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On the asymptotic behaviour of some Markov processes

by

MEHDI MOULINE (Rabat) and BÙI-TRONG-LIỄU (Lille)

The purpose of this paper is essentially to study the relations between the ergodicity of a Markov process and the ergodicity of Markov processes constructed from the former with the help of a measurable mapping (Part II). The conditions under which this construction is possible have been given in [5]. They are slightly modified here; this is the main object of the first part, to which we add some complements. The terminology used in this paper is that of [1] and [5].

In the following, we shall denote by N the set of non-negative integers, N^* the set of positive integers, and 1_B the indicator of the set B.

I. MARKOV PROCESSES CONSTRUCTED FROM ANOTHER WITH THE HELP OF A MEASURABLE MAPPING

I. 0. A σ-algebra $\mathscr{T}$ of subsets of a non-empty set $\mathscr{Y}$ is said to be *separable* if there exists a countable family of subsets of $\mathscr{Y}$ generating $\mathscr{T}$. For example, if $\mathscr{Y}$ is a Polish space, and if $\mathscr{T}$ is the σ-algebra generated by open sets of $\mathscr{Y}$, then $\mathscr{T}$ is separable.

A class $\mathscr{C}$ of subsets of $\mathscr{Y}$ is said to be *semi-compact* if for every sequence $(C_n)_{n\in N^*}$ of elements of $\mathscr{C}$, such that $\bigcap_{n\in N^*} C_n = \emptyset$, there exists an $m\in N^*$ such that $\bigcap_{n\leqslant m} C_n = \emptyset$.

A class $\mathscr{F}$ of subsets of $\mathscr{Y}$ is said to have the *property of approximation* relatively to a probability Q and to a σ-algebra $\mathscr{T}$ if

$$\forall C\in\mathscr{T}, \quad Q(C) = \sup_{\substack{F\in\mathscr{F}\\ F\subset C}} Q(F).$$

I. 1. Let $(\mathscr{X}', \mathscr{B}')$ and $(\mathscr{X}'', \mathscr{B}'')$ be two measurable spaces. We recall that a transition probability P from $(\mathscr{X}', \mathscr{B}')$ to $(\mathscr{X}'', \mathscr{B}'')$ is a mapping $P\colon \mathscr{X}'\times\mathscr{B}'' \to [0,1]$ such that $\forall x\in\mathscr{X}', P(x,\cdot)$ is a probability measure on $\mathscr{B}''$, and $\forall B\in\mathscr{B}'', P(\cdot, B)$ is a real random variable defined on $(\mathscr{X}', \mathscr{B}')$. Let $\big((\mathscr{X}_t, \mathscr{B}_t)\big)_{t\in N^*}$ be a sequence of measurable spaces, $(P_{t,t+1})_{t\in N^*}$ a sequence of transition probabilities such that $\forall t\in N^*, P_{t,t+1}$ is a transition probability from $(\mathscr{X}_t, \mathscr{B}_t)$ to $(\mathscr{X}_{t+1}, \mathscr{B}_{t+1})$. We shall denote by $P_{s,t}$ the transition probability from $(\mathscr{X}_s, \mathscr{B}_s)$ to $(\mathscr{X}_t, \mathscr{B}_t)$ defined by

$$P_{s,t}(x, B) = \int_{\mathscr{X}_{s+1}} P_{s,s+1}(x, dx_{s+1}) \ldots \int_{\mathscr{X}_{t-1}} P_{t-2,t-1}(x_{t-2}, dx_{t-1}) P_{t-1,t}(x, B), \quad \forall (x, B) \in \mathscr{X}_s \times \mathscr{B}_t.$$

The family $\big((\mathscr{X}_t, \mathscr{B}_t), P_{t,t+1}\big)_{t \in N^*}$ is said to be a *Markov process*. In the special case $\mathscr{X}_t = \mathscr{X}$, $\mathscr{B}_t = \mathscr{B}$ and $P_{t,t+1} = P$, $\forall t \in N^*$, we say that the Markov process is *homogeneous*, and we denote it by $\big((\mathscr{X}, \mathscr{B}), P_n\big)_{n \in N^*}$, where $P_n = P_{s,s+n}$, $\forall s \in N^*$.

Let $(\Omega, \mathscr{A}, \Pr)$ be a probability space. A Markov random function $(X_t)_{t \in N^*}$ defined on $(\Omega, \mathscr{A}, \Pr)$ and taking its values in a measurable space $(\mathscr{X}, \mathscr{B})$ is said to be *attached* to the Markov process $\big((\mathscr{X}, \mathscr{B}), P_n\big)_{n \in N^*}$ if it has P as transition probability.

Let $(\mathscr{Y}, \mathscr{T})$ be a measurable space and f be a $\mathscr{T}$-$\mathscr{B}$-measurable mapping from $\mathscr{X}$ *onto* $\mathscr{Y}$. Let us denote by $\mathscr{B}_f$, the sub-σ-algebra $f^{-1}(\mathscr{T})$ of $\mathscr{B}$. As mentionned in the introduction, the purpose of this part is to recall and to modify, slightly, results given by Rosenblatt in [5]. We also give some complements, which will be useful for the second part.

For convenience, we shall define the following hypotheses:

I. 2. (i) $\forall x \in \mathscr{X}$, $\{x\} \in \mathscr{B}$,

(ii) $\forall y \in \mathscr{Y}$, $\{y\} \in \mathscr{T}$,

(iii) $\forall x \in \mathscr{X}$, $P_{\mathscr{B}_f}(x, \cdot)$ is dominated (viz. absolutely continue with respect to) by a positive σ-finite measure μ on $\mathscr{B}_f$, $P_{\mathscr{B}_f}(x, \cdot)$ being the restriction of $P(x, \cdot)$ to $\mathscr{B}_f$.

I. 3. $\mathscr{T}$ is separable and there exists a semi-compact sub-class $\mathscr{F}$ of $\mathscr{T}$ such that, for any probability Q on $\mathscr{T}$, $\mathscr{F}$ has the property of approximation relatively to Q and to $\mathscr{T}$.

Proofs of propositions I. 10 and I. 11 below will be omitted: they are the same as those given in [5], modulo the following result:

I. 4. Proposition. *Let $A \in \mathscr{B}_f$, let ξ be a probability on $\mathscr{B}$ and $\xi_{\mathscr{B}_f}$ be the restriction of ξ on $\mathscr{B}_f$. Under hypothesis* I. 3, *there exists a mapping ν: $A \times \mathscr{T} \to [0, 1]$ such that*

(i) $\forall C \in \mathscr{T}$, $\nu(\cdot, C)$ *is $\mathscr{B}_f^{(A)}$-measurable and* $\forall B \in \mathscr{B}_f^{(A)}$,

$$\int_B \nu(x, C)\, d\xi(x) = \int_B P\big(x, f^{-1}(C)\big)\, d\xi(x),$$

$\mathscr{B}_f^{(A)}$ denoting the σ-algebra $\mathscr{B}_f \cap A$

(ii) $\forall x \in A$, $\nu(x, \cdot)$ *is a probability on* $\mathscr{T}$.

We omit the proof. In the particular case $A = \mathscr{X}$, we have the

I. 5. Corollary. *Under hypothesis* I. 3, *there exists a mapping ν: $\mathscr{X} \times \mathscr{T} \to [0, 1]$ such that*

(i) $\forall C \in \mathscr{T}$, $\nu(\cdot, C)$ *is a version of the conditional expectation* $E_\xi\big[P(\cdot, f^{-1}(C)) \big| \mathscr{B}_f\big]$ *relatively to* ξ.

(ii) $\forall x \in \mathscr{X}$, $\nu(x, \cdot)$ *is a probability on* $\mathscr{T}$.

I. 6. We know (cf. [3]) that: For every family of probabilities dominated by a probability, one can find an equivalent countable sub-family of probabilities. Since the family $\big(P_{\mathscr{B}_f}(x, \cdot)\big)_{x \in \mathscr{X}}$ of probabilities is dominated, it is equivalent to a countable sub-family $\big(P_{\mathscr{B}_f}(x_i, \cdot)\big)_{i \in N^*}$. The dominating (in fact, equivalent) measure can be taken, and shall be taken, as the probability measure

$$\mu = \sum_{i \in N^*} a_i P_{\mathscr{B}_f}(x_i, \cdot), \quad \text{where } a_i > 0 \text{ and } \sum_{i \in N^*} a_i = 1.$$

We shall also make use, in Part II, of the probability measure

$$\bar{\mu} = \sum_{i \in N^*} a_i P(x_i, \cdot) \quad \text{on } \mathscr{B}.$$

I. 7. A set $S \in \mathscr{B}_f$ is said to be a *single entry set* if $\mu(S) > 0$ and if there exists a point $y \in \mathscr{Y}$ such that $P(x, S) = 0$, $\forall x \in \complement f^{-1}(y)$. Such a point y is called a *single entry point* corresponding to S. A set $S \in \mathscr{B}_f$ is said to be a *maximal single entry set* if it is a single entry set and if there does not exist any single entry set S such that $S' \supset S$ and $\mu(S') > \mu(S)$.

I. 8. Let us now recall some results:

1° If S is a single entry set, there exists at least one point $x \in \mathscr{X}$ such that $P(x, S) > 0$.

2° A maximal single entry set S is defined modulo a μ-null set.

3° There are at most a countable class of disjoint maximal single entry sets $(S_k)_{k \in N^*}$ and corresponding entry points.

4° Define $M = \complement\big(\bigcup_{k \in N^*} S_k\big)$. If $\mu(M) = 0$, then $\mathscr{X}$ is the union of an at most countable family of maximal single entry sets.

5° If $\mu(M) = 0$ and if S is a maximal single entry set such that $P(x, S) > 0$, then $P(x, S) = 1$.

6° If $\mu(M) = 0$, and if $y \in \mathscr{Y}$ is such that $\mu\big(f^{-1}(y)\big) > 0$, then $f^{-1}(y)$ is a single entry set.

7° If $\mu(M) = 0$, then $\mathscr{Y}$ is the at most countable set of single entry points.

I. 9. It is to be remarked that if $(X_t)_{t \in N^*}$ is a Markov random function attached to a homogeneous Markov process $\big((\mathscr{X}, \mathscr{B}), P_n\big)_{n \in N^*}$, the random function $(f \circ X_t)_{t \in N^*}$ is not generally Markovian. Rosenblatt in [5] has given the conditions under which $(f \circ X_t)_{t \in N^*}$ is also Markovian.

I. 10. Proposition. *Under hypothesis* I. 2, *and if* $\mu(M) = 0$, *then whatever be the Markov random function $(X_t)_{t \in N^*}$ attached to the Markov process $\big((\mathscr{X}, \mathscr{B}), P_n\big)_{n \in N^*}$, the random function $(f \circ X_t)_{t \in N^*}$ is Markovian.*

I. 11. Consider now the case $0 < \mu(M) < 1$. There exist some single entry sets, for this existence is equivalent to the condition $\mu(M) < 1$. Let S be a single entry set such that we may be able to enter S starting from M, that is to say there exist a positive integer n and n points $y_1, \ldots, y_n \in \mathscr{Y}$ such that: a) $f^{-1}(y_j)$, $j = 2, \ldots, n$, are single entry sets and $f^{-1}(y_1)$ is not; b) y_{j-1} is the single entry point corresponding to $f^{-1}(y_j)$, $j = 2, \ldots, n$ and y_n is the single entry point corresponding to S. The integer n and the points $y_1, \ldots, y$, when they exist, are uniquely defined and constitute a thread of length n entering S.

I. 12. Proposition. *Under the hypotheses* I. 2 *and* I. 3, *a necessary and sufficient condition for* $(f\circ X_t)_{t\in \mathbf{N}^*}$ *to be Markovian whatever be the Markov random function* $(X_t)_{t\in\mathbf{N}^*}$, *attached to the Markov process* $((\mathscr{X}, \mathscr{B}), P_n)_{n\in\mathbf{N}^*}$, *is that:*

(i) *There exists a mapping* $R\colon M\times\mathscr{T}\to[0,1]$ *such that* $\forall C\in\mathscr{T}$, $\forall x_0\in\mathscr{X}$, $R(\cdot, C)$ *is* $\mathscr{B}_f^{(M)}$*-measurable and* $\forall B\in\mathscr{B}_f^{(M)}$,

$$\int_B P(x_0, dx_1)R(x_1, C) = \int_B P(x_0, dx_1)P\big(x_1, f^{-1}(C)\big).$$

(ii) *If* S *is a single entry set and if there exists a thread of length* n *entering* S, *then there exists a mapping* $R_n\colon S\times\mathscr{T}\to[0,1]$ *such that* $\forall C\in\mathscr{T}$, $\forall x_0\in\mathscr{X}$, $R_n(\cdot, C)$ *is* $\mathscr{B}_f^{(S)}$*-measurable and* $\forall B\in\mathscr{B}_f^{(S)}$,

$$\int_B P_{n+1}(x_0, dx_1)R_n(x_1, C) = \int_B P_{n+1}(x_0, dx_1)P\big(x_1, f^{-1}(C)\big).$$

As mentioned above, we omit the proof of I.12 and refer to [5]. We simply recall that the absolute probabilities $\tilde{Q}_t$ and the transition probabilities $Q_{t,t+1}(\cdot,\cdot)$, $t\in\mathbf{N}^*$, of the Markov random function $(f\circ X_t)_{t\in\mathbf{N}^*}$ are constructed as follows:

1° π_t being the law of X_t, $t\in\mathbf{N}^*$, $\tilde{Q}_t$ is defined by $\tilde{Q}_t(C) = \pi_t[f^{-1}(C)]$, $\forall C\in\mathscr{T}$.

2° With regard to the transition probabilities:

If $t = 1$, we define, $\forall y\in\mathscr{Y}$ and $\forall C\in\mathscr{T}$, $Q_{1,2}(y, C) = \nu_1(\cdot, C)$, where $x\in f^{-1}(y)$ and where $\nu_1(\cdot, C)$ is a version of the conditional expectation $E_{\pi_1}[P(\cdot, f^{-1}(C))\,|\,\mathscr{B}_f]$ which is such that $\forall x\in\mathscr{X}$, $\nu_1(x,\cdot)$ is a probability on $\mathscr{T}$ (such a version exists following I. 5).

If $t\geqslant 2$, we distinguish two cases:

a) If $y\in f(M)$, then $f^{-1}(y)\subset M$ (for $f^{-1}(y)$ is an atom of $\mathscr{B}_f$) and we define

$$Q_{t,t+1}(y, C) = R(x, C),\qquad \forall C\in\mathscr{T},\qquad \text{where } x\in f^{-1}(y).$$

b) If $y\notin f(M)$, then $f^{-1}(y)$ is included in a maximal single entry set S. If there exists a thread of length n entering S, we put, $\forall C\in\mathscr{T}$,

$$Q_{t,t+1} = \begin{cases} R_n(x, C) & \text{if}\quad t > n, \text{ where } x\in f^{-1}(y),\\ \nu_t(x, C) & \text{if}\quad t\leqslant n, \text{ where } x\in f^{-1}(y),\end{cases}$$

$\nu_t(\cdot, C)$ denoting a $\mathscr{B}_f^{(S)}$-measurable function such that $\forall B\in\mathscr{B}_f^{(S)}$,

$$\int_B d\pi_t(x)\nu_t(x, C) = \int_B d\pi_t(x)P\big(x, f^{-1}(C)\big)$$

and $\forall x\in\mathscr{X}$, $\nu(\cdot, C)$ is a probability on $\mathscr{T}$ (such a function exists by I. 3).

I. 13. Let us now make a few remarks:

1° $\forall t\geqslant 2$, $\forall B\in\mathscr{B}_f^{(M)}$ and $\forall C\in\mathscr{T}$,

$$\int_B d\pi_t(x_1)P\big(x_1, f^{-1}(C)\big) = \int_B d\pi_t(x_1)R(x_1, C). \tag{I. 13.1}$$

Indeed,

$$\begin{aligned}\int_B d\pi_t(x_1)P\big(x_1, f^{-1}(C)\big) &= \int_B\Big[\int_{\mathscr{X}} d\pi_{t-1}(x)P(x, dx_1)\Big]P\big(x_1, f^{-1}(C)\big)\\ &= \int_{\mathscr{X}} d\pi_{t-1}(x)\Big[\int_B P(x, dx_1)P\big(x_1, f^{-1}(C)\big)\Big]\\ &= \int_{\mathscr{X}} d\pi_{t-1}(x)\Big[\int_B P(x, dx_1)R(x_1, C)\Big]\\ &= \int_B\Big[\int_{\mathscr{X}} d\pi_{t-1}(x)P(x, dx_1)\Big]R(x_1, C)\\ &= \int_B d\pi_t(x_1)R(x_1, C).\end{aligned}$$

If $f^{-1}(y)\in\mathscr{B}_f^{(M)}$ and if $\pi_t[f^{-1}(y)] > 0$, then

$$R(x, C) = \frac{1}{\pi_t[f^{-1}(y)]}\int_{f^{-1}(y)} d\pi_t(x_1)P\big(x_1, f^{-1}(C)\big),\qquad \forall x\in f^{-1}(y). \tag{I.13.2}$$

Indeed, $R(\cdot, C)$ is constant on $f^{-1}(y)$ which is an atom of $\mathscr{B}_f^{(M)}$. We then make use of (I.13.1) by taking $f^{-1}(y)$ in place of B.

2° Similarly, we have: $\forall t\geqslant n+2$, $\forall B\in\mathscr{B}_f^{(S)}$ and $\forall C\in\mathscr{T}$,

$$\int_B d\pi_t(x_1)P\big(x_1, f^{-1}(C)\big) = \int_B d\pi_t(x_1)R_n(x_1, C). \tag{I.13.3}$$

If $f^{-1}(y)\in\mathscr{B}_f^{(S)}$ and if $\pi_t[f^{-1}(y)] > 0$, then

$$R_n(x, C) = \frac{1}{\pi_t[f^{-1}(y)]}\int_{f^{-1}(y)} d\pi_t(x_1)P\big(x_1, f^{-1}(C)\big),\qquad \forall x\in f^{-1}(y). \tag{I.13.4}$$

3° If $\forall x_0\in\mathscr{X}$, $\forall B\in\mathscr{B}_f$ and $\forall C\in\mathscr{T}$,

$$\int_B P(x_0, dx_1)P\big(x_1, f^{-1}(C)\big) = \int_B P(x_0, dx_1)R(x_1, C), \tag{I.13.5}$$

where $R(\cdot, C)$ is $\mathscr{B}_f$-measurable. Then, we have $\forall x_0 \in \mathscr{X}, \forall B \in \mathscr{B}_f, \forall C \in \mathscr{T}$ and $\forall n \in N^*$,

(I.13.6) $$\int_B P_{n+1}(x_0, dx_1) P(x_1, f^{-1}(C)) = \int_B \Big[\int_{\mathscr{X}} P_n(x_0, dx) P(x, dx_1)\Big] P(x_1, f^{-1}(C)) = \int_{\mathscr{X}} P_n(x_0, dx)\Big[\int_B P(x, dx_1) P(x_1, f^{-1}(C))\Big] = \int_{\mathscr{X}} P_n(x_0, dx)\Big[\int_B P(x, dx_1) R(x_1, C)\Big] = \int_B P_{n+1}(x_0, dx_1) R(x_1, C).$$

So, (I.13.5) implies I.12 (ii): it is sufficient to take $R_n = R$.

4° *A sufficient condition for* I.12 (i) *and* I.12 (ii) *to be satisfied is that* $P(\cdot, f^{-1}(C))$ *is* $\mathscr{B}_f$*-measurable,* $\forall C \in \mathscr{T}$.

Indeed, (I.13.5) is then verified, with $P(\cdot, f^{-1}(C))$ in place of $R(\cdot, C)$.

I. 14. Consider now the homogeneous case. $(X_t)_{t \in N^*}$ and $(f \circ X_t)_{t \in N^*}$ having the same meanings as above, it is known that $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t \in N^*}$ is not necessarily homogeneous, although $((\mathscr{X}, \mathscr{B}), P_n)_{n \in N^*}$ is.

A sufficient condition for $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t \in N^*}$ *to be homogeneous is that* $P(\cdot, B)$ *is* $\mathscr{B}_f$*-measurable,* $\forall B \in \mathscr{B}_f$.

In the case where there exists no single entry set, this sufficient condition is also necessary.

I. 15. The following remarks will be useful for Part II.

1° *Suppose that* I.12 (i) *and* (ii) *are verified. Let* $\nu_{s,t}$ *be defined by* $\nu_{s,t}(x, C) = Q_{s,t}(y, C)$ *for* $y \in \mathscr{Y}$, $x \in f^{-1}(y)$ *and* $C \in \mathscr{T}$. *Then* $\nu_{s,t}(\cdot, C)$ *is a version of the conditional expectation* $E_{\pi_s}[P_{t-s}(\cdot, f^{-1}(C)) \mid \mathscr{B}_f]$.

Indeed, $\forall C$ and $C' \in \mathscr{T}$, and $\forall s, t \in N^*$ with $s < t$, we have

$$\Pr\{Y_s \in C', Y_t \in C\} = \Pr\{X_s \in f^{-1}(C'), X_t \in f^{-1}(C)\},$$

that is to say

(I.15.1) $$\int_{C'} d\tilde{Q}_s(y) Q_{s,t}(y, C) = \int_{f^{-1}(C')} d\pi_s(x) P_{t-s}(x, f^{-1}(C)).$$

Since $\nu_{s,t}(\cdot, C)$ is $\mathscr{B}_f$-measurable, we have

$$\int_{C'} d\tilde{Q}_s(y) Q_{s,t}(y, C) = \int_{f^{-1}(C')} d\pi_{s,\mathscr{B}_f}(x) \nu_{s,t}(x, C),$$

$\pi_{s,\mathscr{B}_f}$ denoting the restriction of π_s to $\mathscr{B}_f$. Then

$$\int_{f^{-1}(C')} d\pi_{s,\mathscr{B}_f}(x) \nu_{s,t}(x, C) = \int_{f^{-1}(C')} d\pi_s(x) P_{t-s}(x, f^{-1}(C)).$$

2° *Suppose that* (I.13.5) *is verified. Let* $R'(x, f^{-1}(C)) = R(x, C)$, $C \in \mathscr{T}$; *(*$R'(x, \cdot)$ *is then a positive measure on* $\mathscr{B}_f$*) and*

$$R^{(1)}(x, C) = R(x, C),$$
$$R^{(n)}(x, C) = \int_{\mathscr{X}} R'(x, dx') R^{(n-1)}(x', C).$$

Then $\forall B \in \mathscr{B}_f, \forall t \in N^*$ *and* $\forall (x_0, C) \in \mathscr{X} \times \mathscr{T}$,

(I.15.2) $$\int_B P(x_0, dx) P_t(x, f^{-1}(C)) = \int_B P(x_0, dx) R^{(t)}(x, C).$$

We proceed by induction. Suppose that the equality is true for $n-1$, then $\forall B \in \mathscr{B}_f$,

$$\int_B P(x_0, dx) P_n(x, f^{-1}(C)) = \int_B P(x_0, dx)\Big[\int_{\mathscr{X}} P(x, dx') P_{n-1}(x', f^{-1}(C))\Big] = \int_B P(x_0, dx)\Big[\int_{\mathscr{X}} P(x, dx') R^{(n-1)}(x', C)\Big] = \int_{\mathscr{X}} \Big[\int_B P(x_0, dx) P(x, dx')\Big] R^{(n-1)}(x', C) = \int_{\mathscr{X}} \Big[\int_B P(x_0, dx) R'(x, dx')\Big] R^{(n-1)}(x', C) = \int_B P(x_0, dx) R^{(n)}(x, C).$$

3° $\forall s \geqslant 2, \forall t > s$ *and* $\forall C \in \mathscr{T}$, *we have* μ*-a.s.*

(I.15.3) $$Q_{s,t}(y, C) = R^{(t-s)}(x, C), \quad x \in f^{-1}(y).$$

At first, from I.15.2, we have $\forall B \in \mathscr{B}_f$,

$$\int_B d\bar{\mu}(x) P_n(x, f^{-1}(C)) = \int_B d\bar{\mu}(x) R^{(n)}(x, C) = \int_B d\mu(x) R^{(n)}(x, C),$$

for $R^{(n)}(\cdot, C)$ is $\mathscr{B}_f$-measurable.

Analogously, as in (I.13.3), we have: $\forall B \in \mathscr{B}_f, \forall s \geqslant 2, \forall t > s$ and $\forall C \in \mathscr{T}$,

$$\int_B d\pi_s(x) P_{t-s}(x, f^{-1}(C)) = \int_B d\pi_s(x) R^{(t-s)}(x, C) = \int_B d\pi_{s,\mathscr{B}_f}(x) R^{(t-s)}(x, C).$$

Hence, by I.15.1°,

$$\nu_{s,t}(\cdot, C) = R^{(t-s)}(\cdot, C), \quad \pi_{s,\mathscr{B}_f}\text{-a.s.}$$

and then, $\pi_{s,\mathscr{B}_f}$-a.s.

$$Q_{s,t}(y, C) = R^{(t-s)}(x, C), \quad x \in f^{-1}(y).$$

On the other hand, $\forall s \geqslant 2$, $\pi_{s,\mathscr{B}_f}$ is dominated by μ, so that we have, μ-a.s.

$$Q_{s,t}(y, C) = R^{(t-s)}(x, C), \quad x \in f^{-1}(y).$$

II. PROBLEMS OF ERGODICITY

II. 1. We recall the following definitions (cf. [1]): A Markov process $(\varLambda, \mathscr{L}, q_{t,t+1})_{t\in N^*}$ is said to be:

(i) *strongly ergodic* if $\forall s \in N^*$, $\forall z \in \varLambda$ and $\forall A \in \mathscr{L}$, $\lim_{t\to\infty} q_{s,t}(z, A) = \xi_s(A)$, where ξ_s is a probability on $\mathscr{L}$;

(ii) *weakly ergodic* if $\forall s \in N^*$, $\forall z$ and $z' \in \varLambda$ and $\forall A \in \mathscr{L}$, $\lim_{t\to\infty} [q_{s,t}(z, A) - q_{s,t}(z', A)] = 0$;

(iii) *strongly* (resp. *weakly*) *and uniformly ergodic* if it is strongly (resp. weakly) ergodic and if, moreover, the limit in (i) (resp. (ii)) holds uniformly with respect to z and A.

II. 2. Hypothesis. *We suppose that* (I. 13.5) *is verified, that is to say: there exists a mapping* R: $\mathscr{X} \times \mathscr{T} \to [0, 1]$ *such that* $\forall C \in \mathscr{T}$, $R(\cdot, C)$ *is* $\mathscr{B}_f$*-measurable and that*

$$\int_B P(x_0, dx_1) P(x_1, f^{-1}(C)) = \int_B P(x_0, dx_1) R(x_1, C),$$

$\forall x_0 \in \mathscr{X}$, $\forall C \in \mathscr{T}$ *and* $\forall B \in \mathscr{B}_f$.

By I.13.3°, we see that I.12 (ii) is verified.

II. 3. $\forall C \in \mathscr{T}$, $R(\cdot, C)$ is a version of $E_{P(x_0,\cdot)}[P(\cdot, f^{-1}(C)) \mid \mathscr{B}_f]$, then $R(\cdot, C)$ is also a version of $E_{\overline{\mu}}[P(\cdot, f^{-1}(C)) \mid \mathscr{B}_f]$. We have seen (Part I) that we can choose R in such a way that $\forall x \in \mathscr{X}$, $R(x, \cdot)$ is a probability on $\mathscr{T}$, and that for $s \geqslant 2$, μ-a.s.,

$$Q_{s,t}(y, C) = R^{(t-s)}(x, C), \quad x \in f^{-1}(y),$$

where $R^{(t-s)}(\cdot, C)$ is a version of $E_{\overline{\mu}}[P_{t-s}(\cdot, f^{-1}(C)) \mid \mathscr{B}_f]$.

More precisely, there exists a set $N_f \in \mathscr{B}_f$ (viz. there exists $N \in \mathscr{T}$ such that $N_f = f^{-1}(N)$), with $\mu(N_f) = 0$, such that $\forall s \geqslant 2$, $\forall t \geqslant s$, $\forall C \in \mathscr{T}$,

$$Q_{s,t}(y, C) = R^{(t-s)}(x, C), \quad x \in f^{-1}(y), \quad \text{if } y \notin N.$$

We specially consider the case where, $\forall s \geqslant 2$,

$$\text{(II.3.1)} \quad \begin{aligned} Q_{s,t}(y, C) &= R^{(t-s)}(x, C), \quad x \in f^{-1}(y), \quad \text{if } y \notin N;\\ Q_{s,s+1}(y, C) &= \mu'(C) \quad \text{if } y \in N, \ \mu' \text{ denoting the image of } \mu \text{ by } f. \end{aligned}$$

II. 4. Proposition. *Under Hypothesis* II.2, *if* $(X_t)_{t\in N^*}$ *is a Markov random function attached to a Markov process* $((\mathscr{X}, \mathscr{B}), P_n)_{n\in N^*}$ *strongly and uniformly ergodic, then* $(f\circ X_t)_{t\in N^*}$ *is a Markov random function attached to a strongly and uniformly ergodic Markov process.*

Proof. 1° By hypothesis, $\lim_{t\to\infty} P_t(x, A) = \xi(A)$ uniformly with respect to x and A, where ξ is a probability on $\mathscr{B}$. Let $\mathscr{T}_0 = (C_0^i)_{i\in N^*}$ be an algebra generating $\mathscr{T}$. $\forall C_0^i \in \mathscr{T}_0$, $R^{(t)}(\cdot, C_0^i)$ is a version of the conditional expectation $E_{\overline{\mu}}[P_t(\cdot, f^{-1}(C_0^i)) \mid \mathscr{B}_f]$. Since $0 \leqslant P_t(\cdot, f^{-1}(C_0^i)) \leqslant 1$, $\forall t \in N^*$, we have

$$\lim_{t\to\infty} R^{(t)}(\cdot, C_0^i) = E_{\overline{\mu}}[\lim_{t\to\infty} P_t(\cdot, f^{-1}(C_0^i)) \mid \mathscr{B}_f] = \xi[f^{-1}(C_0^i)], \quad \mu\text{-a.s.}$$

But $\mathscr{T}_0$ is countable, so that there exists a set $N_1 \in \mathscr{B}_f$ such that $\mu(N_1) = 0$ and

$$\text{(II.4.1)} \quad \lim_{t\to\infty} R^{(t)}(x, C_0^i) = \xi[f^{-1}(C_0^i)], \quad \forall x \notin N_1 \text{ and } \forall C_0^i \in \mathscr{T}_0.$$

Let us now prove that this limit is uniform, μ-a.s. Indeed, by hypothesis, $\forall \varepsilon > 0$, $\exists n_\varepsilon \in N^*$ such that $t > n_\varepsilon$ implies

$$-\varepsilon + \xi[f^{-1}(C_0^i)] < P_t(x, f^{-1}(C_0^i)) < \varepsilon + \xi[f^{-1}(C_0^i)], \quad \forall (x, C_0^i) \in \mathscr{X} \times \mathscr{T}_0.$$

Hence, $\forall B \in \mathscr{B}_f$,

$$(-\varepsilon + \xi[f^{-1}(C_0^i)]) \cdot \mu(B) < \int_B d\mu(x) P_t(x, f^{-1}(C_0^i)) < (\varepsilon + \xi[f^{-1}(C_0^i)]) \cdot \mu(B),$$

and, consequently, $\forall B \in \mathscr{B}_f$,

$$(-\varepsilon + \xi[f^{-1}(C_0^i)]) \cdot \mu(B) < \int_B d\mu(x) R^{(t)}(x, C_0^i) < (\varepsilon + \xi[f^{-1}(C_0^i)]) \cdot \mu(B).$$

This implies that

$$\text{(II.4.2)} \quad -\varepsilon + \xi[f^{-1}(C_0^i)] < R^{(t)}(\cdot, C_0^i) < \varepsilon + \xi[f^{-1}(C_0^i)], \quad \mu\text{-a.s.}$$

By (II.4.1) and (II.4.2), there exists a μ-null set $N_2 \in \mathscr{B}_f$ such that

$$\text{(II.4.3)} \quad \lim_{t\to\infty} R^{(t)}(x, C_0^i) = \xi[f^{-1}(C_0^i)], \quad \forall x \notin N_1 \cup N_2 \text{ and } \forall C_0^i \in \mathscr{T}_0$$

and this limit is uniform with respect to (x, C_0^i) on $\complement(N_1 \cup N_2) \times \mathscr{T}_0$.

2° Let us prove now that (II.4.3) *is verified for every* $C \in \mathscr{T}$. For this purpose, let us prove that the class $\mathscr{T}'$ of sets $C \in \mathscr{T}$ such that (II.4.3) is verified, is a monotone class containing $\mathscr{T}_0$ (and, consequently, $\mathscr{T}'$ contains $\mathscr{T}$).

Indeed, it is clear that $\mathscr{T}'$ contains $\mathscr{T}_0$. It remains to prove that $\mathscr{T}'$ is a monotone class. Let $(C_i)_{i\in N^*}$ be a monotone sequence of elements of $\mathscr{T}$ whose limit is C. For every $x \notin N_1 \cup N_2$, we have

$$\text{(II.4.4)} \quad \lim_{t\to\infty} R^{(t)}(x, C) = \lim_{t\to\infty} \lim_{i\to\infty} R^{(t)}(x, C_i),$$

for $R^{(t)}(x, \cdot)$ is a probability on $\mathscr{T}$.

Since the limit $\lim_{t\to\infty} R^{(t)}(x, C_i)$ is uniform with respect to C_i, we have

$$\text{(II.4.5)} \quad \lim_{t\to\infty} R^{(t)}(x, C) = \lim_{i\to\infty}\lim_{t\to\infty} R^{(t)}(x, C_i) = \lim_{i\to\infty} \xi[f^{-1}(C_i)]$$
$$= \xi[f^{-1}(\lim_{i\to\infty} C_i)] = \xi[f^{-1}(C)], \quad \forall x \notin N_1 \cup N_2.$$

This limit is also *uniform* with respect to (x, C) on $\complement(N_1 \cup N_2) \times \mathcal{T}$, for, by (II.4.2), $t > n_\varepsilon$ and $x \notin N_1 \cup N_2$ implies that

$$-\varepsilon < R^{(t)}(x, C_i) - \xi[f^{-1}(C_i)] < \varepsilon, \quad \forall i \in \boldsymbol{N}^*,$$

and

$$-\varepsilon + \xi[f^{-1}(C)] < R^{(t)}(x, C) < \varepsilon + \xi[f^{-1}(C)].$$

Thus, $C \in \mathcal{T}'$, hence $\mathcal{T}' \supset \mathcal{T}$, consequently $\mathcal{T}' = \mathcal{T}$.

3° We examine now separately the three following cases:

(a) $s \geqslant 2$ and $y \notin f(N_1 \cup N_2)$. Then $f^{-1}(y) \subset \complement(N_1 \cup N_2)$ and, by (II.4.5), we have

$$\text{(II.4.6)} \quad \lim_{t\to\infty} Q_{s,t}(y, C) = \lim_{t\to\infty} R^{(t-s)}(x, C) = \xi[f^{-1}(C)], \quad x \in f^{-1}(y),$$

uniformly with respect to (y, C) on $\complement f(N_1 \cup N_2) \times \mathcal{T}$.

(b) $s \geqslant 2$ and $y \in f(N_1 \cup N_2)$. As indicated in (II.3.1), we take $Q_{s,s+1}(y, C) = \mu'(C)$ and, for $t > s+1$, we have

$$Q_{s,t}(y, C) = \int_{\mathcal{Y}} Q_{s,s+1}(y, dy') Q_{s+1,t}(y', C)$$
$$= \int_{\complement f(N_1 \cup N_2)} Q_{s,s+1}(y, dy') Q_{s+1,t}(y', C).$$

By II.4.6 and the Fatou-Lebesgue theorem, we have

$$\text{(II.4.7)} \quad \lim_{t\to\infty} Q_{s,t}(y, C) = \int_{\complement f(N_1 \cup N_2)} d\mu'(y') \lim_{t\to\infty} Q_{s+1,t}(y', C) = \xi[f^{-1}(C)],$$

uniformly with respect to (y, C) on $f(N_1 \cup N_2) \times \mathcal{T}$.

(c) $s = 1$. We have

$$\lim_{t\to\infty} Q_{1,t}(y, C) = \int_{\mathcal{Y}} Q_{1,2}(y, dy') \cdot \lim_{t\to\infty} Q_{2,t}(y', C) = \xi[f^{-1}(C)],$$

by (II.4.6) and (II.4.7), uniformly with respect to (y, C) on $\mathcal{Y} \times \mathcal{T}$.

Thus, the Markov process $((\mathcal{Y}, \mathcal{T}), Q_{t,t+1})_{t\in\boldsymbol{N}^*}$ is strongly and uniformly ergodic.

II.5. Hypotheses. Let $\mathcal{Y}^* = \{y \in \mathcal{Y}: \mu[f^{-1}(y)] > 0\}$. We remark that $\mathcal{Y}^*$ contains an at most countable set of points, $\mathcal{Y}^* = \{y_1, \ldots, y_n, \ldots\}$, for $\mu(\mathcal{X}) = 1$. I.3 (ii) implies that $\mathcal{Y}^* \in \mathcal{T}$. We suppose that $\mu[f^{-1}(\mathcal{Y}^*)] = 1$ and that $\forall x \in \mathcal{X}$, the series $\sum_k P_t(x, f^{-1}(y_k))$ converges uniformly with respect to t.

In the particular case where $\mathcal{Y}$ is a finite or countable set, the condition $\mu[f^{-1}(\mathcal{Y}^*)] = 1$ is always verified.

II.6. Proposition. *If $(X_t)_{t\in\boldsymbol{N}^*}$ is attached to a strongly ergodic Markov process $((\mathcal{X}, \mathcal{B}), P_t)_{t\in\boldsymbol{N}^*}$ and if, moreover, the hypotheses* II.2 *and* II.5 *are satisfied, then $(f\circ X_t)_{t\in\boldsymbol{N}^*}$ is attached to a strongly ergodic Markov process.*

Proof. By hypothesis, there exists no μ-null set $N \in \mathcal{B}_f$ contained in $f^{-1}(\mathcal{Y}^*)$. $\forall y_k \in \mathcal{Y}^*$, there exists a μ-null set $N_k \in \mathcal{B}_f$ such that

$$\lim_{t\to\infty} R^{(t)}(x, \{y_k\}) = E_{\bar\mu}\left[\lim_{t\to\infty} P_t(x, f^{-1}(y_k)) \mid \mathcal{B}_f\right] = \xi[f^{-1}(y_k)], \quad \forall x \notin N_k.$$

Hence, there exists $N = \bigcup_{k=1}^{\infty} N_k \in \mathcal{B}_f$, μ-null, such that

$$\lim_{t\to\infty} R^{(t)}(x, \{y_k\}) = \xi[f^{-1}(y_k)], \quad \forall x \notin N.$$

We must now prove that, $\forall C \subset \mathcal{Y}^*$,

$$\lim_{t\to\infty} R^{(t)}(x, C) = \xi[f^{-1}(C)], \quad \forall x \notin N.$$

This formula is true for every C which is a finite union of $\{y_k\}$. It remains for us to examine the case where C is a countable union of $\{y_k\}$. Let $C = \bigcup_{j=1}^{\infty} \{y_{k_j}\}$ be such a countable union. We have, $\forall x \notin N$,

$$\lim_{t\to\infty} R^{(t)}(x, C) = \lim_{t\to\infty} \sum_{j=1}^{\infty} R^{(t)}(x, \{y_{k_j}\})$$
$$= \lim_{t\to\infty} \sum_{j=1}^{\infty} P_t(x, f^{-1}(y_{k_j}))$$
$$= \sum_{j=1}^{\infty} \lim_{t\to\infty} P_t(x, f^{-1}(y_{k_j}))$$
$$= \sum_{j=1}^{\infty} \lim_{t\to\infty} R^{(t)}(x, \{y_{k_j}\})$$
$$= \sum_{j=1}^{\infty} \xi[f^{-1}(y_{k_j})]$$
$$= \xi[f^{-1}(C)].$$

For the rest of the proof, we use the same argument as that in II.4.3° with the three cases: $s \geqslant 2$ and $y \in \complement f(N)$; $s \geqslant 2$ and $y \in f(N)$; $s \geqslant 1$.

II.7. Proposition. *If $(X_t)_{t\in N^*}$ is attached to a weakly ergodic homogeneous Markov process, and if* II.2 *and* II.5 *are satisfied, then $(f\circ X_t)_{t\in N^*}$ is attached to a weakly ergodic Markov process $\big((\mathscr{Y},\mathscr{T}),Q_{t,t+1}\big)_{t\in N^*}$.*

Proof. We use, for the proof of this proposition, the following criteria for weak ergodicity (cf. [1]):

A Markov process $\big((\Lambda,\mathscr{L}),q_{t,t+1}\big)_{t\in N^*}$ is weakly ergodic if and only if for every $s\in N^*$, $B\in\mathscr{L}$, for every increasing sequence $(t_j)_{j\in N^*}$ of indices such that $q_{s,t_j}(u,B)$ converges to a limit when $j\to\infty$, for some $u\in\Lambda$, $q_{s,t_j}(v,B)$ converges $\forall v\in\Lambda$ to the same limit; moreover, this common limit is independent of s.

1° Let $(x_0,C)\in\mathscr{X}\times\mathscr{T}$, and let $\sigma=(t_i)_{i\in N^*}$ be an increasing sequence of indices such that the sequence $\big(P_{t_i}(x_0,f^{-1}(C))\big)_{i\in N^*}$ is convergent. The criteria cited above implies that $\forall s\in N^*$ and $\forall x\in\mathscr{X}$, the sequence $\big(P_{t_i-s}(x,f^{-1}(C))\big)_{i\in N^*}$ converges to the same limit. Let $l(\sigma,C)$ denote the common limit of these sequences.

An argument analogous to that in II.6.1° shows that

$$\lim_{i\to\infty}R^{(t_i-s)}(\cdot,C)=l(\sigma,C),\qquad \mu\text{-a.s.},$$

and since there exists no μ-null element of $\mathscr{B}_f$, contained in $f^{-1}(\mathscr{Y}^*)$, we have

$$\lim_{i\to\infty}R^{(t_i-s)}(x,C)=l(\sigma,C),\qquad \forall x\in f^{-1}(\mathscr{Y}^*).$$

Using the same argument employed in II.4.3°, with the 3 cases (a), (b), (c), we have $\forall C\in\mathscr{T}$, $\forall\sigma$,

$$\lim_{i\to\infty}Q_{s,t_i}(y,C)=l(\sigma,C),\qquad \forall s\in N^*\text{ and }\forall y\in\mathscr{Y}.$$

2° Now let $s_0\in N^*$, $C\in\mathscr{T}$ and let $\sigma=(t_j)_{j\in N^*}$ be a sequence of indices such that $\big(Q_{s_0,t_j}(y_0,C)\big)_{j\in N^*}$ is convergent for a certain $y_0\in\mathscr{Y}$. Let $l(\sigma,s_0,y_0,C)$ denote this limit. Following the criteria, we must prove that this limit is independent of y_0 and s_0. Let us examine separately the 3 following cases:

(a) $s\geqslant 2$ and $y_0\in\mathscr{Y}^*$. Then, we have

$$Q_{s_0,t_j}(y_0,C)=R^{(t_j-s_0)}(x_0,C),\qquad x_0\in f^{-1}(y_0),$$

and

$$\text{(II.7.1)}\qquad R^{(t_j-s_0)}(x_0,C)\cdot\mu[f^{-1}(y_0)]=\int_{f^{-1}(y_0)}d\mu(x)\cdot P_{t_j-s_0}\big(x,f^{-1}(C)\big).$$

The sequence $\big(P_{t_j-s_0}(x_0,f^{-1}(C))\big)_{j\in N^*}$ being a sequence of numbers contained in the interval $[0,1]$, we can find a convergent subsequence $\big(P_{t_{j_k}-s_0}(x_0,f^{-1}(C))\big)_{k\in N^*}$. Since $\big((\mathscr{X},\mathscr{B}),P_t\big)_{t\in N^*}$ is weakly ergodic, the

limit of this convergent subsequence is independent of x_0 and of s_0. Let $l(\sigma',C)$ denote this limit, σ' being the subsequence $(t_{j_k})_{k\in N^*}$.

By the Fatou-Lebesgue theorem, (II.7.1) implies that

$$\lim_{j\to\infty}R^{(t_{j_k}-s_0)}(x_0,C)\cdot\mu[f^{-1}(y_0)]=\int_{f^{-1}(y_0)}d\bar{\mu}(x)\cdot\lim_{k\to\infty}P_{t_{j_k}-s_0}\big(x,f^{-1}(C)\big),$$

hence

$$l(\sigma,s_0,y_0,C)\cdot\mu[f^{-1}(y_0)]=l(\sigma',C)\cdot\mu[f^{-1}(y_0)].$$

Consequently, since $\mu[f^{-1}(y_0)]>0$,

$$l(\sigma,s_0,y_0,C)=l(\sigma',C).$$

Thus, all the convergent subsequences of the sequence $\big(P_{t_j-s_0}(x_0,f^{-1}(C))\big)_{j\in N^*}$ have the same limit: the sequence itself is then convergent, and

$$\lim_{j\to\infty}P_{t_j-s_0}\big(x_0,f^{-1}(C)\big)=l(\sigma,s_0,y_0,C).$$

By 1° $\forall s\in N^*$ and $\forall y\in\mathscr{Y}$:

$$\lim_{j\to\infty}Q_{s,t_j}(y,C)=\lim_{j\to\infty}Q_{s_0,t_j}(y_0,C).$$

(b) $s_0\geqslant 2$ and $y_0\notin\mathscr{Y}^*$. Then, $f^{-1}(y_0)\subset\complement f^{-1}(\mathscr{Y}^*)$. We have

$$Q_{s_0,t_j}(y_0,C)=\int_{\mathscr{Y}^*}d\mu'(y)Q_{s_0+1,t_j}(y,C).$$

Let $y_1\in\mathscr{Y}^*$, and let $\big(Q_{s_0+1,t_{j_k}}(y_1,C)\big)_{k\in N^*}$ be a convergent subsequence. By (a), $\forall y\in\mathscr{Y}$, the sequence $\big(\big(Q_{s_0+1,t_{j_k}}(y,C)\big)\big)_{k\in N^*}$ converges to the same limit, the Fatou-Lebesgue theorem gives

$$\lim_{k\to\infty}Q_{s_0,t_{j_k}}(y_0,C)=\int_{\mathscr{Y}^*}d\mu'(y)\cdot\lim_{k\to\infty}Q_{s_0+1,t_{j_k}}(y,C),$$

that is to say

$$l(\sigma,s_0,y_0,C)=\lim_{k\to\infty}Q_{s_0+1,t_{j_k}}(y_1,C).$$

Thus, all the convergent subsequences of the sequence $\big(Q_{s_0+1,t_j}(y_0,C)\big)_{j\in N^*}$ have the same limit. Hence, the sequence itself is convergent, and

$$\lim_{j\to\infty}Q_{s_0+1,t_j}(y_1,C)=l(\sigma,s_0,y_0,C).$$

By 1° $\forall s\in N^*$ and $\forall y\in\mathscr{Y}$,

$$\lim_{j\to\infty}Q_{s,t_j}(y,C)=\lim_{j\to\infty}Q_{s,t_j}(y_0,C).$$

(c) $s_0=1$. We have

$$Q_{1,t_j}(y_0,C)=\int_{\mathscr{Y}}Q_{1,2}(y_0,dy)Q_{2,t_j}(y,C).$$

Let $y_2 \in \mathscr{Y}$ and let $(Q_{2,t_{j_k}}(y_2, C))_{k \in \mathbf{N}^*}$ be a convergent subsequence. By (a) and (b), the sequence $(Q_{2,t_{j_k}}(y, C))_{k \in \mathbf{N}^*}$ converges to a limit independent of $y \in \mathscr{Y}$. The Fatou-Lebesgue theorem shows that

$$\lim_{k\to\infty} Q_{1,t_{j_k}}(y_0, C) = \int_{\mathscr{Y}} Q_{1,2}(y_0, dy) \lim_{k\to\infty} Q_{2,t_{j_k}}(y, C),$$

viz.

$$l(\sigma, s_0, y_0, C) = \lim_{k\to\infty} Q_{2,t_{j_k}}(y_2, C).$$

All the convergent subsequences of $(Q_{2,t_j}(y_2, C))_{j \in \mathbf{N}^*}$ having the same limit, the sequence itself is convergent, and we have

$$\lim_{j\to\infty} Q_{2,t_j}(y_2, C) = l(\sigma, s_0, y_0, C).$$

Consequently,

$$\lim_{j\to\infty} Q_{1,t_j}(y, C) = l(\sigma, s_0, y_0, C) = \lim_{j\to\infty} Q_{1,t_j}(y_0, C).$$

Thus, in every case, $l(\sigma, s_0, y_0, C)$ is independent of s_0 and y_0, and the Markov process $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t \in \mathbf{N}^*}$ is weakly ergodic.

II.8. Examples. Here are some examples in the case of a finite state space.

1° $\mathscr{X} = \{1, 2, 3, 4\}$, $\mathscr{B} = \mathscr{P}(\mathscr{X})$ and

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{10} & \frac{2}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}.$$

We can verify that the process $((\mathscr{X}, \mathscr{P}(\mathscr{X})), P_t)_{t \in \mathbf{N}^*}$ is strongly and uniformly ergodic, for $\mathscr{X}$ is finite and

$$\lim_{t\to\infty} P_t = \begin{pmatrix} \frac{11}{47} & \frac{11}{47} & \frac{25}{94} & \frac{25}{94} \\ \frac{11}{47} & \frac{11}{47} & \frac{25}{94} & \frac{25}{94} \\ \frac{11}{47} & \frac{11}{47} & \frac{25}{94} & \frac{25}{94} \\ \frac{11}{47} & \frac{11}{47} & \frac{25}{94} & \frac{25}{94} \end{pmatrix}.$$

Now, consider $f\colon \mathscr{X} \to \mathscr{Y}$ such that $f(1) = f(2) = 1$; $f(3) = f(4) = 2$. We take μ on $\mathscr{B}_f$ such that

$$\mu[f^{-1}(i)] = \sum_{j \in \mathscr{X}} a_j P(j, f^{-1}(i)), \quad \forall i \in \mathscr{Y} \text{ where } a_j > 0, \ \sum_{j \in \mathscr{X}} a_j = 1.$$

Hypothesis II.2, in our case, is satisfied by the mapping $R\colon \mathscr{X} \times \mathscr{P}(\mathscr{Y}) \to [0, 1]$ defined by

$$R(j_0, \{1\}) = \tfrac{7}{12}, \quad R(j_0, \{2\}) = \tfrac{5}{12} \quad \text{for } j_0 = 1, 2,$$
$$R(j_0, \{1\}) = \tfrac{11}{30}, \quad R(j_0, \{2\}) = \tfrac{19}{30} \quad \text{for } j_0 = 3, 4.$$

Then, we have

$$Q_{t,t+1} = \begin{pmatrix} \frac{7}{12} & \frac{5}{12} \\ \frac{11}{30} & \frac{19}{30} \end{pmatrix}, \quad \forall t \geqslant 2.$$

Computations show that

$$\lim_{t\to\infty} Q_{s,t} = \begin{pmatrix} \frac{22}{47} & \frac{25}{47} \\ \frac{11}{30} & \frac{19}{30} \end{pmatrix}, \quad \forall s \geqslant 2,$$

and

$$\lim_{t\to\infty} Q_{1,t} = Q_{1,2} \cdot \lim_{t\to\infty} Q_{2,t} = \lim_{t\to\infty} Q_{2,t},$$

for $Q_{1,2}$ is a stochastic matrix and $\lim_{t\to\infty} Q_{2,t}$ is with all its rows identical.

Thus, $((\mathscr{Y}, \mathscr{P}(\mathscr{Y})), Q_{t,t+1})_{t \in \mathbf{N}^*}$ is also strongly and uniformly ergodic.

2° $\mathscr{X} = \{1, 2, 3\}$, $\mathscr{B} = \mathscr{P}(\mathscr{X})$ and

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

The Markov process $((\mathscr{X}, \mathscr{P}(\mathscr{X})), P_t)_{t \in \mathbf{N}^*}$ is strongly and uniformly ergodic, for $\mathscr{X}$ is finite and $P_t = P$, $\forall t \in \mathbf{N}^*$. Let $\mathscr{Y} = \{1, 2\}$ and $f\colon \mathscr{X} \to \mathscr{Y}$ be defined by $f(1) = f(2) = 1$, and $f(3) = 2$.

We take μ on $\mathscr{B}_f$ such that

$$\mu[f^{-1}(i)] = \sum_{j \in \mathscr{X}} a_j P(j, f^{-1}(i)), \quad \forall i \in \mathscr{Y}, \text{ where } a_j > 0, \ \sum_{j \in \mathscr{X}} a_j = 1.$$

Thus,

$$\mu[f^{-1}(1)] = 1 \quad \text{and} \quad \mu[f^{-1}(2)] = 0.$$

Hypothesis II.2 is satisfied by the mapping $R\colon \mathscr{X} \times \mathscr{P}(\mathscr{Y}) \to [0, 1]$ defined by

$$R(1, \{1\}) = R(2, \{1\}) = 1,$$
$$R(1, \{2\}) = R(2, \{2\}) = 0,$$
$$R(3, \{1\}) + R(3, \{2\}) = 1,$$

where $R(3, \{1\})$ can be arbitrarily chosen.

Hence, we can take

$$Q_{t,t+1} = \begin{pmatrix} 1 & 0 \\ \nu[f^{-1}(1)] & \nu[f^{-1}(2)] \end{pmatrix} \quad \text{for} \quad t \geq 2,$$

where ν is an arbitrarily chosen probability measure on $\mathscr{B}_f$.

a) If ν is such that $\nu[f^{-1}(1)] = 0$ and $\nu[f^{-1}(2)] = 1$, then

$$Q_{t,t+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall t \geq 2,$$

and the process $\big((\mathscr{Y}, \mathscr{P}(\mathscr{Y})), Q_{t,t+1}\big)_{t\in N^*}$ is not strongly ergodic.

b) If ν is such that $\nu[f^{-1}(1)] = \alpha$, where $\alpha \in]0, 1]$, then

$$Q_{t,t+1} = \begin{pmatrix} 1 & 0 \\ \alpha & 1-\alpha \end{pmatrix}, \quad \forall t \geq 2.$$

We have

$$\lim_{t\to\infty} Q_{s,s+t} = \lim_{t\to\infty} \begin{pmatrix} 1 & 0 \\ 1-(1-\alpha)^t & (1-\alpha)^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \forall s \geq 2,$$

and, consequently, the process $\big((\mathscr{Y}, \mathscr{P}(\mathscr{Y})), Q_{t,t+1}\big)_{t\in N^*}$ is strongly ergodic.

We examine now the converse for the proposition II.7.

II.9. Hypotheses. (i) $\forall x \in \mathscr{X}$, $P(x, \cdot)$ is of the form

(II.9.1) $$P(x, B) = \int_B g(x, x')\, d\bar{\mu}(x'),$$

where $B \in \mathscr{B}$, and $g(x, \cdot)$ is $\mathscr{B}_f$-measurable.

(ii) $\forall (x, B) \in \mathscr{X} \times \mathscr{B}$, if $(t_j)_{j\in N^*}$ is an increasing sequence of indices such that the sequence $\big(P_{t_j}(x, B)\big)_{j\in N^*}$ converges, then $\forall x' \in f^{-1}\big(f(x)\big)$, the sequence $\big(P_{t_j}(x', B)\big)_{j\in N^*}$ converges and

(II.9.2) $$\lim_{j\to\infty} P_{t_j}(x, B) = \lim_{j\to\infty} P_{t_j}(x', B).$$

II.10. The following is an interpretation of II.9: II.9 (i) implies the condition of Doeblin (cf. [2]) for the homogeneous Markov process $\big((\mathscr{X}, \mathscr{B}), P_t\big)_{t\in N^*}$. II 9. (ii) implies that $\forall y \in \mathscr{Y}$, if $f^{-1}(y)$ is included in an ergodic set (cf. [2]) decomposable in cyclic subsets $\Gamma_1, \ldots, \Gamma_d$, then $f^{-1}(y)$ is included in one of these cyclic subsets. Indeed, suppose that there exist two distinct cyclic subsets Γ_k and $\Gamma_{k'}$ such that $\Gamma_k \cap f^{-1}(y) \neq \emptyset$ and $\Gamma_{k'} \cap f^{-1}(y) \neq \emptyset$. We know, by [2], that there exist probabilities ${}_k\pi$ and ${}_{k'}\pi$ on $\mathscr{B}$ such that ${}_k\pi(\Gamma_k) = 1$ and ${}_{k'}\pi(\Gamma_{k'}) = 1$ and that

$$\lim_{n\to\infty} P_{nd}(x_k, \Gamma_k) = {}_k\pi(\Gamma_k) = 1,$$

$$\lim_{n\to\infty} P_{nd}(x_{k'}, \Gamma_{k'}) = {}_{k'}\pi(\Gamma_{k'}) = 0,$$

and thus, II.9 (ii) is not be verified.

II.11. Proposition. *Under* II.9 (i), $\forall t \in N^*$ *and* $\forall (x, B) \in \mathscr{X} \times \mathscr{B}$,

$$P_t(x, B) = \int_B g^{(t)}(x, x')\, d\bar{\mu}(x'),$$

where the $g^{(t)}$, $\mathscr{B}_f$*-measurable, are defined by recurrence:*

$$g^{(1)} = g,$$

$$g^{(t)}(x, x') = \int_{\mathscr{X}} g^{(t-1)}(x, x_1)\, g(x_1, x')\, d\bar{\mu}(x_1), \quad \forall x \text{ and } x' \in \mathscr{X}.$$

Proof. We proceed by induction. It is easy to verify the formula for $t = 2$. Suppose now that

$$P_{t-1}(x, B) = \int_B g^{(t-1)}(x, x')\, d\bar{\mu}(x').$$

We have

$$\begin{aligned} P_t(x, B) &= \int_{\mathscr{X}} P_{t-1}(x, dx_1) P(x_1, B) \\ &= \int_{\mathscr{X}} g^{(t-1)}(x, x_1)\, d\bar{\mu}(x_1) \cdot \int_B g(x_1, x')\, d\bar{\mu}(x') \\ &= \int_B \Big[\int_{\mathscr{X}} g^{(t-1)}(x, x_1)\, g(x_1, x')\, d\bar{\mu}(x_1)\Big]\, d\bar{\mu}(x') \\ &= \int_B g^{(t)}(x, x')\, d\bar{\mu}(x'). \end{aligned}$$

II.12. Proposition. *Suppose that* II.5 *and* II.9 *are satisfied. Let* $\big((\mathscr{Y}, \mathscr{T}), Q_{t,t+1}\big)_{t\in N^*}$, *to which* $(f \circ X_t)_{t\in N}$ *is attached, be defined by* (II. 3.1) *with* $N = f^{-1}(\mathscr{Y}^*)$. *If* $\big((\mathscr{Y}, \mathscr{T}), Q_{t,t+1}\big)_{t\in N^*}$ *is strongly ergodic, then* $\big((\mathscr{X}, \mathscr{B}), P_t\big)_{t\in N^*}$ *is also strongly ergodic.*

Proof. By hypothesis, $\forall s \in N^*$, $\forall y \in \mathscr{Y}$ and $\forall C \in \mathscr{T}$,

(II.12.1) $\lim_{t\to\infty} Q_{s,s+t}(y, C) = \pi(C)$, where π is a probability on $\mathscr{T}$.

1° At first, let us examine the limit $\lim_{t\to\infty} P_t(x, B)$, for $x \in \mathscr{X}$ and $B \in \mathscr{B}_f$.

Let $x_0 \in \mathscr{X}$, $y_0 = f(x_0)$ and $C \in \mathscr{T}$.

(a) If $x_0 \in f^{-1}(\mathscr{Y}^*)$, then $f^{-1}(y_0) \subset f^{-1}(\mathscr{Y}^*)$, and

$$Q_{s\,s+t}(y_0, C) = R^{(t)}(x_0, C).$$

Hence,

(II.12.2) $$Q_{s,s+t}(y_0, C) \cdot \mu[f^{-1}(y_0)] = \int_{f^{-1}(y_0)} d\bar{\mu}(x) P_t\big(x, f^{-1}(C)\big).$$

$\big(P_t(x_0, f^{-1}(C))\big)_{t\in N^*}$ being a sequence of numbers of $[0, 1]$, we can find a convergent subsequence $\big(P_{t_j}(x_0, f^{-1}(C))\big)_{j\in N^*}$ whose limit is $l(x_0, \sigma, C)$,

σ denoting the increasing sequence $(t_j)_{j\in N^*}$ of indices. By II.9 (ii), we have $\forall x \in f^{-1}(y_0)$,

(II.12.3) $$\lim_{j\to\infty} P_{t_j}(x, f^{-1}(C)) = l(x_0, \sigma, C).$$

Consequently, the Fatou-Lebesgue theorem, (II.12.1), (II.12.2) and (II.12.3) imply that

$$\lim_{j\to\infty} Q_{s,s+t_j}(y_0, C)\cdot \mu[f^{-1}(y_0)] = \int_{f^{-1}(y_0)} d\bar{\mu}(x)\cdot \lim_{j\to\infty} P_{t_j}(x, f^{-1}(C)),$$

viz.

$$\lim_{j\to\infty} Q_{s,s+t_j}(y_0, C)\cdot \mu[f^{-1}(y_0)] = l(x_0, \sigma, C)\cdot \mu[f^{-1}(y_0)],$$

viz.

$$\pi(C) = l(x_0, \sigma, C), \quad \text{for} \quad \mu[f^{-1}(y_0)] > 0.$$

All the convergent subsequences of the sequence $(P_t(x_0, f^{-1}(C)))_{t\in N^*}$ having the same limit, the sequence itself is then convergent and we have

(II.12.4) $$\lim_{t\to\infty} P_t(x_0, f^{-1}(C)) = \pi(C), \quad \forall x_0 \in f^{-1}(\mathscr{Y}^*) \text{ and } \forall C \in \mathscr{T}.$$

(b) If $x_0 \in \complement f^{-1}(\mathscr{Y}^*)$, we have, for $t \geqslant 2$,

$$P_t(x_0, f^{-1}(C)) = \int_{\mathscr{X}} P(x_0, dx) P_{t-1}(x, f^{-1}(C)).$$

Since $\mu[f^{-1}(\mathscr{Y}^*)] = 1$, we have $P(x, f^{-1}(\mathscr{Y}^*)) = 1$, $\forall x \in \mathscr{X}$. Hence,

$$P_t(x_0, f^{-1}(C)) = \int_{f^{-1}(\mathscr{Y}^*)} P(x_0, dx) P_{t-1}(x, f^{-1}(C)),$$

so that, by (II.12.4) and the Fatou-Lebesgue theorem,

(II.12.5) $$\lim_{t\to\infty} P_t(x_0, f^{-1}(C)) = \int_{f^{-1}(\mathscr{Y}^*)} P(x_0, dx) \lim_{t\to\infty} P_{t-1}(x, f^{-1}(C)) = \pi(C).$$

Thus, (a) and (b) show that $\forall (x, C) \in \mathscr{X}\times\mathscr{T}$,

(II.12.6) $$\lim_{t\to\infty} P_t(x, f^{-1}(C)) = \pi(C).$$

2° We examine now the limit $\lim_{t\to\infty} P_t(x, B)$ for $x \in \mathscr{X}$ and $B \in \mathscr{B}$. Let x and $x' \in \mathscr{X}$, and $y' = f(x')$. By (II.12.6) and II.10, we have

$$\lim_{t\to\infty} P_t(x, f^{-1}(y')) = \pi(\{y'\}) = \lim_{t\to\infty} \int_{f^{-1}(y')} g^{(t)}(x, x_1)\, d\bar{\mu}(x_1).$$

$g^{(t)}(x, \cdot)$ being by hypothesis $\mathscr{B}_f$-measurable, is then constant on $f^{-1}(y')$ which is an atom of $\mathscr{B}_f$. Hence,

$$\pi(\{y'\}) = \lim_{t\to\infty} g^{(t)}(x, x')\cdot \mu[f^{-1}(y')].$$

We then conclude that, $\forall x \in \mathscr{X}$, the sequence $(g^{(t)}(x, \cdot))_{t\in N^*}$ converges μ-a.s. to a limit g' independent of $x \in \mathscr{X}$.

Consequently,

(II.12.7) $$\lim_{t\to\infty} P_t(x, B) = \lim_{t\to\infty} \int_B g^{(t)}(x, x_1)\, d\bar{\mu}(x_1) = \int_B \lim_{t\to\infty} g^{(t)}(x, x_1)\, d\bar{\mu}(x_1) = \int_B g'(x_1)\, d\bar{\mu}(x_1).$$

This last equality proves the strong ergodicity of $((\mathscr{X}, \mathscr{B}), P_{t,t+1})_{t\in N^*}$.

In fact, we have a stronger result than Proposition II.12, but its proof makes use of the latter.

II. 13. Proposition. *Suppose that* II.5 *and* II.9 *are satisfied. Let* $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t\in N^*}$, *to which* $(f\circ X_t)_{t\in N^*}$ *is attached, be defined by* (II.3.1) *with* $N = f^{-1}(\mathscr{Y}^*)$. *If* $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t\in N^*}$ *is weakly ergodic, then* $((\mathscr{X}, \mathscr{B}), P_t)_{t\in N^*}$ *is strongly ergodic.*

Proof. 1° II.9 (i) implies that the Markov process $((\mathscr{X}, \mathscr{B}), P_t)_{t\in N^*}$ verifies the Doeblin condition. The Markov process $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t\geqslant 2, t\in N^*}$ defined by (II.3.1) with $N = f^{-1}(\mathscr{Y}^*)$ is homogeneous. It also verifies the Doeblin condition, for $s \geqslant 2$.

2° We know (cf. [1]) that, for a homogeneous Markov process satisfying the Doeblin condition, weak ergodicity and strong ergodicity are equivalent. Consequently, the Markov process $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t\geqslant 2, t\in N^*}$ is strongly ergodic. Then, $\forall s \geqslant 2$,

$$\forall (y, C) \in \mathscr{Y}\times\mathscr{T}, \quad \lim_{t\to\infty} Q_{s,s+t}(y, C) = \pi(C).$$

3° Let us prove now that $((\mathscr{Y}, \mathscr{T}), Q_{t,t+1})_{t\in N^*}$ is strongly ergodic. Indeed, for $s = 1$, and $\forall (y, C) \in \mathscr{Y}\times\mathscr{T}$,

$$\lim_{t\to\infty} Q_{1,1+t}(y, C) = \lim_{t\to\infty} \int_{\mathscr{Y}} Q_{1,2}(y, dy') Q_{2,1+t}(y', C) = \int_{\mathscr{Y}} Q_{1,2}(y, dy') \lim_{t\to\infty} Q_{2,1+t}(y', C) = \pi(C).$$

4° By III.12, we see that the process $((\mathscr{X}, \mathscr{B}), P_t)_{t\in N^*}$ is strongly ergodic.

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On the zeroes of some random functions

by

R. KAUFMAN (Urbana, Ill.)

Let $F(t)$ be a Fourier series with random coefficients and phases,

$$F(t)=\sum_{n=1}^{\infty}a_nX_n\cos(nt+\Phi_n).$$

Here $(X_n)_1^\infty$ is a sequence of mutually independent Gaussian variables of type $N(0,1)$; $(\Phi(n))_1^\infty$ is a sequence of mutually independent variables, uniformly distributed upon $[0,2\pi]$; and the X's and Φ's are mutually independent. (The basic probability space will be denoted (Ω, P).) About the numbers a_n we suppose

$$a_n>0,\quad \log a_n=-\beta\log n+o(\log n),\quad \text{with}\ \tfrac{1}{2}<\beta\leqslant 1.$$

Our goal is an estimation of the zero-set of F, $Z(\omega)=\{t\colon F(t,\omega)=0\}$.

THEOREM. *Let B be a closed set in $[0,2\pi]$ of Hausdorff dimension d. Then*

$$P\{\dim(Z\cap B)\leqslant d-\beta+\tfrac{1}{2}\}=1,\quad d\geqslant\beta-\tfrac{1}{2},$$

$$P\{Z\cap B=\emptyset\}=1,\quad d<\beta-\tfrac{1}{2},$$

$$P\{\dim(Z\cap B)\geqslant d_1\}>0,\quad 0<d_1<d-\beta+\tfrac{1}{2}.$$

In § 1 we prove a general principle for the lower bound, whose application is dependent upon specific estimates, derived later about F. In § 2 we review some conclusions from [2] about the uniform convergence of F and its modulus of continuity, and we also obtain a technical lemma about the local character of the trajectories of F. In § 3 we obtain an upper bound for the dimension, and in § 4 a lower bound is obtained by combining the work of §§ 1 and 2.

§ 1. Let B be a compact set of real numbers and μ a probability measure in B such that:

(i) $\mu([a,a+h])\leqslant C_1h^d$ for constants $C_1, d>0$ and all intervals $[a,a+h]$.