Hence, by (3.1),

$$\nu(x+y) = \lim_{k\to\infty} \|(x+y)^{2k}\|^{1/2k} \geqslant \frac{1}{2}\|x\| > \frac{1}{\varepsilon}\big(\nu(x)+\nu(y)\big).$$

We summarize the obtained result in the following

THEOREM. *If G is the discrete subgroup of the affine group of the real line as defined in section 1, ε a positive number and x and y the hermitian elements in $l_1(G)$ defined in section 3, then*

$$\varepsilon\nu(x+y) > \nu(x)+\nu(y).$$

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Restrictions and extensions of Fourier multipliers*

by

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Introduction. In this paper we derive certain relations between spaces of Fourier multipliers defined on R^N, Z^N, T^N (definitions and notation are given in section 1). The main result, Theorem (3.7), is for $N = 1$: if $1 < p < \infty$ and $\{m_n\}$ is a multiplier sequence of type (p, p), then the piecewise constant function $m(x) = m_k$ (k is the greatest integer $\leqslant x+\frac{1}{2}$) is a multiplier of type (p, p) for Fourier transforms. In the case $1 \leqslant p \leqslant \infty$, the piecewise linear continuous extension of a sequence of type (p, p) is a function of type (p, p) (see (3.6)).

Sections 2 and 4 contain mostly known results, for which we offer alternate proofs. With one exception the results are due to de Leeuw [3]. Theorem (4.3) is due to Igari [2]. The relations between $M_p^p(R^N)$ and $M_p^p(T^N)$ are given in section 2, and restrictions to Z^N and R^M of elements of $M_p^p(R^N)$ are treated in section 4.

Among the applications of these results are

(i) the Marcinkiewicz multiplier theorem for the line follows from the sequential version (section 4),

(ii) a function m defined on R^N, continuous except at 0, and homogeneous of degree 0 $\big(m(\lambda x) = m(x)$ for $\lambda > 0\big)$ is in $M_p^p(R^N)$ if and only if its restriction to Z^N is a sequence of type (p, p) (section 4).

Questions raised by Professor R. Coifman and Mr. David Shreve led to this work, which has also profited by a comment of Professor Calderón.

1. Preliminaries. We first set down for reference some conventional notation. R^N denotes real N-space, x, y denote points of R^N, with coordinates $x_1, \ldots, x_N, y_1, \ldots, y_N$. $|x| = (x_1^2+\ldots+x_N^2)^{1/2}$, $x\cdot y = x_1y_1+\ldots+x_Ny_N$. $Z^N \subseteq R^N$ is the set of points n with integer coordinates. If $S \subset R^N$, $a \in R$, then $aS = \{as : s \in S\}$, and if $x \in R^N$, then $x+S = \{x+s : s \in S\}$. T^N, the Cartesian product of N copies of the unit circle in the complex

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plane, is identified with $R^N/2\pi Z^N$, and functions on T^N are identified with periodic functions, or with functions defined on $2\pi Q$, where $Q=\{x\in R^N\colon |x_i|<\frac{1}{2}\}$.

For $1\leqslant p<\infty$, $L^p(T^N)$ is identified with $L^p(2\pi Q)$, the space of (equivalence classes of) Lebesgue measurable functions f on $2\pi Q$ for which $\int_{2\pi Q}|f(x)|^p dx$ is finite; $\|f\|_{L^p(2\pi Q)}$ denotes the p-th root of the integral. $L^p\equiv L^p(R^N)$ is defined similarly; $\|f\|_p$ denotes the p-th root of $\int|f(x)|^p dx$ (integrals without limits are taken over all of R^N). l^p is $L^p(Z^N)$ with the counting measure; $\|c\|_{l^p}=\left(\sum|c_n|^p\right)^{1/p}$ (summation with no index is over all $n\in Z^N$).

For $p=\infty$, we define $L^\infty(2\pi Q)$, L^∞, l^∞ in terms of essential suprema

We next give definitions, and recall basic properties of multipliers

Definition. Let $1\leqslant p, q\leqslant\infty$. A sequence $\{m_n\}_{n\in Z^N}$ is a *multiplier* (sequence) *of type* (p,q) if $\sum m_n c_n e^{in\cdot x}$ is the Fourier series of a function in $L^q(2\pi Q)$ whenever $\sum c_n e^{in\cdot x}$ is that of a function in $L^p(2\pi Q)$. The Fourier coefficients c_n of f are defined by

$$c_n=(2\pi)^{-N}\int_{2\pi Q} f(x)e^{-in\cdot x}dx.$$

For more information see [6], Chap. IV, sec. 11.

Notation. $M_p^q(Z^N)$ denotes the linear space of multipliers of type (p,q). For m, m', etc. in $M_p^q(Z^N)$ we let K, K', etc. denote the linear maps assigning to $f\in L^p(2\pi Q)$ the function in $L^q(2\pi Q)$ having the Fourier coefficients $\{m_n c_n\}$, $\{m_n' c_n\}$, etc.

Remarks. By the closed graph theorem, each of these operators is bounded. We norm $M_p^q(Z^N)$ by letting $\|m\|$ denote the operator norm of K.

The space of multipliers then becomes a Banach space. On applying K to $e^{in\cdot k}$, for each $n\in Z^N$, we see that m is a bounded sequence. By the Parseval theorem, $M_q^q(Z^N)=l^\infty$. In case $\sum|m_n|<\infty$, we have $Kf(x)=(2\pi)^{-N}\int K(x-y)f(y)dy$.

It is well-known that every bounded operator of type (p,q) which commutes with translations, corresponds to some $m\in M_p^q(Z^N)$, and conversely. This is true for such operators on L^p and l^p as well, but identification requires the use of tempered distributions. We are interested primarily in finding functions which are multipliers on R^N, and coincide on Z^N with a given multiplier sequence. We will restrict attention to *bounded* functions, and to the cases when p, q are related as follows: $1\leqslant p\leqslant q\leqslant\infty$ with $p=q<\infty$ or p and $q'<\infty$ $(q'=q/(q-1))$. It may be shown that the following definition gives $M_p^q(R^N)\cap L^\infty$, as defined in [1].

Definition. An essentially bounded measurable function m defined on R^N is a *multiplier of type* (p,q) *on* R^N if (and only if) there exists a constant C such that whenever $f\in L^1\cap L^\infty$, $m\hat{f}$ is the Fourier transform (in the L^2-sense) of a function $Kf\in L^q$ such that $\|Kf\|_q\leqslant C\|f\|_p$. We let $\|m\|$ denote the norm of the continuous extension of K (the context will indicate to which space m belongs). Here, $\hat{f}(x)=\int f(y)e^{-ix\cdot y}dy$.

Definition. Let $\{k_n\}$ be a sequence in Z^N. The formal series $m(x)\sim\sum k_n e^{-in\cdot x}$ is a *multiplier of type* (p,q) *on* T^N if there is a constant C such that for every finite sequence $c\in l^p$,

$$\Big(\sum_n\Big|\sum_m k_{n-m}c_m\Big|^q\Big)^{1/q}\leqslant C\Big(\sum|c_n|^p\Big)^{1/p}.$$

Notation. $M_p^q(T^N)$; $\|m\|$ denotes the norm of the operator.

Remarks. The converse of Hölder's inequality shows that $k=\{k_n\}$ is in $l^{p'}\cap l^q$, so the inequality of the definition holds for any $c\in l^p$. If $p\geqslant 2$ or $q\leqslant 2$, the Hausdorff-Young theorem shows that m is a function in $L^r(2\pi Q)$, where $r=\max(p,q')$, with n-th Fourier coefficient k_{-n}. If we regard the trigonometric polynomial $f(x)=\sum c_m e^{-im\cdot x}$ as the Fourier transform of the finite sequence c, we see that $m(x)f(x)$ is the Fourier transform of the sequence $d_n=\sum_m k_{n-m}c_m$.

Properties of multipliers. For the moment we let M_p^q denote any one of the spaces just defined. By use of duality, the appropriate dense subspaces, and Parseval's formula it can be shown that $M_2^2=L^\infty$, and that $M_p^q=M_{q'}^{p'}$, if $p, q'<\infty$ (we will not have occasion to use this result when $p=q=1$). The Riesz interpolation theorem now gives $M_p^p\subseteq M_2^2=L^\infty$. ($M_1^1\subseteq L^\infty$ can be shown directly.)

We will repeatedly use the following properties of multipliers. Proofs can again be made using duality, etc.

(1.1) If $m_k\in M_p^q$, $\|m_k\|\leqslant C$, and $m_k\to m$ pointwise and boundedly as $k\to\infty$, then $m\in M_p^q$ and $\|m\|\leqslant C$.

(1.2) If $m\in M_p^q$ (and is a bounded function), and $h\in L^1$ (or l^1), then the convolution $m*h\in M_p^q$ and $\|m*h\|\leqslant\|m\|\,\|h\|_1$. In the T^N case, the convolution is taken without the factor $(2\pi)^{-N}$.

The following abbreviations will be used in the proofs:

$$\frac{\sin^2 x}{x^2}=\prod_{j=1}^N \sin^2 x_j/x_j^2,\qquad x\in R^N, \tag{1.3}$$

$$r(x)=\prod_{j=1}^N\frac{1}{2}\Big(1-\frac{1}{2}|x_j|\Big)\chi_{4Q}(x),\qquad x\in R^N, \tag{1.4}$$

where χ_S denotes the characteristic function of the set S. In general, A will denote a generic constant depending only on the space dimension.

Finally we mention certain homomorphisms of $M_p^q(R^N)$, $M_p^q(Z^N)$, those of the form $m \to m \circ T$, where T is affine in the appropriate sense.

One uses the operator K corresponding to m if $Tx = Ax + x_0$ is an affine transformation of R^N, $m \in M_p^q(R^N) \cap L^\infty$, to get

$$\|m \circ T\| = |\det A|^{1/q-1/p}\|m\|.$$

We will use translations and dilations.

In the case of Z^N, $\{m_{n-n_0}\}$ has the same norm as m. We will also use the transformations defined for a fixed positive integer k by

$$m \to \{m_{kn}\}_{n \in Z^N} = m'$$

and

$$m \to m'', \quad \text{where } m_n'' = 0, \text{ unless } k|n_i, \text{ for } 1 \leqslant i \leqslant N,$$

in which case we set $m_n'' = m_{(1/k)n}$.

(1.5) Lemma. *If $m \in M_p^q(Z^N)$, so do m', m'', and* $\|m'\| \leqslant \|m\| = \|m''\|$.

Proof. For $f \in L^p(2\pi Q)$ let $Sf(x) = f(kx)$,

$$Tf(x) = k^{-N} \sum_{0 \leqslant n_i < k} f\left(\frac{x+2\pi n}{k}\right).$$

Then $Sf(x) \sim \sum c_n e^{ikn\cdot x}$, $Tf(x) \sim \sum c_{kn} e^{in\cdot x}$. Also $TSf = f$, $\|T\| = 1$, and S is an isometry of $L^p(2\pi Q)$, $1 \leqslant p \leqslant \infty$, for

$$\int_{2\pi Q} |Sf(x)|^p\, dx = \int_{2\pi kQ} |f(x)|^p dx \cdot k^{-N}.$$

Now if we let K, K', K'' denote the operators corresponding to the sequences m, m', m'' we can apply them to trigonometric polynomials, to obtain $K' = TKS$, $K'' = SKT$. Hence K', K'' are bounded, $\|m'\| \leqslant \|m\|$, $\|m''\| \leqslant \|m\|$. Since now $K = TK''S$, we also have $\|m\| \leqslant \|m''\|$.

2. Periodic multipliers. In addition to de Leeuw's result that the periodic elements of $M_p^p(R^N)$ can be identified with those of $M_p^p(T^N)$, we prove that a multiplier on R^N with support in a closed cube can be extended periodically to a multiplier on R^N.

(2.1) Theorem (de Leeuw). *If m is periodic on R^N, then $m \in M_p^p(R^N)$ if and only if m is in $M_p^p(T^N)$ as a function on T^N. The norms are the same.*

Proof. Suppose $m \in M_p^p(T^N)$, with norm A. Then in particular m is bounded. Let

$$K_n = (2\pi)^{-N} \int_{2\pi Q} m(x) e^{in\cdot x} dx.$$

We know from the definition that $\{K_n\} \in l^{p'}$. Suppose $f \in L^p$. Set

$$Kf(x) = \sum K_m f(x-m).$$

Since $\|f\|_p^p = \int_Q \sum |f(x+n)|^p dx$, $\{f(x-n)\} \in l^p$ for almost all x. Thus Kf is defined almost everywhere as the sum of an absolutely convergent series, and

$$\int |Kf(x)|^p dx = \int_Q \sum_n \Big|\sum_m K_m f(x+n-m)\Big|^p dx$$
$$\leqslant \int_Q A^p \sum |f(x+n)|^p dx = A^p \|f\|_p^p.$$

We must show that $(Kf)^\wedge = m\hat{f}$ for $f \in L^1 \cap L^\infty$. Let $s_j(x) = 2jr(2jx)$ (r is the "roof" function (1.4), which has integral 1). Let

$$\sigma_j(x) = \sum_n s_j(x+2\pi n)$$

denote its periodic extension. For large j,

$$\sigma_j * m(x) = \int_{2\pi Q} \sigma_j(y) m(x-y)\, dy$$
$$= \int s_j(y) m(x-y)\, dy = s_j * m(x) \to m(x) \qquad \text{a.e.}$$

The convolution $\sigma_j * m$ is then over T^N, $s_j * m$ over R^N. By (1.2), $\sigma_j * m \in M_p^p(T^N)$ and $\|\sigma_j * m\| \leqslant \|m\| = A$.

Let K_j denote the operator on L^p corresponding to $\sigma_j * m$,

$$K_j f(x) = \sum_n s_{j,-n} K_n f(x-n),$$

where $s_{j,n} = \sin^2 n/2j/(n/2j)^2$ is $(2\pi)^N$ times the n-th Fourier coefficient of σ_j (the factor occurs because $\sigma_j * m$ is $(2\pi)^N$ times the usual convolution on T^N). The absolute convergence of the series for $Kf(x)$, and the convergence boundedly of $s_{j,n}$ to 1 give that $K_j f \to Kf$ pointwise.

Since $f \in L^1 \cap L^\infty$ and $\sum_n |s_{j,n}| < \infty$, we can interchange integral and sum to obtain

$$(K_j f)^\wedge(x) = \sum_n s_{j,-n} K_n e^{-in\cdot x} \hat{f}(x) = (s_j * m)(x)\hat{f}(x).$$

The right-hand terms form a Cauchy sequence in L^2. It follows that $K_j f$ converges in L^2 and that the limit is Kf, as desired.

Next suppose m is periodic, in $M_p^p(R^N)$, with norm B. Let

$$f(x) = \sum c_m \chi_Q(x-n) \quad \text{and} \quad g(x) = \sum d_n \chi_Q(x-n),$$

where c, d are finite sequences. Then

$$\left|(2\pi)^{-N}\int (Kf)^\wedge(x)\overline{g^\wedge}(x)\,dx\right| = \left|\int Kf(x)\bar{g}(x)\,dx\right|$$
$$\leqslant B\|f\|_p\|g\|_{p'} = B\|c\|_{l^p}\|d\|_{l^{p'}}.$$

But

$$(2\pi)^{-N}\int (K_j f)^\wedge\,\bar{\hat{g}}\,dx = (2\pi)^{-N}\sum_{m,n} c_m d_n\int m(x)\,|\chi_Q(x)|^2 e^{i(n-m)\cdot x}\,dx$$

$$= \sum_{m,n} c_m d_n (2\pi)^{-N}\int_{2\pi Q} m(x)\,e^{i(n-m)\cdot x}\sum_{\nu}\frac{\sin^2(x+2\pi\nu)/2}{\big((x+2\pi\nu)/2\big)^2}\,dx = \sum_{m,n} c_m d_n K_{n-m},$$

since

$$\sum_n \frac{\sin^2\pi(x-n)}{\big(\pi(x-n)\big)^2} \equiv 1.$$

Thus $m\in M_p^p(T^N)$ and $\|m\|\leqslant B$.

(2.2) Remark. In [5], Titchmarsh proved that the kernel sequence $\{(n+\frac{1}{2})^{-1}\}$ gives a bounded operator on l^p for $1<p<\infty$. The multiplier in $M_p^p(T^1)$ corresponding to this is a multiple of the periodic function defined for $|t|<\pi$,

$$m(t) = ie^{it/2}\operatorname{sgn} t.$$

Hence $m\in M_p^p(R)$. Now positive dilations leave this space invariant, and since

$$i\operatorname{sgn} t = \lim_{k\to 0} m(kt),$$

we apply (1.1) to get the continuity of the Hilbert transform from that of a discrete analog.

(2.3) Theorem. *Let m in $M_p^q(R^N)$, $p\leqslant q$, be a bounded function with support in $\{x\colon |x_i|\leqslant\pi\}$ and norm C. The periodic extension of m is in $M_p^q(T^N)$ with norm bounded by AC.*

Proof. We need to show that the restriction to Z^N of

$$K(x) = (2\pi)^{-N}\int_{2\pi Q} m(y)\,e^{iy\cdot x}\,dy$$

is a kernel of type (p, q).

Let

$$Ec(x) = \sum c_n\chi_Q(x-n),\qquad (Pf)_n = \int_{Q+n} f(x)\,dx.$$

Then

$$\|Ec\|_{L^p} = \|c\|_{l^p},\qquad \|Pf\|_{l^q}\leqslant \|f\|_{L^q}.$$

If c is a finite sequence, then

$$(PKEc)_n = \int_{Q+n}\int K(x-y)\,Ec(y)\,dy\,dx = \sum_m c_m\int_Q\int_Q K(n-m+x-y)\,dx\,dy$$
$$= \sum_m c_m k_{n-m}.$$

A direct computation shows that

$$k_n = (2\pi)^{-N}\int_{2\pi Q} m(y)\,e^{in\cdot y}\frac{\sin^2 y/2}{(y/2)^2}\,dy\qquad \text{(cf. (1.3))}.$$

Now let $g\in C_0^\infty$ be 1 on $2\pi Q$ and 0 outside $3\pi Q$, and put

$$h(x) = g(x)\frac{(x/2)^2}{\sin^2 x/2}.$$

h is then the Fourier transform of an L^1 function with norm A, so mh has norm no larger than AC, with kernel K_1 satisfying $(PK_1Ec)_n = \sum_m K(n-m)c_m$. Thus

$$\Big(\sum_n\Big|\sum_m K(n-m)\,c_m\Big|^q\Big)^{1/q}\leqslant AC\Big(\sum_n |c_n|^p\Big)^{1/p}.$$

(2.4) Corollary. *If $1<p<\infty$, then the Banach algebras $M_p^p(T^N)$ and $\chi_{2\pi Q}M_p^p(R^N)$ are isomorphic.*

3. Extensions from Z^N.

(3.1) Theorem. *If $p\leqslant q$ and $m\in M_p^q(Z^N)$, then the function*

$$\sum m_n\Big(\prod_{i=1}^{N}\frac{\sin^2\pi(x_i-n_i)}{\big(\pi(x_i-n_i)\big)^2}\Big)$$

is in $M_p^q(R^N)$, and its norm there is no larger than a constant multiple of the norm of m.

Proof. Let K denote the operator determined by m. That is, for $f\in L^p(2\pi Q)$ and periodic,

$$Kf(x)\sim\sum m_n c_n e^{n\cdot x},$$
$$c_n = (2\pi)^{-N}\int_{2\pi Q} f(x)\,e^{-in\cdot x}\,dx.$$

For $n\in Z^N$ let $|n| = |n_1|+\ldots+|n_N|$, and put

$$K_r(x) = \sum m_n r^{|n|}e^{in\cdot x},\qquad 0<r<1.$$

Then by (1.2),

$$\Big(\int_{2\pi Q}\Big|(2\pi)^{-N}\int_{2\pi Q}K_r(x-y)f(y)\,dy\Big|^q\,dx\Big)^{1/q}\leqslant C\Big(\int_{2\pi Q}|f(x)|^p\,dx\Big)^{1/p},$$

where C is the norm of m in $M_p^q(Z^N)$.

Let $S(x)$ denote the periodic function equal to $r(4x/\pi)(4/\pi)^N$ in $2\pi Q$ (cf. (1.4)). The Fourier coefficients S_n of S are non-negative and $\sum S_n = (2/\pi)^N$. Now

$$S(x)K_r(4x) = \sum_n\Big(\sum_\nu S_{n-4\nu}m_\nu r^{|\nu|}\Big)e^{in\cdot x}, \tag{3.2}$$

so by (1.2)

$$\Big(\int_{2\pi Q}\Big|\int_{2\pi Q}S(y)K_r(4y)f(x-y)\,dy\Big|^q\,dx\Big)^{p/q}\leqslant 4^{Np}C^p\int_{2\pi Q}|f(x)|^p\,dx. \tag{3.3}$$

Define $K_r'(x)$ on R^N to be $S(x)K_r(4x)$ in $2\pi Q$ and 0 elsewhere. For f in $L^p(R^N)$ let f_n be the periodic function equal to $f(x+\pi n)$ in $2\pi Q$. Note that for $x\in\pi Q$,

$$K_r'*f(x+\pi n) = \int_{2\pi Q}S(y)K_r(4y)f_n(x-y)\,dy.$$

If we now raise this to the q power (in absolute value), integrate over πQ, apply (3.3) and sum over n, we get, using the sub-additivity of $t^{p/q}$,

$$\|K_r'*f\|_{L^q}^p\leqslant 2^N4^{Np}C^p\int|f(x)|^p\,dx.$$

Thus $\hat K_r'\in M_p^q(R^N)$ and has norm $\leqslant 2^{N/p}4^NC$. Finally (using (1.1))

$$\hat K_r'(4x) = \sum_n r^{|n|}m_n\frac{\sin^2\pi(x-n)}{(\pi(x-n))^2},$$

which converges pointwise to the desired limit as $r\to 1$.

The function S used in the proof of (3.1) was chosen so that the multiplier obtained from m would be an extension.

(3.4) Corollary. *Let S have support in $\pi\bar Q$ and suppose its extension by periodicity from $2\pi Q$ has an absolutely convergent Fourier series. Then*

$$\sum_n m_n\hat S(x-n)$$

is in $M_p^q(R^N)$ when $m\in M_p^q(Z^N)$, and its norm is bounded by $AM\|m\|$, where M is the sum of the moduli of the Fourier coefficients of S.

To prove this, form K_r as before and apply the preceding argument to $S(x)K_r(x)$ (cf. (3.2)).

(3.5) Theorem. *Let $R(x) = (1-|x_1|)\ldots(1-|x_N|)$ in $2Q$ and 0 elsewhere in R^N. If $m\in M_p^q(Z^N)$, where $q\geqslant p$, then $Rm(x) = \sum_n m_nR(x-n)$ is in $M_p^q(R^N)$, and $\|Rm\|\leqslant A\|m\|$.*

Proof. Since R is a product of functions of one variable, it is enough to give the proof for $N=1$. We will apply (3.4) to a dilation of the inverse Fourier transform of R. Consider

$$S(x) = \frac{2}{\pi}\frac{\sin^2 2x}{(2x)^2},$$

let $S_k = \chi_{\pi(k+Q)}S$ in $\pi k+2\pi Q$ be periodic of period 2π. Now

$$S_{kn} = \frac{1}{2\pi}\int_{(k-\frac12)\pi}^{(k+\frac12)\pi}\frac{2}{\pi}\frac{\sin^2 2x}{(2x)^2}e^{-inx}\,dx$$

$$= -\frac{1}{\pi^2n^2}\int_{(k-\frac12)\pi}^{(k+\frac12)\pi}\Big(\frac{2\cos 4x}{x^2}-\frac{2\sin 4x}{x^3}+\frac{3}{2}\frac{\sin^2 2x}{x^4}\Big)e^{-inx}\,dx$$

if $n\neq 0$. Hence it is clear that

$$|S_{kn}|\leqslant\frac{C}{(1+k^2)(1+n^2)}.$$

By (3.4) the norm of $\sum_n m_n\hat S_k(x-n)$ is $O(1/1+k^2)\|m\|$, so

$$\sum m_n\hat S(x-n) = \sum m_nR\Big(\frac{x-n}{4}\Big)$$

is a multiplier. To complete the proof we note that the sequence m', with $m_n' = m_{n/4}$ if $4|n$ and 0 otherwise, has the same norm as m.

(3.6) Remark. For $N=1$, Rm is the continuous piecewise linear extension of the sequence m to a function on R. For $N>1$, note that if all the m_n at the vertices of a cube with side one have the same value, $Rm(x)$ has that value inside the cube.

(3.7) Theorem. *If $m\in M_p^q(Z^N)$, with $p\leqslant q$, and $1<p<\infty$ (or $1<q<\infty$), then*

$$Em(x) = \sum m_n\chi_Q(x-n)$$

is in $M_p^q(R^N)$ and $\|Em\|\leqslant A'\|m\|$, where A' depends on p (or q) and N.

Proof. The proof for $N=1$ and (3.6) are used to prove the theorem for $N>1$. If $m\in M_p^q(Z)$, the sequence m' with $m_n' = m_{n/2}$ if n is even, and zero otherwise, has the same norm as m, so if m'' is defined by $m_n'' = m_n'+m_{n-1}'$, $\|m''\|\leqslant 2\|m\|$. By (3.5) the continuous piecewise linear extension Rm'', of m'' is in $M_p^q(R)$; $Rm''(x) = m_n$ if $2n\leqslant x\leqslant 2n+1$.

By (2.4) the function χ of period 2 which is 1 for $0<x<1$ and 0 for $-1<x<0$ is in $M_p^p(R)$ if $1<p<\infty$ (which we may assume without loss). Thus $f=\chi Rm''$ belongs to $M_p^q(R)$ and $\|f\|\leqslant A_p\|m\|$. Now $f(x)=m_n$ if $2n<x<2n+1$, for each n, and $f(x)=0$ almost everywhere else. Since $Em(x)=f(x/2)+f(1+x/2)$ a.e. the theorem follows.

4. Restrictions to Z^N, R^M.

(4.1) THEOREM (de Leeuw). *If $m_n=\lim_{k\to 0} m_k(n)$, where $m_k\in M_p^p(R^N)$ is continuous at each $n\in Z^N$ and $\|m_k\|\leqslant C$, then $m\in M_p^p(Z^N)$ and $\|m\|\leqslant AC$.*

Proof. Let K_k denote the operator corresponding to m_k, and take $f(x)=\sum c_n e^{in\cdot x}$ to be a trigonometric polynomial. We will choose functions g_s such that

(4.2) $$F_s(y)=s^N\sum g_s(y+2\pi m)K_k(g_sf)(y+2\pi m)$$

is a good approximation for $\sum m_k(n)c_ne^{in\cdot y}$.

Put $\hat g_s(x)=2^N(2\pi)^N r(sx)$, $s>0$ (see (1.4)). We have

$$\operatorname{supp}\hat g_s=(4/s)\bar Q,$$

$$\operatorname{supp}\hat g_s*\hat g_s\subseteq 2\bar Q \text{ for } s \text{ large enough},$$

$$\hat g_s(0)=(2\pi)^N,\qquad \hat g_s*\hat g_s(0)=(2\pi)^{2N}(4/3s)^N,$$

$$0\leqslant g_s(x)\leqslant 2^N s^{-N}.$$

It follows that for s large enough

$$\hat g_s(n)=0=\hat g_s*\hat g_s(n)\quad\text{if } n\neq 0,$$

and

$$\sum_n g_s(x+2\pi n)\equiv 1.$$

Now

$$F_s(y)=s^N\sum_m g_s(y+2\pi m)(2\pi)^{-N}\int m_k(x)(g_sf)^\wedge(x)e^{ix\cdot(y+2\pi m)}\,dx$$

$$=s^N\sum_n c_ne^{in\cdot y}\sum_m(2\pi)^{-N}\int g_s(y+2\pi m)e^{ix\cdot(y+2\pi m)}m_k(x+n)\hat g_s(x)\,dx$$

$$=s^N\sum_n c_ne^{in\cdot y}\int(2\pi)^{-2N}\sum_m\hat g_s(m-x)e^{iy\cdot m}m_k(x+n)\hat g_s(x)\,dx,$$

by the Poisson summation formula. If s is large, then

$$\sum\hat g_s(m-x)e^{iy\cdot m}\hat g_s(x)=g_s(-x)\hat g_s(x),$$

so

$$F_s(y)=(4/3)^N\sum m_k(n)c_ne^{in\cdot y}+$$
$$+s^N\sum c_ne^{in\cdot y}\int[m_k(x+n)-m_k(n)](2\pi)^{-2N}\hat g_s(x)^2\,dx.$$

The second term is bounded by

$$(4/3)^N\sum|c_n|\sup_{x\in(4/s)Q}|m_k(x+n)-m_k(n)|$$

which may be made small by taking s large enough. Hence

$$\Big(\int_{2\pi Q}\Big|\sum m_k(n)c_ne^{in\cdot y}\Big|^p dy\Big)^{1/p}\leqslant o(1)+(3/4)^Ns^N\Big(\int_{2\pi Q}|F_s(y)|^p\,dy\Big)^{1/p}.$$

Set

$$A_{r,s}=\sup_y\Big(\sum g_s(y+2\pi m)^r\Big)^{1/r}.$$

We use Hölder's inequality in (4.2) to get

$$\|F_s\|_{L^p(2\pi Q)}\leqslant CA_{p',s}A_{p,s}\|f\|_{L^p(2\pi Q)}.$$

But $g_s(x)^p\leqslant g_s(x)^{p-1}g_s(x)\leqslant(2^Ns^{-N})^{p-1}g_s(x)$, so $A_{p',s}A_{p,s}\leqslant(2/s)^N$, and it follows that the norm of $\{m_k(n)\}$ is dominated by $(3/2)^NC$.

COROLLARY. *The Banach algebra $M_p^p(Z^N)$ is isomorphic to a subspace of $M_p^p(R^N)$, $1\leqslant p<\infty$.*

Remark. The proof of (4.1) shows that $M_p^q(R^N)\cap C$ is trivial if $q<p$.

(4.3) THEOREM (Igari). *If m is bounded on R^N, continuous almost everywhere, and the sequences*

$$m_k=\{m(kn)\}$$

are in $M_p^q(Z^N)$ $(p\leqslant q)$ with

$$\liminf_{k\to 0}k^{N(1/p-1/q)}\|m_k\|\leqslant C,$$

then $m\in M_p^q(R^N)$ and $\|m\|\leqslant AC$.

Proof. Let

$$m_k'(x)=\sum m(kn)\frac{\sin^2\pi(x/k-n)}{(\pi(x/k-n))^2}.$$

By (3.1) $\|m_k'\|\leqslant AC$ for a sequence of k tending to 0, and $m_k'(x)$ converges to $m(x)$ if x is a point of continuity of m.

This theorem may be used to derive the Marcinkiewicz multiplier theorem for functions defined on R, from the corresponding theorem for sequences [4, 6]. For the sum of $|m(2^{-j}(n+1))-m(2^{-j}n)|$ over $2^k\leqslant n<2^{k+1}$ is bounded by the variation of m on $(2^{-j+k},2^{-j+k+1})$.

(4.4) THEOREM (de Leeuw). *If $m\in M_p^p(R^N)$ is regulated, the restriction R_Mm of m to R^M is in $M_p^p(R^M)$ and $\|R_Mm\|\leqslant A\|m\|$.*

Proof. This is a consequence of (4.1) and of

(4.5) LEMMA. *If $m\in M_p^p(Z^N)$, the restriction R_Mm of m to Z^M is in $M_p^p(Z^M)$ and $\|R_Mm\|\leqslant\|m\|$.*

To use this we let

$$m_\varepsilon(x) = c\varepsilon^{-N} \int_{|x-y|<\varepsilon} m(y)\,dy;$$

c is chosen so that $m_\varepsilon(x) \to m(x)$ for each x. Then $\|m_\varepsilon\| \leqslant \|m\|$, so by (4.1) and the lemma, the sequences $\{m_\varepsilon(kn)\}_{n\epsilon Z^M}$ have norms in $M_p^p(Z^M)$ bounded by $A_N\|m\|$. The continuity of m_ε and (4.3) yield $\|R_M m\| \leqslant A_M A_N \|m\|$ and the theorem.

To prove the lemma we let K be the operator corresponding to m, $Ef(x_1, \ldots, x_N) = f(x_1, \ldots, x_{N-1})$, where $f \epsilon L^p(T^{N-1})$ and

$$Pf(x_1, \ldots, x_{N-1}) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(x_1, \ldots, x_{N-1}, t)\,dt \qquad \text{a.e.}$$

for $f \epsilon L^p(T^N)$. E has norm $(2\pi)^{1/p}$; an application of Jensen's inequality shows that P has norm $(2\pi)^{-1/p}$. The composite PKE is the operator corresponding to $R_{N-1}m$, and so on.

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