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Convergence of monotone nets in ordered topological vector spaces*

by

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1. Introduction. The main question dealt with in this paper is "When does a bounded or an order bounded monotone net in an ordered topological vector space converge?" A definitive answer to the question is provided by Theorem 1 for order bounded monotone nets in locally convex spaces if the order is provided by the positive cone of a total biorthogonal system. A partial answer to the question is given by Theorem 5 for monotone bounded sequences in certain conjugate spaces. Results on the relationship of regularity of cones to normality of cones are presented in Section 4.

All vector spaces in this paper will be assumed to be over the real field unless otherwise stated. An *ordered vector space* is a real vector space E equipped with a transitive, reflexive, antisymmetric relation $\leqslant$ satisfying the following conditions:

(a) If x, y, z are elements of E and $x \leqslant y$, then $x+z \leqslant y+z$.

(b) If x, y are elements of E and $\lambda \geqslant 0$, then $x \leqslant y$ implies $\lambda x \leqslant \lambda y$.

The *positive cone* K in an ordered vector space E is defined by $K = \{x \in E\colon \theta \leqslant x\}$. It has the properties: $K+K \subset K$, $\lambda K \subset K$ for each $\lambda \geqslant 0$, and $K \cap (-K) = \{\theta\}$, where θ is the zero vector. A set K with the above three properties is called a *cone*. If K is a cone in a real vector space E, then a relation $\leqslant$ is defined on E by $x \leqslant y$ if $y - x \in K$ with respect to which E is an ordered vector space with positive cone K. If x and y are elements of an ordered vector space and $x \leqslant y$, then the *order interval between x and y* is the set $[x, y] = \{z \in E\colon x \leqslant z \leqslant y\}$. A set in an ordered vector space is *order bounded* if it is a subset of an order interval. When an ordered vector space is also a topological vector space, the resulting structure is an *ordered topological vector space*. A net $\{x_\alpha\}_{\alpha \in A}$ in an ordered vector space is *increasing* (*decreasing*) if $x_\alpha \leqslant x_\beta$ $(x_\beta \leqslant x_\alpha)$ when $\alpha \leqslant \beta$. The positive cone K of an ordered topological vector space is *regular* (*sequentially regular*) if each order bounded increasing net (sequence)

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in K converges to an element of K. The positive cone K of an ordered topological space is *fully regular* (*fully sequentially regular*) if each increasing and topologically bounded net (sequence) in K converges to an element of K. The positive cone of an ordered topological vector space is *normal* if there exists a local base $\mathscr{V} = \{V\}$ of neighborhoods for the topology of E with the property that for each $V \in \mathscr{V}$, $V = \bigcup \{[x, y]: x, y \in V\}$. For a locally convex ordered topological space it is known ([13], p. 215) that K is normal if and only if there is a generating family $\mathscr{P}$ of seminorms for the topology such that if $p \in \mathscr{P}$ and $\theta \leqslant x \leqslant y$, then $p(x) \leqslant p(y)$.

If E is a linear topological space, a pair of sequences $\{x_i, f_i\}_{i\in\omega}$, where $\{x_i\} \subset E$ and $\{f_i\} \subset E'$ (the topological dual of E), is called a *biorthogonal system* if $f_i(x_j) = \delta_{ij}$. For a biorthogonal system $\{x_i, f_i\}$ let $K = \{x \in E: f_i(x) \geqslant 0, \text{ all } i \in \omega\}$, and observe that K is a cone if and only if $\{f_i\}$ is total over E. Thus if $\{x_i, f_i\}$ is a biorthogonal system with $\{f_i\}$ total over E, the *positive cone of* $\{x_i, f_i\}$ is defined to be the above set K. If a biorthogonal system $\{x_i, f_i\}$ has the property that for each $x \in K \equiv \{y \in E: f_i(y) \geqslant 0, \text{ all } i \in \omega\}$, $x = \sum_{i=1}^{\infty} f_i(x) x_i$, then $\{x_i, f_i\}$ is said to be a *basis on* K. Observe that if $\{x_i, f_i\}$ is a basis on K, then $\{f_i\}$ is total over E assuming that E is a Hausdorff space. If a biorthogonal system $\{x_i, f_i\}$ for a Hausdorff space E has the property that for each $x \in E$, $x = \sum_{i=1}^{\infty} f_i(x) x_i$, then $\{x_i, f_i\}$ is called a *Schauder basis*.

2. Lemmas. A number of facts needed for the proof of the first main theorem are of interest in themselves and are presented below as lemmas.

LEMMA 1. *A cone K in a locally convex Hausdorff space E is weakly sequentially regular if and only if it is sequentially regular.*

Proof. Suppose $\theta \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_0$ and K is weakly sequentially regular. Let $z_1 = x_1$, and $z_n = x_n - x_{n-1}$, $n = 2, 3, \ldots$ Let $\sum_{i=1}^{\infty} z_{n_i}$ be a subseries of the series $\sum_{i=1}^{\infty} z_i$. Observe that

$$\theta \leqslant \sum_{i=1}^{m} z_{n_i} \leqslant \sum_{i=1}^{m+1} z_{n_i} \leqslant x_0 \quad \text{for} \quad m = 1, 2, \ldots$$

The assumed weak sequential regularity of K implies the existence of a weak sum in K of the subseries $\sum_{i=1}^{\infty} z_{n_i}$. In particular, the series $\sum_{i=1}^{\infty} z_i$ has a weak sum $x \in K$ and by the theorem of Orlicz-Pettis [7] x is the sum of the series $\sum_{i=1}^{\infty} z_i$, i.e., $x = \lim x_n$;

A proof of the next lemma may be found in [12], 3.1, pp. 90, 91.

LEMMA 2. *If $\{x_a\}_{a\in\Delta}$ is a net which is increasing (decreasing) in a topological vector space E ordered by a closed cone K and if x_0 is a cluster point of $\{x_a\}_{a\in\Delta}$, then $x_0 = \sup_{a\in\Delta} x_a$ ($x_0 = \inf_{a\in\Delta} x_a$). The supremum (infimum) is unique.*

LEMMA 3. *If $(E, \mathscr{T})$ is a topological vector space ordered by a closed cone K and if C is a compact subset of $(E, \mathscr{T})$, the supremum (infimum) of each increasing (decreasing) net in C exists and the net converges to it with respect to $\mathscr{T}$.*

Proof. Let $\{x_a\}_{a\in\Delta}$ be an increasing net in C. Since C is compact, the net has a cluster point x_0 in C. By Lemma 2, $x_0 = \sup_{a\in\Delta} x_a$. It follows that x_0 is the only cluster point of $\{x_a\}_{a\in\Delta}$ in C. It is well known that if a net has a unique cluster point in a compact set, it must converge to that cluster point.

LEMMA 4. *If $\sum_{i=1}^{\infty} x_i$ is a series in a locally convex Hausdorff space E over the real or complex numbers such that for each permutation τ of the set of positive integers ω the sequence $\{\sum_{i=1}^{n} x_{\tau(i)}\}_{n\in\omega}$ is Cauchy, then for each bounded set B of the Banach space (m) of real or complex bounded sequences with supremum norm the set*

$$S(B, \mathscr{F}) \equiv \Big\{\sum_{i\in\sigma} b_i x_i: \ b = \{b_i\} \in B \ \text{ and } \ \sigma \in \mathscr{F}\Big\}$$

is totally bounded in E, where $\mathscr{F}$ is the family of finite subsets of ω.

Proof. By hypothesis there exists $M > 0$ such that $\|b\| \leqslant M$ if $b \in B$. First it will be shown that if $n \in \omega$, then

$$S_n(B, \mathscr{F}) \equiv \Big\{\sum_{i\in\sigma} b_i x_i: \ b = \{b_i\} \in B, \ \sigma \subset [1, n]\Big\}$$

is totally bounded in E. Since $S_n(B, \mathscr{F})$ is a subset of the finite-dimensional subspace spanned by $x_1, x_2, \ldots, x_n$, it is totally bounded in E if and only if it is bounded in E. If $f \in E'$ and $b \in B$ and $\sigma \subset [1, n]$, then

$$\Big|f\Big(\sum_{i\in\sigma} b_i x_i\Big)\Big| \leqslant \|b\| \sum_{i=1}^{\infty} |f(x_i)| \leqslant M \sum_{i=1}^{\infty} |f(x_i)| < +\infty,$$

so $S_n(B, \mathscr{F})$ is weakly bounded and hence bounded.

Now let U be a neighborhood of θ and V be a closed balanced, convex neighborhood of θ such that $V + V \subset U$. The hypothesis on $\sum_{i=1}^{\infty} x_i$ implies that it is Moore-Smith Cauchy, i.e., there exists $N \in \omega$ (dependent on V)

such that if $\sigma \in \mathscr{F}$ and $\sigma \cap [1, N] = \emptyset$, then $\sum_{i\in\sigma} x_i \in (1/4M)V$. Let p denote the Minkowski functional of V. If $b \in B$ and $\sigma \in \mathscr{F}$ and $\sigma \cap [1, N] = \emptyset$ then, by [8],

$$p\Big(\sum_{i\in\sigma} b_i x_i\Big) \leqslant 4M \sup_{\sigma' \subset \sigma} p\Big(\sum_{i\in\sigma'} x_i\Big) \leqslant 1,$$

so $\sum_{i\in\sigma} b_i x_i \in V$. Since $S_N(B, \mathscr{F})$ is totally bounded, there exist elements $y_1, y_2, \ldots, y_m$ in E such that

$$S_N(B, \mathscr{F}) \subset \bigcup_{i=1}^{m} (y_i + V).$$

If then $\sum_{i\in\sigma} b_i x_i \in S(B, \mathscr{F})$, let $\sigma_1 = \sigma \cap [1, N]$ and $\sigma_2 = \sigma \cap [N+1, \infty)$. Then for some j, $1 \leqslant j \leqslant m$, $\sum_{i\in\sigma_1} b_i x_i \in y_j + V$. Since $\sum_{i\in\sigma_2} b_i x_i \in V$, it follows that $\sum_{i\in\sigma} b_i x_i \in (y_j + V) + V \subset y_j + U$.

LEMMA 5. *Let $\{x_i, f_i\}$ be a biorthogonal system with $\{f_i\}$ total for a Hausdorff topological vector space E. Let S be a subset of E with the property that $\sum_{i=1}^{\infty} f_i(x) x_i$ converges to x for each $x \in S$. If E is ordered by the positive cone K of $\{x_i, f_i\}$ and B is a non-empty subset of S, then $\sup B$ exists in S if and only if the series $\sum_{i=1}^{\infty} \big(\sup_{x\in B} f_i(x)\big) x_i$ converges to an element of S. Also $\inf B$ exists in S if and only if the series $\sum_{i=1}^{\infty} \big(\inf_{x\in B} f_i(x)\big) x_i$ converges to an element of S. The sums of these series are the supremum and infimum of B respectively.*

Proof. If

$$z = \sum_{i=1}^{\infty} \big(\inf_{x\in B} f_i(x)\big) x_i,$$

the biorthogonality of $\{x_i, f_i\}$ implies that $f_i(z) = \inf_{x\in B} f_i(x)$, so $z \leqslant x$ if $x \in B$. Also if $z' \leqslant x$ for all $x \in B$, then $f_i(z') \leqslant f_i(x)$ for all $x \in B$, i.e., $f_i(z') \leqslant \inf_{x\in B} f_i(x) = f_i(z)$, so $z' \leqslant z$ and hence $z = \inf B$. Conversely, if $z = \inf B$ exists in S, then

$$z = \sum_{i=1}^{\infty} f_i(z) x_i.$$

Since $z = \inf B$, $f_i(z) \leqslant \inf_{x\in B} f_i(x)$. Indeed, equality must hold for each $i \in \omega$. Otherwise for some i_0, $f_{i_0}(z) < \inf_{x\in B} f_i(x)$. Define $z' \in E$ by

$$z' = \sum_{i\neq i_0} f_i(z) x_i + \big(\inf_{x\in B} f_{i_0}(x)\big) x_{i_0}.$$

Clearly, $f_i(z') \geqslant f_i(z)$ for all $i \in \omega$, so $z \leqslant z'$. Also $f_i(z') \leqslant \inf_{x\in B} f_i(x)$, so $z' \leqslant x$ for all $x \in B$ and it follows that $z' \leqslant z$ and hence, since K is a cone, $z = z'$. This however is not so since $f_{i_0}(z') \neq f_{i_0}(z)$.

3. Regularity of cones of biorthogonal systems.

THEOREM 1. *Let E be a locally convex Hausdorff space and $\{x_i, f_i\}$ a biorthogonal system for E with $\{f_i\}$ total. Let E be ordered by the positive cone K of $\{x_i, f_i\}$. Then the following statements are equivalent:*

(i) *For each $x \in K$ and each bounded sequence $\{b_i\}$ of non-negative real numbers the series $\sum_{i=1}^{\infty} b_i f_i(x) x_i$ converges.*

(ii) *$[\theta, x]$ is compact for each $x \in K$ (i.e. $[\theta, x]$ is complete and totally bounded for each $x \in K$).*

(iii) *$[\theta, x]$ is weakly compact for each $x \in K$.*

(iv) *K is weakly regular.*

(v) *K is sequentially weakly regular.*

(vi) *K is regular.*

(vii) *K is sequentially regular.*

(viii) *$[\theta, x]$ is $\sigma(E, E')$-sequentially complete and bounded for each $x \in K$.*

(ix) *The supremum of each order bounded subset of K exists and is an element of K.*

Moreover, when the above equivalences are in effect, each order bounded $\sigma(E, E')$ Cauchy net in E converges strongly to an element of E.

Proof. To show that (i) implies (ii), assume (i) and let $x \in K$. Let

$$S = \Big\{\sum_{i\in\sigma} b_i f_i(x) x_i \colon \sigma \in \mathscr{F} \text{ and } b \in B\Big\},$$

where B is the unit ball of the space m of bounded sequences of reals with supremum norm. Since $\sum_{i=1}^{\infty} f_i(x) x_i$ is unconditionally convergent, it follows, by Lemma 4, that S and hence $\bar{S}$ is totally bounded. Since $[\theta, x]$ is a subset of $\bar{S}$, it is also totally bounded. It suffices now to show that if $x \in K$ and (i) holds, then $[\theta, x]$ is complete. For such an x let $\{y_a\}_{a\in A}$ be a Cauchy net in $[\theta, x]$. Let $a_i = \lim_a f_i(y_a)$ and let $y = \sum_{i=1}^{\infty} a_i x_i$. This series converges since $\theta \leqslant y_a \leqslant x$ implies $0 \leqslant a_i \leqslant f_i(x)$ and hence for each $i \in \omega$ there exists b_i, $0 \leqslant b_i \leqslant 1$, with $a_i = b_i f_i(x)$, so

$$\sum_{i=1}^{\infty} a_i x_i = \sum_{i=1}^{\infty} b_i f_i(x) x_i$$

converges by the hypothesis (i). Next let b_i^a be that number between 0 and 1 such that $f_i(y_a) = b_i^a f_i(x)$, and let $b^a = \{b_i - b_i^a\}_{i\in\omega}$. Note that $\|b^a\| \equiv \sup_{i\in\omega} |b_i - b_i^a| \leqslant 2$, $a \in A$. Let U be a neighborhood of θ and V a closed, convex, balanced neighborhood of θ such that $V+V \subset U$. Since $\sum_{i=1}^{\infty} f_i(x)x_i$ is unordered convergent, there exists a positive integer N such that if $\sigma \in \mathscr{F}$ and $\sigma \cap [1, N] = \emptyset$, then $4\left(\sum_{i\in\sigma} f_i(x)x_i\right) \in V$. Let p be the Minkowski functional of V. Then [8], if $\sigma \cap [1, N] = \emptyset$,

$$p\Big(\sum_{i\in\sigma} (b_i - b_i^a) f_i(x)x_i\Big) \leqslant 2\|b^a\| \sup_{\sigma' \subset \sigma} p\Big(\sum_{i\in\sigma'} f_i(x)x_i\Big) \leqslant \sup_{\sigma' \subset \sigma} p\Big(4 \sum_{i\in\sigma'} f_i(x)x_i\Big) \leqslant 1,$$

so it follows that

$$\Big(\sum_{i=n}^{\infty} a_i x_i - \sum_{i=n}^{\infty} f_i(y_a)x_i\Big) \in V$$

if $n \geqslant N$. Since $a_i = \lim_a f_i(y_a)$ and the summations are finite, there exists $a_0 \in A$ such that if $a \geqslant a_0$, then

$$\sum_{i=1}^{N} a_i x_i - \sum_{i=1}^{N} f_i(y_a)x_i \in V,$$

so $y - y_a \in U$ if $a \geqslant a_0$ and hence $[\theta, x]$ is complete.

It is obvious that statement (ii) implies statement (iii). That statement (iii) implies statement (iv) is a consequence of Lemma 3. Note that since

$$K = \bigcap_{i=1}^{\infty} \{x: f_i(x) \geqslant 0\}$$

and each f_i is continuous, it follows that K is closed and that since $\{f_i\}$ is total, K is a proper cone. That (iv) implies (v) is obvious and (v) and (vii) are equivalent by Lemma 1. To see that (vii) implies (i) let $x \in K$, and let $\{b_i\}$ be a sequence of non-negative real numbers such that $\lambda = \sup_{i\in\omega} b_i < +\infty$. Using the biorthogonality of $\{x_i, f_i\}$,

$$\theta \leqslant \sum_{i=1}^{n} b_i f_i(x)x_i \leqslant \lambda x$$

for each $n \in \omega$. If (vii) holds, then (i) clearly follows. That (vi) implies (iv) is obvious and that (ii) implies (vi) is given by Lemma 3. The above arguments constitute a proof of the equivalence of statements (i) through (vii).

It is clear that (iii) implies (viii) since if $[\theta, x]$ is $\sigma(E, E')$ compact it is $\sigma(E, E')$ complete and $\sigma(E, E')$ totally bounded and hence $\sigma(E, E')$ sequentially complete and $\sigma(E, E')$-bounded and hence bounded. Now

assume (viii), let $x \in K$, and let $\{b_i\}$ be a sequence of non-negative real numbers such that $\lambda = \sup_{i\in\omega} b_i < +\infty$. Since $[\theta, x]$ is bounded and $\sigma(E, E')$-sequentially complete, $[\theta, \lambda x]$ is also. Using biorthogonality we have for arbitrary $\sigma \in \mathscr{F}$, $\theta \leqslant \sum_{i\in\sigma} b_i f_i(x)x_i \leqslant \lambda x$. Thus $\{\sum_{i\in\sigma} b_i f_i(x)x_i : \sigma \in \mathscr{F}\}$ is a subset of a bounded set, so is bounded. It follows that

$$\sum_{i=1}^{\infty} |b_i f_i(x) f(x_i)| < +\infty$$

for each $f \in E'$. Thus for any increasing sequence of indices $n_1 < n_2 < \ldots < n_m < \ldots$ we have that $\{\sum_{i=1}^{m} b_{n_i} f_{n_i}(x) x_{n_i}\}$ is a $\sigma(E, E')$-Cauchy sequence in $[\theta, \lambda x]$ and hence each subseries of $\sum_{i=1}^{\infty} b_i f_i(x)x_i$ has a weak sum in $[\theta, \lambda x]$ so the series itself converges to an element of $[\theta, \lambda x]$ by the Orlicz-Pettis theorem [7].

Finally, assume (i) and let B be a subset of K such that $x \leqslant z$ for all $x \in B$ and some $z \in K$. Let $a_i = \sup_{x\in B} f_i(x)$, $i = 1, 2, \ldots$ Thus $0 \leqslant a_i \leqslant f_i(z)$ for $i \in \omega$, so $a_i = b_i f_i(z)$, where $0 \leqslant b_i \leqslant 1$, $i \in \omega$. Thus

$$\sum_{i=1}^{\infty} a_i x_i = \sum_{i=1}^{\infty} b_i f_i(z) x_i$$

converges by (i) and by Lemma 5,

$$\sup B = \sum_{i=1}^{\infty} a_i x_i,$$

i.e., (i) implies (ix). Conversely assume that (ix) holds. Let $x \in K$ and $\{b_i\}$ be a sequence of non-negative real numbers such that $\lambda = \sup_{i\in\omega} b_i < +\infty$. Letting

$$y_n = \sum_{i=1}^{n} b_i f_i(x)x_i, \qquad n \in \omega,$$

by biorthogonality it follows that $\theta \leqslant y_n \leqslant \lambda x$, $n \in \omega$. Hence, by (ix), $z = \sup_{n\in\omega} y_n$ exists in K, so, by Lemma 5,

$$z = \sum_{j=1}^{\infty} \sup_{n\in\omega} \big(f_j(y_n)\big) x_j.$$

Since $0 \leqslant f_j(y_n) \leqslant f_j(y_{n+1}) \leqslant f_j(\lambda x)$, $j, n \in \omega$, it follows using biorthogonality that $\sup_n f_j(y_n) = b_j f_j(x)$, so

$$z = \sum_{j=1}^{\infty} b_j f_j(x) x_j,$$

i.e., (ix) implies (i).

To verify the last statement of the theorem let $\{z_a\}_{a\in A}$ be a $\sigma(E, E')$-Cauchy net in E such that there exist $z_1, z_2 \in E$ with $\{z_a\}_{a\in A} \subset [z_1, z_2]$. From (iii) $[z_1, z_2]$ is weakly complete, so there exists $z \in [z_1, z_2]$ such that $\{z_a\}$ converges to z for the $\sigma(E, E')$-topology. Now since $[z_1, z_2]$ is strongly compact, by (ii), $\{z_a\}$ has a strong cluster point in $[z_1, z_2]$. This strong cluster point must also be a weak cluster point, so it must be z. Thus $\{z_a\}$ has the unique strong cluster point z in the strongly compact set $[z_1, z_2]$ so $\{z_a\}$ must converge strongly to z.

Fullerton [2], Theorem 3, has shown that if K is the positive cone of an unconditional basis $\{x_i, f_i\}$ for a complete locally convex vector space E, then for each $x \in K$ the order interval $[\theta, x]$ is homeomorphic to a Hilbert cube of countable dimension and hence is compact and metrizable. In his proof he uses only that $\{x_i, f_i\}$ is a biorthogonal system with $\{f_i\}$ total, that $\{x_i, f_i\}$ is an unconditional basis on its positive cone K, and that $[\theta, x]$ is complete for each $x \in K$. Consequently, using Fullerton's proof one may show that condition (i) implies that $[\theta, x]$ is compact and metrizable for each $x \in K$. When K is sequentially complete condition (i) is equivalent to

(i)′ $\sum_{i=1}^{\infty} f_i(x) x_i$ is unconditionally convergent for each $x \in K$.

The equivalence of (i) and (ix) was suggested to the author by a recently announced result of Ceĭtlin [1]; namely, if E is a real sequentially complete locally convex space ordered by the positive cone of a Schauder basis, then the space is a conditionally complete vector lattice if and only if the basis is unconditional.

4. Relations between normality and regularity. Let $\{e_i\}$ denote the unit vector basis in m and for each $i \in \omega$ define f_i on m by $f_i(b) = b_i$, where $b = \{b_i\} \in m$. Clearly, $\{e_i, f_i\}$ is a biorthogonal system for m with $\{f_i\}$ total and its positive cone K is normal since the usual supremum norm for m is monotone on K. Let $x_n = \sum_{i=1}^{n} e_i$. The sequence $\{x_n\}$ is increasing in K and bounded above by $e = \{b_i\}$, where $b_i = 1$, $i \in \omega$. If $n \neq m$, $\|x_n - x_m\| = 1$, so $\{x_n\}$ does not converge. This shows that the positive cone of a biorthogonal system may be normal but not sequentially regular.

Lemma 6. *Let E be a locally convex Hausdorff space with a biorthogonal system $\{x_i, f_i\}$ whose positive cone K is normal and such that $\{x_i, f_i\}$ is a basis on K. Then $\sum_{i=1}^{\infty} f_i(x) x_i$ converges unconditionally to x for each $x \in K$.*

Proof. Let $\mathscr{P}$ be a generating family of seminorms for the topology of E each of which is monotone on K. Let $x \in K$, $p \in \mathscr{P}$, and $\varepsilon > 0$ be given. Then there exists $N \in \omega$ such that

$$p\Big(\sum_{i=N}^{\infty} f_i(x) x_i\Big) < \varepsilon.$$

Now if $\sigma \in \mathscr{F}$ and $\sigma \supset [1, N]$, then

$$\theta \leqslant x - \sum_{i\in\sigma} f_i(x) x_i \leqslant \sum_{i=N}^{\infty} f_i(x) x_i,$$

so by the monotonicity of p on K, $p\big(x - \sum_{i\in\sigma} f_i(x) x_i\big) < \varepsilon$, and hence $\sum_{i=1}^{\infty} f_i(x) x_i$ is unordered convergent to x and hence unconditionally convergent to x.

Lemma 7. *If E is a barrelled Hausdorff space with an unconditional basis $\{x_i, f_i\}$, then its positive cone K is normal.*

Proof. Let $\mathscr{P}$ be a generating family of seminorms for the topology of E. For each $\sigma \in \mathscr{F}$ and $x \in E$ let $S_\sigma(x) = \sum_{i\in\sigma} f_i(x) x_i$. Clearly, $\{S_\sigma\}_{\sigma\in\mathscr{F}}$ is a family of continuous linear operators from E into E. Since $\{x_i, f_i\}$ is an unconditional basis for E, it follows that $\{S_\sigma(x)\colon \sigma \in \mathscr{F}\}$ is for each $x \in E$ bounded and hence, since E is barrelled, the family $\{S_\sigma\}_{\sigma\in\mathscr{F}}$ is equicontinuous. For each $p \in \mathscr{P}$ and $x \in E$ define $p'(x) = \sup_{\sigma\in\mathscr{F}} p\big(S_\sigma(x)\big)$. Each such p' is continuous and $p(x) \leqslant p'(x)$ for each $x \in E$. Thus, the family $\mathscr{P}' = \{p'\colon p \in \mathscr{P}\}$ is also a generating family of seminorms for the topology of E. For each $p' \in \mathscr{P}'$ and $x \in E$ and $\sigma \in \mathscr{F}$ it follows that if $\sigma' \subset \sigma$, then

$$p'\Big(\sum_{i\in\sigma'} f_i(x) x_i\Big) \leqslant p'\Big(\sum_{i\in\sigma} f_i(x) x_i\Big).$$

It will now be shown that if $\theta \leqslant x \leqslant y$, then $p'(x) \leqslant p'(y)$ for each $p' \in \mathscr{P}'$. Since $\theta \leqslant x \leqslant y$, $0 \leqslant f_i(x) \leqslant f_i(y)$ for $i \in \omega$. Let $m \in \omega$. By the Hahn-Banach theorem there exists $f \in E'$ such that $|f(z)| \leqslant p'(z)$ for all $z \in E$ and

$$p'\Big(\sum_{i=1}^{m} f_i(x) x_i\Big) = f\Big(\sum_{i=1}^{m} f_i(x) x_i\Big).$$

Let $\sigma(m) = \{i \in [1, m]\colon f(x_i) \geqslant 0\}$. Suppose $\sigma(m) \neq \emptyset$. Then

$$p'\Big(\sum_{i=1}^{m} f_i(x) x_i\Big) = f\Big(\sum_{i=1}^{m} f_i(x) x_i\Big) \leqslant \sum_{i\in\sigma(m)} f_i(x) f(x_i)$$
$$\leqslant \sum_{i\in\sigma(m)} f_i(y) f(x_i) \leqslant p'\Big(\sum_{i\in\sigma(m)} f_i(y) x_i\Big) \leqslant p'\Big(\sum_{i=1}^{n} f_i(y) x_i\Big)$$

for n large enough, so $\sigma(m) \subset [1, n]$. Passing to the limit with respect to n, it follows that

$$p'\Big(\sum_{i=1}^{m} f_i(x) x_i\Big) \leqslant p(y).$$

On the other hand, if $\sigma(m) = \emptyset$, then $f(x_i) < 0$ for $i = 1, 2, \ldots, m$ and since

$$0 \leqslant p'\Big(\sum_{i=1}^{m} f_i(x)x_i\Big) = \sum_{i=1}^{m} f_i(x)f(x_i)$$

with $f_i(x) \geqslant 0$ for all $i \in \omega$, it follows that $f_i(x) = 0$, $i = 1, 2, \ldots, m$. Hence,

$$\sum_{i=1}^{m} f_i(x)x_i = \theta$$

so

$$p'\Big(\sum_{i=1}^{m} f_i(x)x_i\Big) \leqslant p'(y)$$

in this case also. It has been shown that if $p' \in \mathscr{P}'$ and $\theta \leqslant x \leqslant y$, then

$$p'\Big(\sum_{i=1}^{m} f_i(x)x_i\Big) \leqslant p'(y)$$

for all $m \in \omega$ and hence $p'(x) \leqslant p'(y)$.

Ceĭtlin [1] has shown that a Schauder basis in a sequentially complete bornological space is unconditional if and only if its positive cone is generating and normal. Now a sequentially complete bornological space is barrelled ([5], p. 184). Thus Ceĭtlin's result is a corollary of the following theorem:

THEOREM 2. *A Schauder basis $\{x_i, f_i\}$ for a sequentially complete barrelled space E is unconditional if and only if its positive cone K is generating and normal.*

Proof. If K is normal, then $\{x_i, f_i\}$ is an unconditional basis on K by Lemma 6 and hence on E since K is generating. Conversely, if the basis is unconditional, then K is normal by Lemma 7. Furthermore, if $x \in E$, then $x = y - z$, where $y = \sum_{i \in \sigma} f_i(x)x_i$ with $\sigma = \{i \in \omega\colon f_i(x) \geqslant 0\}$ and $-z = \sum_{i \in \omega/\sigma} f_i(x)x_i$. The sequential completeness is used only to assure the convergence of the above two subseries.

The following theorem is known for Banach spaces [6]. The author acknowledges learning of its validity for Fréchet spaces from Mark D. Levin who presented a proof in seminar. The proof below is for complete metric linear spaces.

THEOREM 3. *If E is a complete metric linear space ordered by a cone K, then*

(i) *K is normal if and only if $[\theta, x]$ is bounded for each $x \in K$, and*

(ii) *if K is a closed sequentially regular cone, then K is normal.*

Proof. If K is normal and $x \in K$, then $[\theta, x]$ is bounded ([12], p. 62). To show the converse of (i) note that K is normal if and only if $\bar{K}$ is normal ([12], p. 63). Hence the converse of (i) will be shown by supposing K to be closed and not normal and showing the existence of $x \in K$ such that $[\theta, x]$ is not bounded. If K is not normal, there exists a closed, balanced neighborhood V of θ such that if W is a neighborhood of θ there exist elements x, y such that $\theta \leqslant x \leqslant y$ with $y \in W$ and $x \notin V$ ([12], p. 62, Proposition 1.3d). Let $\{W_i\}_{i \in \omega}$ be a local base for the topology of E such that $W_{i+1} + W_{i+1} \subset W_i$, $i \in \omega$. Thus, for each $i \in \omega$ there exist x_i', y_i' such that $\theta \leqslant x_i' \leqslant y_i'$ and $y_i' \in (1/i)\, W_i$ and $x_i' \notin V$. Let $y_i = iy_i'$ and $x_i = ix_i'$. Thus $y_i \in W_i$ and $x_i \notin iV$. It is clear that $\{x_i\}$ is not bounded. However, the sequence $\{\sum_{i=1}^{n} y_i\}_{n \in \omega}$ is a Cauchy sequence in the closed and hence complete cone K, so there exists $y \in K$ such that $y = \sum_{i=1}^{\infty} y_i$. Now $\theta \leqslant x_i \leqslant y$ for all $i \in \omega$ so $[\theta, y]$ is not bounded. To prove (ii) assume that K is not normal. Then select sequences $\{x_i\}$ and $\{y_i\}$ just as in the proof of the converse of (i). Thus

$$\theta \leqslant \sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} y_i \leqslant y$$

for $n = 1, 2, \ldots$ However, the increasing sequence $\{\sum_{i=1}^{n} x_i\}_{n \in \omega}$ in K, bounded above by $y \in K$, cannot converge since $\{x_i\}$ is not bounded. In particular, the general term x_i of the series $\sum_{i=1}^{\infty} x_i$ does not go to zero.

It is known for Banach space ([14], pp. 220 and 221) that a cone is normal if and only if it is weakly normal. The preceding theorem enables one to establish this result for Fréchet spaces.

COROLLARY 1. *A cone K in a Fréchet space E is normal if and only if it is weakly normal.*

Proof. If K is normal, it is weakly normal ([14], Corollary, 3, p. 220). Conversely, if K is weakly normal, its dual cone K' generates E' ([14], Corollary 3, p. 220). Let $x \in K$ and $f \in E'$. Then $f = f_1 - f_2$, where $f_1, f_2 \in K'$. Thus if $y \in [\theta, x]$, then $0 \leqslant f_i(y) \leqslant f_i(x)$, $i = 1, 2$, so $|f(y)| \leqslant f_1(x) + f_2(x)$ for all $y \in [\theta, x]$. This shows that $[\theta, x]$ is weakly bounded and hence bounded for each $x \in K$. By Theorem 3, K is normal.

The following theorem elevates to Fréchet spaces a result recently obtained for Banach spaces [9]:

THEOREM 4. *Let K be the positive cone of a biorthogonal system $\{x_i, f_i\}$ for a Fréchet space E with $\{f_i\}$ total over E. Then K is regular if and only if $\{x_i, f_i\}$ is a basis on K and K is normal.*

Proof. Suppose that K is regular. Then by Theorem 3, K is normal. Also if $x \in K$, then by the equivalence of (vi) and (i) of Theorem 1, $\sum_{i=1}^{\infty} f_i(x) x_i$ converges to some element. This sum must be x because of the totality of $\{f_i\}$ and biorthogonality of $\{x_i, f_i\}$. Conversely, suppose $\{x_i, f_i\}$ is a basis on K and K is normal. Then, by Lemma 6, $\{x_i, f_i\}$ is an unconditional basis on K and since E is complete, condition (i) of Theorem 1 is satisfied, so K is regular by Theorem 1.

In general linear topological spaces if a closed cone K is normal and fully sequentially regular, then it is sequentially regular, for if $\theta \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant \ldots \leqslant x$, then $[\theta, x]$ is topologically bounded ([12], p. 62), so $\{x_n\}$ is bounded, so by full sequential regularity it converges. Krasnoselskiĭ ([6], p. 37) shows that for Banach spaces a fully sequentially regular closed cone is normal and hence sequentially regular. It is a corollary of a theorem of Schaeffer [13] that if E is a weakly sequentially complete locally convex space ordered by a normal closed cone K, then K is fully sequentially regular. From this and Lemma 7 it follows that *the positive cone of an unconditional basis for a weakly sequentially complete barrelled space is fully sequentially regular.*

5. Regularity of dual cones. A real barrelled space E with the property P: the canonical embedding of E in E'' is $\sigma(E'', E')$-sequentially dense in E', will be called a *P-space*. Examples of P-spaces are reflexive spaces, quasi-reflexive spaces, and spaces whose topological dual E' is $\beta(E', E)$-separable ([14], p. 143). Banach spaces which are P-spaces have been studied by McWilliams [10, 11].

Karlin [4] showed that if E is a Banach space with separable strong dual E' and if E is ordered by a closed cone K with the property that there exists a constant $M > 0$ such that for each $F \in E''$ there exist $F_1, F_2 \in K''$ with $F = F_1 - F_2$ and $\|F_i\| \leqslant M$, $i = 1, 2$, then K' is sequentially regular. The following generalization of Karlin's theorem is proved by using Karlin's proof with several modifications:

THEOREM 5. *If E is a P-space ordered by a closed generating cone K, then K' is fully sequentially regular for the $\beta(E', E)$-topology. If E is both a Banach space and a P-space ordered by a closed cone K, then the following are equivalent:*

(i) *K generates E;*

(ii) *K' is $\beta(E', E)$ fully sequentially regular;*

(iii) *K' is $\beta(E', E)$ sequentially regular;*

(iv) *K' is $\beta(E', E)$ normal.*

Proof. Let $\{y_n\}$ be a sequence in K' such that $y_n(x) \leqslant y_{n+1}(x)$ for each $x \in K$ and each $n \in \omega$ and such that $\{y_n\}$ is $\beta(E', E)$ bounded. It follows that $\{y_n(x)\}$ is bounded for each $x \in E$. Define a functional $\bar{y}$ by $\bar{y}(x) = \lim_n y_n(x)$ for each $x \in E$. If $x \in E$, $x = x' - x''$, where $x', x'' \in K$, so $\lim_n y_n(x) = \lim_n y_n(x') - \lim_n y_n(x'')$ exists. Clearly, $\bar{y}$ is a linear functional and $\bar{y}$ is continuous, being the pointwise limit of continuous linear functionals on a barrelled space. Next, it will be shown that $\bar{y}$ is the $\sigma(E', E'')$ limit of $\{y_n\}$. Let $x'' \in E''$. Since E is $\sigma(E'', E')$-sequentially dense in E'', there exists a sequence $\{x_m\} \subset E$ such that $x''(y) = \lim_m y(x_m)$ for each $y \in E'$. Now if it can be shown that $\lim_n \lim_m y_n(x_m) = \lim_m \lim_n y_n(x_m)$, then $\lim_n x''(y_n) = \lim_m \bar{y}(x_m) = x''(\bar{y})$, so $\bar{y}$ would be the $\sigma(E', E'')$ limit of $\{y_n\}$. The interchange of limits is justified as follows. Define T on E into l by $T(x) = \{y_n(x) - y_{n-1}(x)\}_{n \in \omega}$, where $y_0(x) = 0$. Note that if $x \in K$, then

$$\sum_{n=1}^{\infty} |y_n(x) - y_{n-1}(x)| = \sum_{n=1}^{\infty} y_n(x) - y_{n-1}(x) = \lim_n y_n(x) < +\infty.$$

Since K generates E, it follows that T is well defined on E into l and T is clearly linear. Now define T_n on E into l' by $T_n(x) = \big(y_1(x), y_2(x) - y_1(x), \ldots, y_n(x) - y_{n-1}(x), 0, 0, \ldots\big)$. It is clear that T_n is continuous and $\lim_n T_n(x) = T(x)$ for each $x \in E$, so since E is a barrelled space, T is continuous. Hence T is $\sigma(E, E') - \sigma(l', m)$ continuous. Since $x''(y) = \lim_m y(x_m)$ for each $y \in E'$, $\{x_m\}$ is $\sigma(E, E')$-Cauchy so $\{T(x_m)\}_{m \in \omega}$ is $\sigma(l', m)$-Cauchy. Now a $\sigma(l', m)$-Cauchy sequence in l' must be a Cauchy sequence in the norm topology of l'. Consequently, given $\varepsilon > 0$ there is an integer m such that

$$\sum_{n=1}^{N} |\{y_n(x_{m_1}) - y_{n-1}(x_{m_1})\} - \{y_n(x_{m_2}) - y_{n-1}(x_{m_2})\}| < \varepsilon$$

for all N and all $m_1 \geqslant m$, $m_2 \geqslant m$. Passing to the limit with m_2 we obtain

$$\sum_{n=1}^{N} |\{y_n(x_{m_1}) - y_{n-1}(x_{m_1})\} - \{x''(y_n) - x''(y_{n-1})\}| \leqslant \varepsilon$$

for all N and $m_1 \geqslant m$. Hence, if $m_1 \geqslant m$, then

$$\Big| \sum_{n=1}^{\infty} \{y_n(x_{m_1}) - y_{n-1}(x_{m_1})\} - \sum_{n=1}^{\infty} \{x''(y_n) - x''(y_{n-1})\} \Big| \leqslant \varepsilon,$$

that is,

$$\Big| \lim_{m_1 \to \infty} \sum_{n=1}^{\infty} \{y_n(x_{m_1}) - y_{n-1}(x_{m_1})\} - \sum_{n=1}^{\infty} \{x''(y_n) - x''(y_{n-1})\} \Big| \leqslant \varepsilon .$$

But,

$$\lim_{m\to\infty}\lim_{n\to\infty} y_n(x_m) = \lim_{m\to\infty}\sum_{n=1}^{\infty}\{y_n(x_m)-y_{n-1}(x_m)\}$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty} y_n(x_m) = \sum_{n=1}^{\infty}\{x''(y_n)-x''(y_{n-1})\}$$

so

$$|\lim_{m\to\infty}\lim_{n\to\infty} y_n(x_m)-\lim_{n\to\infty}\lim_{m\to\infty} y_n(x_m)| \leqslant \varepsilon .$$

Since ε was an arbitrary positive number, the interchange of limits is justified and $\bar{y}$ is the $\sigma(E', E'')$-limit of the monotone increasing sequence $\{y_n\}$. By Lemma 1, $\bar{y}$ is the $\beta(E', E)$ limit of $\{y_n\}$. This concludes the proof of the first sentence of the theorem and hence proves that (i) implies (ii). A proof that (ii) implies (iii) is given in [6], p. 37. By Theorem 3, (iii) implies (iv). Finally, if K' is normal for $\beta(E', E)$, then ([14], p. 221) K is a strict b-cone and hence generates E.

COROLLARY 2. *If E is a P-space ordered by the positive cone K of a biorthogonal system $\{x_i, f_i\}$ with $\{f_i\}$ total and if K is generating, then for each $f\in K'$ and each bounded sequence $\{b_i\}$ of non-negative real numbers the series $\sum_{i=1}^{\infty} b_i f(x_i) f_i$ converges for the $\beta[E', E)$ topology. If, moreover, $\{x_i, f_i\}$ is a Schauder basis, then K' is not only fully sequentially regular but regular as well.*

Proof. Let $f\in K'$ and $\{b_i\}$ a sequence of non-negative real numbers with $\lambda = \sup_{i\in\omega} b_i < +\infty$. The sequence $\{\sum_{i=1}^{m} b_i f(x_i) f_i\}$ is increasing in K' since $b_i f(x_i) \geqslant 0$ and $f_i\in K'$ for $i\in\omega$ and K' is a cone. Furthermore, the sequence is $\beta(E', E)$ bounded since if $x\in K$, then

$$0 \leqslant \Big(\sum_{i=1}^{m} b_i f(x_i) f_i\Big)\, x = f\Big(\sum_{i=1}^{m} b_i f_i(x) x_i\Big) \leqslant \lambda f(x)$$

because

$$\theta \leqslant \sum_{i=1}^{m} b_i f_i(x) x_i \leqslant \lambda x .$$

Now if $x\in E$, then $x = x' - x''$ with $x', x''\in K$, so

$$\Big|\Big(\sum_{i=1}^{m} b_i f(x_i) f_i\Big) x\Big| \leqslant \lambda\big(f(x')+f(x'')\big).$$

This shows that $\{\sum_{i=1}^{m} b_i f(x_i) f_i\}_{m\in\omega}$ is $\sigma(E', E)$ bounded and, since E is barrelled, $\beta(E', E)$ bounded. Now using the fact that K' is fully sequentially regular for $\beta(E', E)$ it follows that $\sum_{i=1}^{\infty} b_i f(x_i) f_i$ converges for $\beta(E', E)$ to some element. In particular, $\sum_{=1}^{\infty} f(x_i) f_i$ is unconditionally convergent for $\beta(E', E)$ to some element $g\in E'$. Since $\{x_i\}$ is total over E' and $\{x_i, f_i\}$ is biorthogonal, $g = f$. That K' is regular follows from Theorem 1, because K' (since $\{x_i, f_i\}$ is a Schauder basis) is the positive cone of $\{f_i, J(x_i)\}$. Here J denotes the canonical embedding map of E into E''.

COROLLARY 3. *If E is both a Banach space and a P-space ordered by the positive cone K of a Schauder basis $\{x_i, f_i\}$, then K generates E if and only if $\{f_i, J(x_i)\}$ is an unconditional basis on K' for the $\beta(E', E)$ topology (J denotes the canonical map of E into E'').*

Proof. If K generates E, then the conclusion follows by the preceding corollary. Conversely, suppose $\{f_i, J(x_i)\}$ is a $\beta(E', E)$ unconditional basis on K'. Now since $\{x_i, f_i\}$ is a Schauder basis, it follows that the dual cone of K is the positive cone of $\{f_i, J(x_i)\}$, i.e., $K' = \{f\in E'\colon f(x_i) \geqslant 0$, all $i\in\omega\}$. Since E' is a Banach space, in particular E' is $\beta(E', E)$ complete, condition (i) of Theorem 1 is satisfied for $\{f_i, J(x_i)\}$ and hence K' is regular. By the equivalences of Theorem 5, it follows that K generates E.

The following generalized theorem of James ([3], Corollary 2, p. 523) recently announced by [15], p. 38, is an easy corollary of this approach.

COROLLARY 4. *If E is a sequentially complete barrelled space with an unconditional Schauder basis $\{x_i, f_i\}$, then E' is $\beta(E', E)$ separable if and only if $\{f_i, J(x_i)\}$ is a $\beta(E', E)$ unconditional basis for E'.*

Proof. To see the non-trivial implication note ([14], p. 143) that if E' is $\beta(E', E)$ separable, then E is a P-space. By Theorem 2, the positive cone K of $\{x_i, f_i\}$ is generating and normal. By Corollary 2, $\{f_i, J(x_i)\}$ is a $\beta(E', E)$-unconditional basis on K'. Since K is normal, K' generates E', so $\{f_i, J(x_i)\}$ is a $\beta(E', E)$-unconditional basis for E'.

Added in proof. Using Eberlein's theorem, Nguen Van Khue, *Tests for unconditional bases* (Russian), Izv. Vyss. Ucebn. Zaved., Matematika, 2 (69) (1968), p. 68-74, has shown that if $\{\psi_i, f_i\}$ is a biorthogonal system for a real Banach space E with both $\{\psi_i\}$ and $\{f_i\}$ total and K is the cone of $\{\psi_i, f_i\}$, then $\{\psi_i\}$ is an unconditional basis for E if and only if K is generating and each order interval is weakly compact.

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Perfect sets in some groups

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Let G be a compact, metric, totally disconnected abelian group, and $G_1 \supset \ldots \supset G_n \supset G_{n+1} \supset \ldots$ a decreasing sequence of open subgroups meeting in $\{0\}$. Let $(H_n)_1^\infty$ be a sequence of positive numbers; a closed subset X of G is said to have *positive H-capacity* if X supports a Borel probability measure μ with the property

$$\mu(b+G_n) \leqslant KH_n, \quad 1 \leqslant n < \infty, \; b \in G.$$

In the first paragraph we prove an abstract lemma relating "economical coverings" of a set with additive set functions; it follows that capacity and metric covering properties are connected much as in a Euclidean space.

Next we specialize to the group of p-adic integers, as the multiplication in this ring yields an abundance of continuous endomorphisms. An analogue of C^1 mappings is introduced, in terms of which a p-adic analogue of the construction in [4] is accomplished.

I. Let S be a set and A a collection of subsets with this property:

(1) For each choice $\{T_1, \ldots, T_r\}$ from A of a covering of S (namely $S = \bigcup_{i=1}^{r} T_i$) there is a choice $\{T_1', \ldots, T_t'\} \subseteq \{T_1, \ldots, T_r\}$ of pairwise disjoint subsets such that $S = \bigcup_{j=1}^{t} T_j'$.

Moreover, let $h \geqslant 0$ be a real function on A such that

(2) $\sum_{i=1}^{r} h(T_i) \geqslant 1$ whenever each T_i is in A and $S = \bigcup_{i=1}^{r} T_r$.

LEMMA. *There is a non-negative finitely additive set function σ, so defined on all the subsets of S that $\sigma(S) = 1$, and $\sigma(T) \leqslant h(T)$ for each T in A.*

Proof. The argument is based on [3]. First, the covering property (2) is valid for multiple coverings: writing I_T for the characteristic function of T,