'+1}}}, \ldots, \bar C_1x_{t_{k+j_{m'+\bar m''}}}, w_1, \ldots, w_s\end{pmatrix},$$

where summations $\sum_{\mathfrak s,\mathfrak t}$ and $\sum_{\mathfrak i,\mathfrak j}$ are extended over the same permutations $\mathfrak s, \mathfrak t$ and $\mathfrak i, \mathfrak j$ as in (22) and (27). Let $\mathfrak p, \mathfrak f$ be arbitrary permutations (of the integers $1, \ldots, k+\bar n'+n''$ and $1, \ldots, k+\bar n'$ respectively) of the form

$$\begin{aligned}&\mathfrak p = (p_1, \ldots, p_{k+\bar n'+n''}), \quad p_1 < \ldots < p_{k+\bar n'}, p_{k+\bar n'+1} < \ldots < p_{k+\bar n'+n''};\\ &\mathfrak f = (f_1, \ldots, f_{k+\bar n'}), \quad f_1 < \ldots < f_k, f_{k+1} < \ldots < f_{k+\bar n'}.\end{aligned}\tag{29}$$

By putting $s_l = p_{f_l}$ $(l = 1, \ldots, k)$, $s_{k+i_l} = p_{f_{k+l}}$ $(l = 1, \ldots, \bar n')$, $s_{k+i_{\bar n''+l}} = p_{k+\bar n'+l}$ $(l = 1, \ldots, n'')$, any permutation

$$(s_1, \ldots, s_k, s_{k+i_1}, \ldots, s_{k+i_{\bar n'+n''}}) \tag{30}$$

of the integers $1, \ldots, k+\bar n''$ appearing in (28) can be expressed in terms of permutations $\mathfrak p$ and $\mathfrak f$, i.e.

$$(p_{f_1}, \ldots, p_{f_{k+\bar n'}}, p_{k+\bar n,+1}, \ldots, p_{k+\bar n'+n''}). \tag{31}$$

Conversely, any two permutations $\mathfrak{p}$ and $\mathfrak{f}$ defined by (29), uniquely define (31) which, in turn, is nothing but the permutation (30). Similarly, on defining permutations $\mathfrak{q}$ and $\mathfrak{g}$ (of the integers $1, \ldots, k'+m'+\overline{m}''$ and $1, \ldots, k+m'$, respectively)

$$\begin{aligned} &\mathfrak{q} = (q_1, \ldots, q_{k+m'+\overline{m}''}), \quad q_1 < \ldots < q_{k+m'}, q_{k+m'+1} < \ldots < q_{k+m'+\overline{m}''},\\ (32) \qquad &\mathfrak{g} = (g_1, \ldots, g_{k+m'}), \quad g_1 < \ldots g_k, g_{k+1} < \ldots < g_{k+m'},\end{aligned}$$

any permutation $(t_1, \ldots, t_k, t_{k+j_1}, \ldots, t_{k+j_{m'+\overline{m}''}})$ appearing in (28) can be written as

$$(q_{g_1}, \ldots, q_{g_{k+m'}}, q_{k+m'+1}, \ldots, q_{k+m'+\overline{m}''}),$$

where

$$q_{g_l} = t_l \quad (l = 1, \ldots, k), \qquad q_{g_{k+l}} = t_{k+j_l} \quad (l = 1, \ldots, m'),$$
$$q_{k+m'+l} = t_{k+j_{m'+l}} \quad (l = 1, \ldots, \overline{m}'').$$

Since the corresponding ± 1 coefficients satisfy the equality

$$\operatorname{sgn}\mathfrak{s}\operatorname{sgn}\mathfrak{i} = \operatorname{sgn}\mathfrak{p}\cdot\operatorname{sgn}\mathfrak{f},$$
$$\operatorname{sgn}\mathfrak{t}\operatorname{sgn}\mathfrak{j} = \operatorname{sgn}\mathfrak{q}\cdot\operatorname{sgn}\mathfrak{g},$$

the sum (28) can be written as

$$(33) \quad S_{m^*+k}^{n^*+k}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*+k}\\ x_1, \ldots, x_{m^*+k}\end{pmatrix}$$
$$= \sum_{\mathfrak{p},\mathfrak{q}} \operatorname{sgn}\mathfrak{p}\operatorname{sgn}\mathfrak{q}\left[\sum_{\mathfrak{f},\mathfrak{g}}\operatorname{sgn}\mathfrak{f}\cdot\operatorname{sgn}\mathfrak{g}\begin{vmatrix}\omega p_{f_1} C_2\overline{C}_1 x_{q_{g_1}}, \ldots, \omega p_{f_1} C_2\overline{C}_1 x_{q_{g_k}}\\ \cdots\cdots\cdots\cdots\\ \omega p_{f_k} C_2\overline{C}_1 x_{q_{g_1}}, \ldots, \omega p_{f_k} C_2\overline{C}_1 x_{q_{g_k}}\end{vmatrix}\times\right.$$
$$\left.\times D_{m'}^{n'}\begin{pmatrix}\omega p_{f_{k+1}} C_2, \ldots, \omega p_{f_{k+\overline{n}'}} C_2, \psi_1, \ldots, \psi_s\\ x_{q_{g_{k+1}}}, \ldots\ldots\ldots, x_{q_{g_{k+m'}}}\end{pmatrix}\right]\times$$
$$\times T_{m''}^{n''}\begin{pmatrix}\omega p_{k+n'+1}, \ldots, \omega p_{k+\overline{n}'+n''}\\ \overline{C}_1 x_{q_{k+m'+1}}, \ldots, \overline{C}_1 x_{q_{k+m'+\overline{m}''}}, w_1, \ldots, w_s\end{pmatrix},$$

where the permutations $\mathfrak{p}, \mathfrak{q}$ and $\mathfrak{f}, \mathfrak{g}$ are the same as in (29) and (32). The expression in the square brackets can be written, by virtue of (8), (24) and the definition of the permutations $\mathfrak{f}, \mathfrak{g}$ as

$$D_{m'+k}^{n'+k}\begin{pmatrix}\omega p_1 C_2, \ldots, \omega p_{\overline{n}'} C_2, \psi_1, \ldots, \psi_s\\ x_{q_1}, \ldots\ldots\ldots\ldots, x_{q_{m'}}\end{pmatrix}.$$

Furthermore, due to skew-symmetricity of $T_{m''}^{n''}$, the quasi-inverse $\overline{C}_1$ of A_1 can be replaced by any quasi-inverse C_1 of A_1, since, by (23), C_1 can be written as

$$C_1 = \overline{C}_1 + \sum_{i=1}^{s} w_i\cdot\sigma'_{\overline{n}'+i} + \sum_{i=1}^{s}\sum_{j=1}^{m'}(\sigma'_{n'+i}A_1 u'_j) w_i\cdot\zeta'_j.$$

Thus the sum (33) becomes

$$(34) \quad S_{m^*+k}^{n^*+k}\begin{pmatrix}\omega_1, \ldots, \omega_{n^*+k}\\ x_1, \ldots, x_{m^*+k}\end{pmatrix}$$
$$= \sum_{\mathfrak{p},\mathfrak{q}}\operatorname{sgn}\mathfrak{p}\cdot\operatorname{sgn}\mathfrak{q}\, T_{m'+k}^{\overline{n}'+k+s}\begin{pmatrix}\omega p_1 C_2, \ldots, \omega p_{\overline{n}'+k} C_2, \psi_1, \ldots, \psi_s\\ x_{q_1}, \ldots\ldots\ldots\ldots, x_{q_{m'+k}}\end{pmatrix}\times$$
$$\times T_{m''}^{n''}\begin{pmatrix}\omega p_{\overline{n}'+k+1}, \ldots\ldots\ldots\ldots, \omega p_{\overline{n}'+n''+k}\\ C_1 x_{q_{m'+k+1}}, \ldots, C_1 x_{q_{m'+\overline{m}''+k}}, w_1, \ldots, w_s\end{pmatrix}.$$

Finally, by putting $n^*+k = n$, $m^*+k = n$ ($n^* = \overline{n}'+n''$, $m^* = m'+\overline{m}''$, $\overline{n}' = \overline{n}-s$, $\overline{m}'' = m''-s$), we obtain formula (17). This completes the proof.

References

[1] F. W. Atkinson, *Normal solubility of linear equations in normed spaces* (Russian), Mat. Sb. 28 (1951), p. 3-14.

[2] A. Buraczewski, *The determinant theory of generalized Fredholm operators*, Studia Math. 22 (1963), p. 265-307.

[3] R. Sikorski, *Determinant systems*, ibidem 18 (1959), p. 161-186.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI, GHANA.

Reçu par la Rédaction le 20. 2. 1969