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# A covering lemma with applications to differentiability of measures and singular integral operators

by

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## § 0. INTRODUCTION

In 1945 Besicovitch made use of a lemma of geometric type, involving spheres in $\boldsymbol{R}^n$, to obtain from it several results on differentiability of measures [1], [2]. A little later A. P. Morse [14] generalized such results substituting spheres by certain families of nearly spherical starshaped sets (1). Later on Cotlar [6], [7] presented and used in several problems dealing with singular integrals, a lemma of the type introduced by Besicovitch. Such a lemma, dealing with cubes in $\boldsymbol{R}^n$, is introduced by him as a sharpening of a result used by Wiener and others, but the geometric character (uniformly finite overlapping of the covering) of the result seems to belong to him independently of Besicovitch. Moreover, he was the first in making use of this kind of lemma in singular integrals, substituting by it the covering lemma used by Calderón and Zygmund in [5].

In the first section of this paper we present a geometric covering lemma of the same type. We give two versions of it, one in $\boldsymbol{R}^n$ which is extremely simple and already sufficient for the applications to singular integrals we present here and to obtain satisfactory results in differentiation theory. The second version, more general and powerful in differentiation theory, is a little more complicated. The method of proof we use here, a sort of generalization of the one-dimensional result by means of a combinatorial theorem (Ramsey's theorem) is interesting in itself and might be of use in other situations. In the second section we make use of the covering lemma to obtain a new version of the Vitali's covering theorem and some new results on differentiation of measures. In the third section we present an application of the same covering lemma to obtain a theorem on singular integral operators of convolution type which includes

(1) For detailed information on these and other results on differentiation theory we refer te the excellent expository article of Bruckner [3].

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and generalizes some results on operators associated to kernels with mixed homogeneity obtained by Jones [13] and Fabes and Rivière [8], [9]. For another study of some properties of such operators we refer also to [10], [11], [12].

The results presented here are included in my Ph. D. Dissertation at the University of Chicago and my Doctoral Thesis at the University of Madrid. My work has been mainly done at the University of Chicago under the direction of Professor Alberto P. Calderón. I wish to acknowledge here his kind and generous advice. Many helpful suggestions were made by Professor Alberto M. Dou, my director at the University of Madrid. To him also I am most grateful. Many conversations with R. R. Coifman and N. M. Rivière have been particularly helpful.

## § 1. THE COVERING LEMMA

**1.1.** LEMMA. *Let $\{R_i\}_{i=1,2,\ldots,k}$ be a finite sequence of non-decreasing, open intervals of $\boldsymbol{R}^n$ centered at 0. Let $S$ be a bounded set of $\boldsymbol{R}^n$. For every $x \in S$ we take an integer $i(x)$, $1 \leqslant i(x) \leqslant k$, and the set $R_x = R_{i(x)} + x$. Then one can choose a finite number of elements of $S$, $x_1, x_2, \ldots, x_l$, such that $S \subset \bigcup_{j=1}^{l} R_{x_j}$ and every $y \in \boldsymbol{R}^n$ is at most in $2^n$ of these sets $\{R_{x_j}\}$.*

Proof. Choose $x_1$ such that $i(x_1)$ is largest possible. Assume $x_1, \ldots, x_m$ already chosen. Take then $x_{m+1} \in S - \bigcup_{j=1}^{m} R_{x_j}$ such that $i(x_{m+1})$ is largest possible. Since $S$ is bounded and the $R_{x_j}$'s we thus obtain are such that $R^*_{x_j} = x_j + \frac{1}{2} R_{i(x_j)}$ are obviously disjoint, we end this selection process in a finite number of steps, $l$, obtaining $S \subset \bigcup_{j=1}^{l} R_{x_j}$. We now prove that any $y \in \boldsymbol{R}^n$ is at most in $2^n$ sets $R_{x_j}$. To see this, through $y$ draw $n$ hyperplanes parallel to the coordinate hyperplanes and consider the $2^n$ closed quadrants so obtained. In each quadrant there is at most one $x_j$ with $y \in R_{x_j}$, for if there were two, the larger $R_{x_j}$ would contain the centre of the smaller one and this is excluded by construction. This proves the lemma.

**1.2.** LEMMA. *Let $(M, \varrho)$ be a metric space. Call $B(x, r) = \{m \in M: \varrho(m, x) < r\}$. Assume that the following properties are satisfied:*

($\alpha$) *Every ball contains at most a finite number of disjoint balls of a given positive radius.*

($\beta$) *There is a positive integer $\xi$ such that for every $p \in M$ there are at most $\xi - 1$ balls $B(x_i, r_i)$, $i = 1, 2, \ldots, \xi - 1$, such that*

(1) $p \in B(x_i, r_i) - B(x_i, \frac{1}{3} r_i)$, $i = 1, 2, \ldots, \xi - 1$,

(2) $r_{i+1} \leqslant \frac{4}{3} r_i$, $i = 1, \ldots, \xi - 2$,

(3) $x_j \notin B(x_i, r_i)$ *for* $j > i$.



($\gamma$) *There is a positive integer $\theta$ such that for every ball $B(x, r)$ there are at most $\theta - 1$ balls $B(x_i, r_i)$, $i = 1, \ldots, \theta - 1$, such that*

(1) $B(x, r) \cap B(x_i, r_i) \neq \emptyset$, $i = 1, 2, \ldots, \theta - 1$,

(2) $r_i \geqslant \frac{3}{4} r$, $i = 1, \ldots, \theta - 1$,

(3) $x_j \notin B(x_i, r_i)$ *for* $j > i$.

*Then, given $S \subset M$ bounded, and a mapping $x \in S \to B_x = B(x, r(x))$, $r(x) > 0$, one can select from $(B_x)_{x \in S}$ a sequence $\{B_k\}$ such that*

(a) $S \subset \bigcup B_k$.

(b) *Every $m \in M$ is at most in $\xi$ balls $B_k$.*

(c) *The family $\{B_k\}$ can be split into $\theta$ disjoint families.*

**1.3.** Remark. It is easy to see that $\boldsymbol{R}$ with the metric of the absolute value satisfies all conditions of Lemma 1.2. Also $\boldsymbol{R}^n$ with the Euclidean distance or with $\varrho(x, y) = \max(|x_i - y_i|)$ satisfies these properties as one can see with an easy geometrical argument. This is the lemma of Besicovitch in [1]. The fact that one has a metric is not very important, as one can realize from the proof of the lemma. One can substitute the above conditions on the balls by appropriate conditions on the sets one wants to consider in order to be able to follow the same procedure. If for every $x$ of $S \subset \boldsymbol{R}^n$, $S$ bounded, one has a set $C_x$ which contains a Euclidean ball $B(x, r)$ and is contained in another $B(x, R)$ with $R/r = a$ independent of $x \in S$ and, moreover, for every point $p$ of $C_x$ the convex hull of $\{p\} \cup B(x, r)$ is contained in $C_x$, then one can select from $(C_x)_{x \in S}$ a sequence $\{C_k\}$ which satisfies properties (a), (b), (c) of the conclusion of Lemma 1.2. The reason is that such sets satisfy properties analogous to those described in the conditions of the lemma. This result is contained in [14].

Proof of Lemma 1.2. Suppose $\sup r(x) = \infty$ for $x \in S$. Then, since $S$ is bounded, one can select a single ball $B$, which satisfies (a), (b), (c). Let us assume $\sup r(x) = a_0 < \infty$ and take $x_1 \in S$ such that $r(x_1) > \frac{3}{4} a_0$. Suppose $x_1, \ldots, x_m$ already chosen. If $S = \bigcup_{1}^{m} B_k$, where $B_k = B(x_k, r(x_k))$, we stop. Otherwise define $a_m = \sup r(x)$ for $x \in S - \bigcup_{1}^{m} B_k$ and take $x_{m+1} \in S - \bigcup_{1}^{m} B_k$ such that $r(x_{m+1}) > \frac{3}{4} a_m$. We will show that $\{B_k\}$ satisfies (a), (b), (c) of the conclusion. If the process stops in a finite number of steps, $S \subset B_k$ is obvious. Assume then that $\{B_k\}$ is an infinite sequence. Then we have $r(x_k) \to 0$ as $k \to \infty$. In fact, the sets $B(x_k, \frac{1}{3} r(x_k))$ are clearly disjoint and so, if $r(x_k) \not\to 0$ we have in a bounded set an infinite number of disjoint balls of a fixed positive radius which contradicts ($\alpha$). Assume now $s \in S - \bigcup_{1}^{\infty} B_k$. Then $r(s) > 0$ would mean that $s$ has been overlooked in the selection of the $x_k$'s. Thus $S \subset B_k$ and (a) is proved.

Property (b) is an immediate consequence of condition ($\beta$) of the lemma, since the balls $B(x_k, \frac{1}{3} r(x_k))$ are disjoint and the $B_k$'s clearly satisfy (2), (3) of ($\beta$). Property (c) is also an easy consequence of condition ($\gamma$). To see this consider any ball $B_h$ with $h > \theta$. We claim that $B_h$ intersects at most $\theta - 1$ balls $B_k$ with $k < h$. This is clear from ($\gamma$) and from the choice of the balls $B_k$. Now in order to split $\{B_k\}$ we consider the families $A_i$, $i = 1, \ldots, \theta$. We set $B_1 \in A_1, \ldots, B_\theta \in A_\theta$. We know that $B_{\theta+1}$ is disjoint from at least one of the previous sets. We add $B_{\theta+1}$ to this family. Also $B_{\theta+2}$ intersects at most $\theta - 1$ of the previous sets. Thus there is a family such that $B_{\theta+2}$ does not intersect any of its sets. In this way we obtain $\theta$ disjoint families and $\{B_k\} = \bigcup_1^\theta A_h$.

The following covering lemma will be proved by means of two interesting results in combinatorial theory due to Ramsey [15]:

**1.4.** THEOREM. *Let $M$ be a set with infinitely many elements and $n$ and $r$ positive integers. Let all subsets of $r$ elements of $M$ be arbitrarily distributed into $n$ classes. Then there exists an infinite subset $D$ of $M$ such that all subsets of $r$ elements of $D$ are in the same class.*

**1.5.** THEOREM. *Let $r, m, n$ be positive integers. There exists a positive integer $a = a(r, n, m)$ such that if we take a set $M$ of $p$ elements, $p \geqslant a$, form the subsets of $r$ elements of $M$ and distribute them arbitrarily into $n$ classes, then there is a subset $N$ of $M$ with $m$ elements such that all subsets of $r$ elements of $N$ are in the same class.*

**1.6.** LEMMA. *Let $M_i$, $i = 1, 2, \ldots, n$, be $n$ metric spaces satisfying conditions ($\alpha$), ($\beta$), ($\gamma$) of 1.2 with corresponding integers $\xi_i$, $\theta_i$.*

*Let $S$ be a set in $P = M_1 \times M_2 \times \ldots \times M_n$, bounded with respect to the metric $\varrho = \max \varrho_i$. Let*

$$s \in S \to (r_1(s), r_2(s), \ldots, r_n(s)) \in (R^+)^n$$

*be any mapping such that $r_i(s) > 0$ and for any two points $s_1, s_2 \in S$ we have either $r_i(s_1) \geqslant r_i(s_2)$ for all $i$ or else $r_i(s_1) \leqslant r_i(s_2)$ for all $i$. Consider*

$$C_s = \{m \in P\colon\ \varrho_i(m, s) < r_i(s),\ i = 1, \ldots, n\},$$

*where $m = (m_1, \ldots, m_n)$, $s = (s_1, \ldots, s_n)$.*

*Then one can choose from $(C_s)_{s \in S}$ a sequence $\{C_k\}$ such that:*

(a) *$S \subset \bigcup C_k$.*

(b) *Every $m \in P$ is at most in $\mu = \mu(\xi_1, \ldots, \xi_n)$ sets $C_k$.*

(c) *The family $\{C_k\}$ can be split into $\nu = \nu(\theta_1, \ldots, \theta_n)$ disjoint families.*

Proof. (For the sake of clarity the proof will be presented for $n = 2$, but it is easy to see that the same reasoning applies to an arbitrary $n$.) Suppose $\sup r_i(s) = \infty$ for $s \in S$, $i = 1, 2$. Then one can select a single $C_s$ covering $S$. Suppose $a_0^i = \sup r_i(s) < \infty$ for $s \in S$, $i = 1, 2$. Choose $s_1 \in S$ such that $r_i(s_1) > \frac{3}{4} a_0^i$ for $i = 1, 2$. This can be done because of the comparability condition imposed on the $r_i(s)$. Assume $s_1, \ldots, s_m$ chosen. If $S \subset \bigcup_1^m C_k$, where $C_k = C_{s_k}$, we stop. Otherwise, if $a_m^i = \sup r_2(s)$ for $s \in S - \bigcup_1^m C_k$, we choose $s_{m+1}$ such that $r_i(s_{m+1}) > \frac{3}{4} a_m^i$ for $i = 1, 2$. We claim that $\{C_k\}$ satisfies (a), (b), (c).

If $\{C_k\}$ is finite, we obviously have (a). Assume $\{C_k\}$ infinite. Then necessarily at least one of the sequences $\{r_1(s_k)\}$, $\{r_2(s_k)\}$ tends to zero as $k \to \infty$. In fact, otherwise there is an $\varepsilon > 0$ and an inifnite subsequence $\{s'_k\}$ of $\{s_k\}$ such that $r_1(s'_k) > \varepsilon$, $r_2(s'_k) > \varepsilon$. Consider $s'_j, s'_h, j > h$. Then we have $s'_j \notin C_{s'_h}$. Call $m_l^1, m_l^2$ the projections of $s'_j$ on $M_1$, $M_2$ respectively and $R_l^1, R_l^2$ the projections of $C_{s'_l}$ on $M_1$, $M_2$. It is then clear, since $s'_j \notin C_{s'_h}$, that we have $m'_j \notin R'_h$ or $m_j^2 \notin R_h^2$. In the first case we write $(h, j) \in A_1$, in the second $(h, j) \in A_2$. By Theorem 1.4 we have an infinite sequence $\{s'_{k_l}\}$ of $\{s'_k\}$ such that for all $h > j$ we have $m^1_{k_j} \notin R^1_{k_h}$ or else $m^2_{k_j} \notin R^2_{k_j}$. Suppose we are in the first case. Then in $M_1$ we have a bounded set $\{m^1_{k_j}\}$, $j = 1, 2, \ldots$, for each point of $\{m^1_{k_j}\}$ a ball $R^1_{k_j} = B_1(m^1_{k_j}, r_1(s'_{k_j}))$ such that $r_1(s'_{k_j}) > \varepsilon$ and the balls $B_1(m^1_{k_j}, \frac{1}{3} r_1(s'_{k_j}))$ are disjoint. This contradicts property ($\alpha$) of $M_1$. Thus we have $r_1(s_k) \to 0$ or $r_2(s_k) \to 0$ as $k \to \infty$. Assume now $s \in S - \bigcup_1^\infty C_k$. Then, since $r_i(s) > 0$, it is clear that $s$ has been overlooked in the choice of the $s_k$'s. Thus we obtain (a).

For property (b) we can apply the same reasoning using now Ramsey's Theorem 1.5. Suppose $p \in P$ is in more than $\nu = a(2, 2, \xi)$ sets $C_k$, where $a$ is the function defined in 1.5 and $\xi = \max(\xi_1, \xi_2) + 1$; call these sets $C_{k_1}, \ldots, C_{k_{\nu+1}}$ and $R^i_{k_1}, \ldots, R^i_{k_{\nu+1}}$ their corresponding projections on $M_i$ and assume $k_h < k_j$ for $h < j$. Then, for $h < j$ we have $s_{k_j} \notin C_{k_h}$ and if $m^i_{k_j}, m^i_{k_h}$ are the projections of $s_{k_j}, s_{k_h}$ on $M_i$ we have $m^1_{k_j} \notin R^1_{k_h}$ or $m^2_{k_j} \notin R^2_{k_h}$. As before in the first case we write $(h, j) \in A_1$ and in the second $(h, j) \in A_2$. By Theorem 1.5 we have $\xi \geqslant \xi_1 + 1$ indices such that all their binary combinations are for example in $A_1$. But one easily sees that this contradicts property ($\beta$) of $M_1$. So (b) is proved.

For (c) one applies also Theorem 1.5 and the same reasoning as in the proof of (c) of 1.2.

Suppose finally $\sup r_1(s) = \infty$, $\sup r_2(s) = a_0^2 < \infty$ for $s \in S$. Then, if $S^i$ is the projection of $S$ over $M_i$ and $l$ is the diameter of $S^1$, we select $s_1$ such that $r_1(s_1) > l$, $r_2(s_1) > \frac{3}{4} a_0^2$ and if $a_m^i = \sup r_i(s)$ for $s \in S - \bigcup_1^m C_k$ we take $s_{m+1} \in S - \bigcup_1^m C_k$ such that $r^1(s_{m+1}) > \min(l, \frac{3}{4} a'_m)$, $r^2(s_{m+1}) > \frac{3}{4} a_m^2$. The same considerations lead to the conclusion also in this case.

**1.7.** Corollary. *Let $(R_a)_{a\in A}$ be a collection of open (or closed) intervals of $\boldsymbol{R}_n^n$ centered at 0 such that if $a_1, a_2 \in A$ we have $R_{a_1} \subset R_{a_2}$ or $R_{a_2} \subset R_{a_1}$. Let $S$ be a bounded set of $\boldsymbol{R}^n$ and $i\colon S \to A$ any mapping from $S$ to the index set $A$ and $R_x = x + R_{i(x)}$. Then there is a sequence $\{x_k\} \subset S$ such that if $R_k = x_k + R_{i(x_k)}$, we have:*

(a) $\bigcup R_k \subset S$.

(b) *Every $y \in \boldsymbol{R}_n^n$ is at most in $\xi = \xi(n)$ of the sets $\{R_k\}$.*

(c) *The sequence $\{R_k\}$ can be split into $\theta = \theta(n)$ disjoint families.*

## § 2. APPLICATION TO DIFFERENTIATION THEORY

For the sake of clarity we will present some consequences of the covering lemma stated in Corollary 1.7. It will be obvious that similar results on differentiation hold for spaces $P$ for which one has families of sets satisfying the properties of the sets $C_s$ of Lemma 1.6.

**2.1.** Theorem. *Let $(R_a)_{a\in A}$ be a collection of closed intervals as in Corollary 1.7 containing arbitrarily small intervals and $\mu$ any non-negative measure whose domain of definition includes all Borel sets of $\boldsymbol{R}^n$. Let $E$ be a $\mu$-measurable set of $\boldsymbol{R}^n$ such that $\mu(E) < \infty$ and assume that for every $x \in E$ there is a sequence $\{x + R_k\}$, $R_k \in (R_a)_{a\in A}$, contracting to $x$ as $k \to \infty$. Then there is a disjoint sequence $\{T_k\}$ of sets $T_k$ of the form $y + R_h$ with $y \in E$ and $R_h \in (R_a)_{a\in A}$ such that $\mu(E - \bigcup T_k) = 0$.*

Proof. If $Q(r)$ is the open cubic interval of centre 0 and semiedge $r$ and $\partial Q(r)$ denotes its boundary, it is clear that there is an increasing sequence $\{r_k\}$, $r_k \to \infty$ as $k \to \infty$, such that $\mu\big(\partial Q(r_k) \cap E\big) = 0$ for every $k$. Otherwise we would obtain that $\mu\big(Q(r) \cap E\big) = \varphi(r)$ is a bounded non-decreasing function of $r$ with an uncountable set of discontinuities. Thus, by considering $E \cap \big(Q(r_{k+1}) - \overline{Q(r_k)}\big)$ we can assume $E \subset B$, $B$ open and bounded. For each point $x \in E$ we have, for some $R_x \in (R_a)_{a\in A}$, $x + R_x \subset B$ and so one can select a sequence $\{R_k\}$ of intervals of the form $x + R_x$, $R_x \in (R_a)_{a\in A}$, $x \in E$, such that $B \supset \bigcup R_k \supset E$ and $\{R_k\}$ satisfies (c) of 1.7. If we consider the $\theta$ disjoint families we obtain, it is obvious that for at least one of them $\{R_k'\}$ we have

$$\mu\big(E \cap (\bigcup R_k')\big) \geqslant \frac{1}{\theta}\mu(E)$$

and so

$$\mu(E - \bigcup R_k') \leqslant \left(1 - \frac{1}{\theta}\right)\mu(E).$$

Thus we can select a finite number $h_1$ of rectangles of $\{R_k'\}$ such that

$$\mu(E - \bigcup_1^{h_1} R_k') \leqslant \eta\mu(E), \quad \text{where} \quad \left(1 - \frac{1}{\theta}\right) < \eta < 1.$$



Now we apply the same process to $E - \bigcup_1^{h_1} R_k'$ taking intervals disjoint from $\bigcup_1^{h_1} R_k'$ and we obtain $\{R_k'\}$, $k = 1, 2, \ldots, h_1, h_1+1, \ldots, h_2$ disjoint such that

$$\mu(E - \bigcup_1^{h_2} R_k') \leqslant \eta^2\mu(E).$$

In this way we obtain $\{T_k\}$ disjoint with $\mu(E - \bigcup T_k) = 0$.

**2.2.** Remark. When we have around each $x \in E$ not just a sequence, but a continuously contracting family of intervals, then by the consideration at the beginning of our proof we can take the sequence $\{T_k\}$ such that $\mu(\partial T_k) = 0$ and so disregard the boundary of $T_k$.

**2.3.** Definition. Let $(R_a)_{a\in A}$ be a family of intervals as in 2.1 and $\nu, \mu$ two non-negative measures defined on the Borel sets of $\boldsymbol{R}^n$. Consider for $x \in \boldsymbol{R}^n$ all the sequences $\{R_k\}$, where $R_k$ is of the form $x + R_a$, contracting to $x$ as $k \to \infty$ and the numbers

$$\underline{D}(x, \nu/\mu) = \inf\left[\varliminf_{k\to\infty} \frac{\nu(R_k)}{\mu(R_k)}\right], \qquad \overline{D}(x, \nu/\mu) = \sup\left[\varlimsup_{k\to\infty} \frac{\nu(R_k)}{\mu(R_k)}\right],$$

where sup and inf are taken over all such sequences $\{R_k\}$ and one puts $\underline{D}(x, \nu/\mu) = \overline{D}(x, \nu/\mu) = 0$ if for some $R_k$ one has $\mu(R_k) = 0$.

Define $D(x, \nu/\mu) = \overline{D}(x, \nu/\mu) = \underline{D}(x, \nu/\mu)$ whenever $\overline{D}(x, \nu/\mu) < \infty$ and $\overline{D}(x, \nu/\mu) = \underline{D}(x, \nu/\mu)$. The number $D(x, \nu/\mu)$ is called the *derivative at $x$* of $\nu$ with respect to $\mu$ and $(R_a)_{a\in A}$.

**2.4.** Theorem. *Let $\mu, \nu$ be two non-negative complete measures which are defined on the Borel sets of $\boldsymbol{R}^n$. Assume $\mu, \nu$ regular (i.e. $\varrho$ is regular when for $M$ $\varrho$-measurable, $\varrho(M) < \infty$, and $\varepsilon > 0$ there is an open set $G \supset M$ such that $\varrho(G - M) < \varepsilon$). Suppose further $\mu, \nu$ are finite on bounded sets.*

*Let $(R_a)_{a\in A}$ be as in the preceding definition. Then we have the following:*

(a) *If $\overline{D}(x, \nu/\mu)$, $\underline{D}(x, \nu/\mu)$ are measurable, then $D(x, \nu/\mu)$ exists $\mu$-a.e. in $\boldsymbol{R}^n$.*

(b) *Assume $\mathscr{C}^0(\boldsymbol{R}^n)$ is dense in $L^1(\boldsymbol{R}^n, \mu)$. Then, if $\nu = \nu_1 + \nu_2$ with $\nu_1$ $\mu$-continuous and $\nu_2$ $\mu$-singular (Lebesgue decomposition of $\nu$ with respect to $\mu$) we have:*

(i) $D(x, \nu_2/\mu) = 0$ *$\mu$-a.e. in $\boldsymbol{R}^n$.*

(ii) $D(x, \nu_1/\mu)$ *exists $\mu$-a.e. in $\boldsymbol{R}^n$, is $\mu$-integrable and for every $P$, Borel set, one has:*

$$\nu_1(P) = \int_P D(x, \nu_1/\mu)\, d\mu(x).$$

Proof. The proof of (a) follows a standard pattern. We first show that $\bar{D}(x, \nu/\mu)$ is finite $\mu$-a.e. Let $E$ be any open bounded set and define, for $\infty \geqslant M > 0$,

$$B_M = \{x \in E\colon \bar{D}(x, \nu/\mu) \geqslant M\}.$$

Take $G \supset E$, $G$ open, $\nu(G) < \infty$. Then, for every $x \in B_M$ there is an $R_x \subset G$ of the form $x + R_\alpha$ so that $\nu(R_x) > M\mu(R_x)$. Using 1.7, we have $\{R_k\}$, $\bigcup R_k \supset B_M$, such that every $y \in \boldsymbol{R}^n$ is at most in $\xi$ sets $R_k$, so that $\nu(R_k) > M\mu(R_k)$. Thus we obtain

$$\nu(G) \geqslant \nu(\bigcup R_k) \geqslant \frac{1}{\xi} \sum \nu(R_k) \geqslant \frac{M}{\xi} \sum \mu(R_k)$$
$$\geqslant \frac{M}{\xi} \mu(\bigcup R_k) \geqslant \frac{M}{\xi} \mu(B_M) \geqslant \frac{M}{\xi} \mu(B_\infty).$$

Hence, since $M$ is arbitrarily large, $\mu(B_\infty) = 0$. So $\bar{D}(x, \nu/\mu)$ is finite $\mu$-a.e. in $\boldsymbol{R}^n$.

Now we prove $\bar{D}(x, \nu/\mu) = \underline{D}(x, \nu/\mu)$, $\mu$-a.e. in $\boldsymbol{R}^n$. For any open bounded set $E$ and for $h, k \in \boldsymbol{Z}^+$ we write

$$A_{hk} = \left\{x \in E : \bar{D}(x, \nu/\mu) > \frac{h+1}{k} > \frac{h}{k} > \underline{D}(x, \nu/\mu)\right\}.$$

We will see that $\mu(A_{hk}) = 0$ and so $D(x, \nu/\mu)$ exists $\mu$-a.e. First we remark that if $A$ is a $\mu$-measurable bounded set such that $\bar{D}(x, \nu/\mu) \geqslant a$ for $x \in A$, then we have $\nu(A) \geqslant a\mu(A)$. In fact, take any $G \supset A$, $G$ open. Then, for $x \in A$, there is a sequence $\{R_x^k\}$ contracting to $x$, of sets of the form $x + R_\alpha$ so that $R_x^k \supset G$, $\nu(R_x^k) \geqslant a\mu(R_x^k)$. Then, using 2.1, we get a disjoint sequence $\{R_k\}$ such that $\mu(A - \bigcup R_k) = 0$, $R_k \subset G$. Hence,

$$\nu(G) \geqslant \nu(\bigcup R_k) = \sum \nu(R_k) \geqslant a \sum \mu(R_k) = a\mu(\bigcup R_k) \geqslant a\mu(A).$$

Thus $\nu(A) \geqslant a\mu(A)$ as we wished to show.

Now, for $\varepsilon > 0$, we take a $G$ open bounded, $G \supset A_{hk}$, $\mu(G) \leqslant \mu(A_{hk}) + \varepsilon$. If $x \in A_{hk}$, there is again, as before, a sequence $\{R_x^i\}$ contracting to $x$ so that

$$\nu(R_x^i) < \frac{h}{k} \mu(R_x^i).$$

Thus we obtain $\{R_i\}$ disjoint, $R_i \subset G$,

$$\nu(R_i) < \frac{h}{k} \mu(R_i)$$



such that $\mu(A_{hk} - \bigcup R_i) = 0$. Thus $\mu(A_{hk}) = \mu[(\bigcup R_i) \cap A_{hk}]$ and

$$\nu(\bigcup R_i) = \sum \nu(R_i) < \frac{h}{k} \sum \mu(R_i) \leqslant \frac{h}{k} \mu(G) \leqslant \frac{h}{k} [\mu(A_{hk}) + \varepsilon].$$

Using the preceding fact we also have

$$\nu(\bigcup R_i) \geqslant \nu[(\bigcup R_i) \cap A_{hk}] \geqslant \frac{h+1}{k} \mu[(\bigcup R_i) \cap A_{hk}] = \frac{h+1}{k} \mu(A_{hk}).$$

Hence $\mu(A_{hk}) \leqslant h\varepsilon$ and so $\mu(A_{hk}) = 0$.

In order to prove (b) i) we show that $D(x, \nu_2/\mu) = 0$ in $B$, $\mu$-a.e, for any $B$ open and bounded. Since $\mu$ is regular and $\nu_1$ is $\mu$-continuous, $\nu_1$ is also regular and so is $\nu_2$. Both $\nu_1$, $\nu_2$ are non-negative. Further, since $\nu_2$ is $\mu$-singular, there is a set $N$ such that $\mu(N) = 0$, $\nu_2(N') = 0$, where $N'$ denotes the complement of $N$. Consider $H_a = \{x \in B - N\colon \bar{D}(x, \nu_2/\mu) > a\}$ for any $a > 0$. Take any open bounded set $G \supset H_a$ and for every $x \in H_a$ take $R_x \subset G$ of the form $x + R_\alpha$ such that $\nu_2(R_k) > a\mu(R_k)$. We select a sequence $\{R_k\}$ with $\bigcup R_k \supset H_a$ and such that every point of $\boldsymbol{R}^n$ is at most in $\xi$ sets $R_k$. So we get

$$\nu_2(G) \geqslant \frac{1}{\xi} \sum \nu_2(R_k) \geqslant \frac{a}{\xi} \sum \mu(R_k) \geqslant \frac{a}{\xi} \mu^*(H_a),$$

where $\mu^*$ is the exterior measure associated to $\mu$. Now $G$ can be chosen so that $\nu_2(G)$ is arbitrarily small, since $\nu_2$ is regular and $\nu_2(H_a) = 0$. Thus $\mu(H_a) = 0$ and $D(x, \nu_2/\mu) = 0$ $\mu$-a.e.

For the proof of (b) ii) we first define

$$\mu(x, \nu/\mu) = \sup \frac{\nu(R)}{\mu(R)},$$

where the sup is taken over all those $R$ of the form $x + R_\alpha$ such that $\mu(R) \neq 0$. We show that, if for any $\lambda > 0$, we call $A_\lambda = \{x : M(x, \nu/\mu) > \lambda\}$, then

$$\mu^*(A_\lambda) \leqslant c \frac{\nu(\boldsymbol{R}^n)}{\lambda},$$

with $c > 0$, constant independent of $\lambda$.

In fact, if $x \in A_\lambda \cap E$, for $E$ open bounded, we have an $R = x + R_\alpha$ so that $\nu(R)/\mu(R) > \lambda$. Thus, using 1.7, we can obtain a sequence $\{R_k\}$ so that no point $y \in \boldsymbol{R}^n$ is in more than $\xi$ sets $R_k$ and $\bigcup R_k \supset A_\lambda \cap E$. Hence

$$\mu^*(A_\lambda \cap E) \leqslant \sum \mu(R_k) \leqslant \frac{1}{\lambda} \sum \nu(R_k) \leqslant \frac{\xi}{\lambda} \nu(\bigcup R_k) \leqslant \frac{\xi}{\lambda} \nu(\boldsymbol{R}^n).$$

Therefore

$$\mu^*(A_\lambda) \leqslant \frac{\xi}{\lambda}\nu(\boldsymbol{R}^n).$$

By the Radon-Nikodym theorem, since $\nu_1(\boldsymbol{R}^n) < \infty$ and $\nu_1$ is $\mu$-continuous, there is an $f \in L^1(\boldsymbol{R}^n, \mu)$ so that for any Borel set $P$ we have

$$\nu_1(P) = \int_P f(x)\,d\mu(x).$$

We shall show that $D(x, \nu/\mu) = f(x)$ $\mu$-a.e.

First one proves, as before, that $\overline{D}(x, \nu_1/\mu)$ is finite $\mu$-a.e. Let now be $f = g+h$, where $g \in \mathscr{C}_0(\boldsymbol{R}^n)$ and $h \in L^1(\boldsymbol{R}^n, \mu)$, $\|h\|_1 \leqslant \varepsilon$. It is obvious that if

$$\nu_{1g}(P) = \int_P g(x)\,d\mu(x),$$

we then have

$$\overline{D}(x, \nu_{1g}/\mu) = \underline{D}(x, \nu_{1g}/\mu) = g(x) \quad \text{for every } x.$$

We prove now that if $H_\lambda = \{x\colon |\overline{D}(x, \nu_1/\mu) - f(x)| > \lambda > 0\}$, then $\mu(H_\lambda) = 0$. In fact,

$$H_\lambda = \{x\colon |\overline{D}(x, (\nu_{1g}+\nu_{1h})/\mu) - (g(x)+h(x))| \lambda\}$$
$$\subset \left\{x\colon |\overline{D}(x, |\nu_{1h}|/\mu) > \frac{\lambda}{2}\right\} \cup \left\{x\colon |h(x)| > \frac{\lambda}{2}\right\}$$
$$\subset \left\{x\colon M(x, |\nu_{1h}|/\mu > \frac{\lambda}{2}\right\} \cup \left\{x\colon |h(x)| > \frac{\lambda}{2}\right\} = A_\lambda \cup B_\lambda,$$

where $|\nu_{1h}|$ denotes the total variation of $\nu_{1h}$. One therefore has

$$\mu^*(A_\lambda) \leqslant \frac{2\xi}{\lambda}\|h\|_1, \qquad \mu(B_\lambda) \leqslant \frac{2}{\lambda}\|h\|_1.$$

Hence

$$\mu^*(H_\lambda) \leqslant \frac{2(\xi+1)}{\lambda}\varepsilon$$

and so $\mu(H_\lambda) = 0$. In the same way one can proceed with $\underline{D}(x, \nu/\mu)$ and so one obtains $D(x, \nu/\mu) = f(x)$ $\mu$-a.e. in $\boldsymbol{R}^n$.

## § 3. APPLICATION TO SINGULAR INTEGRAL OPERATORS

**3.1.** Theorem. (a) *Let $\varrho\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to [0, \infty)$ be a translation invariant metric in $\boldsymbol{R}^n$ (we also call $\varrho(x) = \varrho(0, x)$) satisfying:*

(1) *The balls $B(0, r)$, $r > 0$, are compact convex bodies symmetric with respect to the coordinate hyperplanes.*



(2) *For all $r > 0$, $|B(0, 2r)| \leqslant c|B(0, r)|$, where $|\ |$ represents Lebesgue measure and $c$ is a positive constant independent of $r$ (we will denote in general by $c$ a positive constant, not necessarily the same at each occurrence, independent of certain elements of the expression in which it appears).*

(3) *The function of $r$, $|B(0, r)|$, is a continuous function of $r$ and the balls $B(0, r)$ contract to $0$ as $r \to 0$ and expand as $r$ increases so that they contain any given bounded set for $r$ sufficiently large.*

(4) *We have $\varrho(tx) \geqslant t\varrho(x)$ for $0 \leqslant t \leqslant 1$ and any $x$.*

(b) *Let $k$ be a function $k\colon \boldsymbol{R}^n \to C$, $k \in L^1_{\text{loc}}(\boldsymbol{R}^n - \{0\})$ such that:*

(1) *For $0 < \varepsilon < \eta$, $\left|\int_{\varepsilon<\varrho(x)<\eta} k(x)\,dx\right| \leqslant c$, $c$ independent of $\varepsilon, \eta$ and $\int_{\varepsilon<\varrho(x)<1} k(x)\,dx$ converges as $\varepsilon \to 0$.*

(2) *For any $a > 0$, $\int_{\varrho(x)<a} \varrho(x)|k(x)|\,dx \leqslant ca$, $c$ independent of $a$.*

(3) $\int_{\varrho(x)\geqslant 4\varrho(y)} |k(x) - k(x-y)|\,dx \leqslant c$, *$c$ independent of $y$.*

*Define, for $f \in L_0^\infty(\boldsymbol{R}^n)$,*

$$K_{\varepsilon\eta}f(x) = \int_{\varepsilon<\varrho(y)<\eta} k(y)f(x-y)\,dy, \qquad \varepsilon, \eta > 0.$$

*Then $K_{\varepsilon\eta}$ is an operator of weak type $(1, 1)$ on $L_0^\infty$, (i.e.*

$$|E_t(K_{\varepsilon\eta}f)| = |\{x\colon |K_{\varepsilon\eta}f(x)| > t > 0\}| \leqslant c_1\frac{\|f\|_1}{t})$$

*and of type $(p, p)$ on $L_0^\infty$ (i.e. $\|K_{\varepsilon\eta}f\|_p \leqslant c_p\|f\|_p$) for $1 < p < \infty$, uniformly in $\varepsilon, \eta$ (i.e. the constants $c_1$, $c_p$ are independent of $\varepsilon, \eta$).*

*Furthermore $K_{\varepsilon\eta}f$ converges in $L^p$ as $\varepsilon \to 0$, $\eta \to \infty$.*

Proof. We first prove that $K_{\varepsilon\eta}$ is of type $(2, 2)$ uniformly in $\varepsilon, \eta$. Consider

$$h(x) = \begin{cases} k(x) & \text{if } \varepsilon < \varrho(x) < \eta,\ \varepsilon > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

We will show that $\int |h(x-y) - h(x)|\,dx < c$, $c$ independent of $y$. If $\varrho(x) > 4\varrho(y)$ we have:

$$h(x-y) - h(x) = \begin{cases} k(x-y) - k(x) & \text{if } \frac{4}{3}\varepsilon < \varrho(x) < \frac{4}{5}\eta, \\ 0 & \text{if } \varrho(x) < \frac{4}{5}\varepsilon \text{ or } \varrho(x) > \frac{4}{3}\eta. \end{cases}$$

Thus, if we write

$$s_1 = \{x\colon 4\varrho(y) < \varrho(x),\ \tfrac{4}{3}\varepsilon \geqslant \varrho(x) \geqslant \tfrac{4}{5}\varepsilon\},$$
$$s_2 = \{x\colon 4\varrho(y) < \varrho(x),\ \tfrac{4}{3}\varepsilon < \varrho(x) < \tfrac{4}{5}\eta\},$$
$$s_3 = \{x\colon 4\varrho(y) < \varrho(x),\ \tfrac{4}{5}\eta \leqslant \varrho(x) \leqslant \tfrac{4}{3}\eta\},$$

we obtain

$$\int_{\varrho(x)>4\varrho(y)} |h(x)-h(x-y)|\,dx \leqslant \int_{s_2} |k(x-y)-k(x)|\,dx + \int_{s_1} [|k(x)|+ |k(x-y)|]\,dx + \int_{s_3} [|k(x)|+|k(x-y)|]\,dx$$

Now

$$\int_{s_2} |k(x-y)-k(x)|\,dx \leqslant c$$

by condition (b) (3),

$$\int_{s_1} |k(x)|\,dx \leqslant \int_{4/3\varepsilon\geqslant\varrho(x)\geqslant 4/5\varepsilon} |k(x)|\,dx \leqslant \frac{5}{4\varepsilon}\int_{4/3\varepsilon\geqslant\varrho(x)} \varrho(x)|k(x)|\,dx \leqslant c,$$

$$\int_{s_1} |k(x-y)|\,dx \leqslant \int_{\varrho(x)>4\varrho(y)} |k(x-y)-k(x)|\,dx + \int_{s_1} |k(x)|\,dx \leqslant c$$

by (b) (3) and the preceding inequality. The same is true for the third integral on $s_3$. Thus

$$\int_{\varrho(x)>4\varrho(y)} |h(x-y)-h(x)|\,dx < c$$

holds.

We next prove that the Fourier transform of $h$ is bounded in $\boldsymbol{R}^n$ uniformly in $\varepsilon, \eta$. Consider

$$\hat{h}(x) = \int h(y)e^{-2\pi i(x,y)}\,dy.$$

If $x=0$, by condition (b) (1) we have $|\hat{h}(0)|\leqslant c$. Suppose $x\neq 0$ and take $z\in\boldsymbol{R}^n$ such that $(x,z)=\frac{1}{2}$ and $\varrho(z)=\min\varrho(u)$ for $|(u,x)|\geqslant\frac{1}{2}$. We can write

$$\begin{aligned}2\hat{h}(x) &= \int e^{-2\pi i(x,y)}[h(y)-h(y-z)]\,dy\\ &= \int_{\varrho(y)>4\varrho(z)} e^{-2\pi i(x,y)}[h(y)-h(y-z)]\,dy + \int_{\varrho(y)\leqslant 4\varrho(z)} h(y)[e^{-2\pi i(x,y)}-1]\,dy +\\ &\quad + \int_{\varrho(y)\leqslant 4\varrho(z)} h(y)\,dy - \int_{\varrho(y)\leqslant 4\varrho(z)} h(y-z)[e^{-2\pi i(x,y)}+1]\,dy +\\ &\quad + \int_{\varrho(y-z)\leqslant 5\varrho(z)} h(y-z)\,dy - \int_{\substack{\varrho(y)>4\varrho(z)\\ \varrho(y-z)\leqslant 5\varrho(z)}} h(y-z)\,dy = \sum_{j=1}^{6} I_j.\end{aligned}$$

Now $|I_1|$ is bounded as shown above.



Also $|I_2|$ is bounded in $\boldsymbol{R}^n$, uniformly in $\varepsilon, \eta$. In fact, if we have $\varrho(z)\leqslant\varrho(y)\leqslant 4\varrho(z)$, we also have

$$|e^{-2\pi i(x,y)}-1|\leqslant 2\leqslant 2\,\frac{\varrho(y)}{\varrho(z)}$$

and so

$$\int_{\varrho(z)\leqslant\varrho(y)\leqslant 4\varrho(z)} |e^{-2\pi i(x,y)}-1|\,|k(y)|\,dy \leqslant \frac{2}{\varrho(z)}\int_{\varrho(y)\leqslant 4\varrho(z)} \varrho(y)|k(y)|\,dy \leqslant c.$$

If $\varrho(y)\leqslant\varrho(z)$ we take $t_1, y_1$ such that $0\leqslant t_1\leqslant 1$, $y=t_1y_1$, $\varrho(y_1)=\varrho(z)$. It is clear from the conditions on $\varrho$ that this is always possible if $y\neq 0$. Then we have

$$|e^{-2\pi i(x,y)}-1|\leqslant c|(x,y)| = ct_1|(x,y_1)|\leqslant ct_1\tfrac{1}{2}$$

because of the choice of $z$. Since $t_1\varrho(z)=t_1\varrho(y_1)\leqslant\varrho(t_1y_1)=\varrho(y)$, we have

$$|e^{-2\pi i(x,y)}-1|\leqslant c\,\frac{\varrho(y)}{\varrho(z)}$$

and so

$$\int_{\varrho(y)\leqslant\varrho(z)} |e^{-2\pi i(x,y)}-1|\,|k(y)|\,dy \leqslant \frac{c}{\varrho(z)}\int_{\varrho(y)\leqslant\varrho(z)} \varrho(y)|k(y)|\,dy \leqslant c.$$

Also $|I_3|$ is bounded by condition (b) (1).

For $|I_4|$ we can proceed as with $|I_2|$, since

$$|e^{-2\pi i(x,y)}+1| = |e^{-2\pi i(x,y-z)}e^{-2\pi i(x,z)}+1| = |e^{-2\pi i(x,y-z)}-1|.$$

Condition (b) (1) implies the boundedness of $|I_5|$.

For $|I_6|$ we have

$$\begin{aligned}|I_6| &= \Bigg|\int_{\substack{\varrho(y)>4\varrho(z)\\ \varrho(y-z)\leqslant 5\varrho(z)}} h(y-z)\,dy\Bigg| \leqslant \int_{3\varrho(z)\leqslant\varrho(y-z)\leqslant 5\varrho(z)} |k(y-z)|\,dy\\ &\leqslant \frac{1}{3\varrho(z)}\int_{\varrho(y-z)\leqslant 5\varrho(z)} \varrho(y-z)|k(y-z)|\,dy\leqslant c.\end{aligned}$$

Hence $|\hat{h}(x)|\leqslant c$ with $c$ independent of $\varepsilon, \eta, x$. So we have, for $f\in L_0^\infty$,

$$\|K_{\varepsilon\eta}f\|_2 = \|h*f\|_2 = \|\hat{h}\hat{f}\|_2\leqslant c\|f\|_2$$

which proves that $K_{\varepsilon\eta}$ is of type (2, 2) on $L_0^\infty$ uniformly in $\varepsilon, \eta$.

(Note that most of the properties of the metric $\varrho$ are not needed for this result. In particular, it is not necessary that the balls satisfy any symmetry condition with respect to the coordinate hyperplanes.)

By using the preceding fact we will now prove that $K_{\varepsilon\eta}$ is also of weak type (1, 1) uniformly in $\varepsilon, \eta$.

We first remark the following: Consider $B(0, r)$, $r > 0$, and the intersections

$$\left(\pm a_1(r), 0, \ldots, 0\right), \ldots, \left(0, 0, \ldots, 0, \pm a_n(r)\right)$$

of its boundary with the coordinate axes. Call $C(0, r)$ the interval

$$C(0, r) = \left\{x \in \boldsymbol{R}^n \colon |x_i| \leqslant \frac{a_i(r)}{2}\right\}.$$

An easy geometrical argument shows that $C(0, r) \subset B(0, r)$, $|B(0, r)| \leqslant 4^n |C(0, r)|$, $|C(0, r)|$ is continuous in $r$, $C(0, r)$ contracts to $0$ as $r \to 0$ and contains any fixed given bounded set for $r$ sufficiently large. Moreover $C(0, r_1) \subset C(0, r_2)$ if $r_1 \leqslant r_2$.

Call for brevity $K_{\varepsilon\eta} = A$. Let $t > 0$ be given and take $f \in \mathscr{C}_0(\boldsymbol{R}^n)$ (i.e. continuous function with compact support). We want to prove $|E_t(Af)| \leqslant c \dfrac{\|f\|}{t}$ with $c$ independent of $f, t, \varepsilon, \eta$.

Consider $f = f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \geqslant t, \\ 0 & \text{if} \quad |f(x)| < t. \end{cases}$$

Then $|f_2(x)| \leqslant t$ for any $x \in \boldsymbol{R}^n$ and we have

$$|E_t(Af)| \leqslant |E_{t/2}(Af_1)| + |E_{t/2}(Af_2)|$$

from the sublinearity of $A$ and the definition of $E_t(Ah)$. Now we get

$$|E_{t/2}(Af_2)| = \int\limits_{E_{t/2}(Af_2)} dx \leqslant \frac{2^2}{t^2} \int |Af_2(x)|^2 dx$$

$$\leqslant \frac{c}{t^2} \int |f_2(x)|^2 dx \leqslant \frac{c}{t} \|f_2\|_1 \leqslant \frac{c}{t} \|f\|_1$$

by using the fact that $A$ is of type $(2, 2)$.

We will try to estimate $|E_{t/2}(Af_1)|$. For any $x \in \operatorname{supp} f_1$ we obviously have some $C_x$ of the form $x + C(0, r)$ such that

$$\frac{t}{2} < \frac{1}{|C_x|} \int\limits_{C_x} |f(y)| dy < 2t.$$

Furthermore, from the continuity of the integral, we see that for every $x \in \operatorname{supp} f_1$ we have a neighborhood $V$ of $x$ such that for every point $\bar{x} \in V$ we have for $C_{\bar{x}} = \bar{x} + C(0, r)$ with the same $r$ as before

$$\frac{t}{2} < \frac{1}{|C_{\bar{x}}|} \int\limits_{C_{\bar{x}}} |f(y)| dy < 2t.$$



By the Heine-Borel theorem we obtain the following:

There is a finite number of $r$'s, $r_1, r_2, \ldots, r_s$, such that for every $x \in \operatorname{supp} f_1$ there is one of these $r$'s, say $r_i$, such that, if $C_x = x + C(0, r_i)$, we have

$$\frac{t}{2} < \frac{1}{|C_x|} \int\limits_{C_x} |f(y)| dy < 2t.$$

Hence we can apply Lemma 1.1 and we obtain a finite sequence $\{C_k\}$, $k = 1, \ldots, j$ of intervals such that $\operatorname{supp} f_1 \subset \bigcup_1^j C_k$ and every $x \in \boldsymbol{R}^n$ is at most in $\theta = 2^n$ of the sets $\{C_k\}$. (Note that the fact that the intervals are taken closed or open is irrelevant.)

Define now $E_1 = C_1, E_2 = C_2 - C_1, \ldots, E_k = C_k - \bigcup_1^{k-1} C_m$. The sets $E_k$ are disjoint and $\bigcup_1^j E_k = \bigcup_1^j C_k$. Let $\chi_P$ be the characteristic function of $P$ and let $\varphi_i = f_1 \chi_{E_i}$ and

$$g_i(x) = \begin{cases} \dfrac{1}{|C_i|} \displaystyle\int\limits_{C_i} \varphi_i(y) dy & \text{if} \quad x \in C_i, \\ 0 & \text{if} \quad x \notin C_i. \end{cases}$$

Call

$$g = \sum_1^j g_i, \qquad l_i = \varphi_i - g_i, \qquad l = \sum_1^j l_i.$$

Then we have

$$f_1 = \sum_1^j \varphi_i = g + l \quad \text{for any} \quad x \in \boldsymbol{R}^n,$$

$$|g(x)| = \theta \max |g_i(x)| \leqslant 2\theta t,$$

and

$$\|g\|_1 \leqslant \sum_1^j \int |g_i(x)| dx \leqslant \sum_1^j \int |f(y)| dy \leqslant \theta \|f\|_1.$$

Thus we obtain, as with $f_2$,

$$|E_{t/4}(Ag)| = \frac{c\|f\|}{t}.$$

On the other hand, we clearly have

$$\operatorname{supp} l_i \subset C_i, \qquad \int l_i(x) dx = 0 \quad \text{for} \quad i = 1, 2, \ldots, j.$$

Consider, if $C_i = x_i + C(0, s_i)$, $s_i > 0$, the ball $B_i = x_i + B(0, 4s_i)$. Then we obtain, if $B_i'$ is the complement of $B_i$, and $h$ is defined as in the proof of the type (2, 2),

$$\int_{B_i'} |Al_i(x)|\,dx = \int_{\varrho(x-x_i)>4s_i} \left|\int h(x-y)l_i(y)\,dy\right| dx$$
$$= \int_{\varrho(x)>4s_i} \left|\int h(x-y)l_i(y+x_i)\,dy\right| dx$$
$$= \int_{\varrho(x)>4s_i} \left|\int_{\varrho(y)\leqslant s_i} [h(x-y)-h(x)]l_i(y+x_i)\,dy\right| dx$$
$$\leqslant \int_{\varrho(y)\leqslant s_i} |l_i(y+x_i)| \int_{\varrho(x)>4s_i} |h(x-y)-h(x)|\,dx\,dy = c\|l_i\|_1$$

since $h$ satisfies

$$\int_{\varrho(x)>4\varrho(y)} |h(x-y)-h(x)|\,dx \leqslant c$$

as we have already seen.

In this way we get our last estimate for $|E_{t/4}(Al)|$,

$$|E_{t/4}(Al)| = |E_{t/4}(Al) \cap D| + |E_{t/4}(Al) \cap D'|,$$

where $D = \bigcup_1^j B_k$ and so

$$|D| \leqslant \sum_1^j |B_k| \leqslant c \sum_1^j |C_k| \leqslant c \sum_1^j \frac{2}{t} \int_{C_k} |f(y)|\,dy \leqslant \frac{c}{t}\|f\|_1$$

and, finally,

$$|D' \cap E_{t/4}(Al)| \leqslant \frac{4}{t}\int_{D'} |Al(x)|\,dx \leqslant \frac{4}{t}\sum_1^j \int_{B_k'} |Al_k(x)|\,dx$$
$$\leqslant \frac{c}{t}\sum_1^j \int |l_k(x)|\,dx \leqslant \frac{c}{t}\|f\|_1.$$

Adding up all our estimates we obtain

$$|E_t(Af)| \leqslant c\frac{\|f\|}{t}.$$

The weak type (1, 1) on $L_0^\infty(\boldsymbol{R}^n)$ is easily obtained by approximation. Marcinkiewicz interpolation theorem (cf. [16], II, p. 111) yields the result on the type $(p, p)$ for $1 < p \leqslant 2$ and a duality argument for the convolution operator $K_{\varepsilon\eta}$ the same result for $1 < p < \infty$.

We will finally prove the convergence of $K_{\varepsilon\eta}f$ in $L^p$ as $\varepsilon \to 0$, $\eta \to \infty$.



We first remark that the condition $t\varrho(x) \leqslant \varrho(tx)$ for all $x$, $0 \leqslant t \leqslant 1$, implies $|x| \leqslant c\varrho(x)$ for $\varrho(x) \leqslant 1$ with $c$ independent of $x$. In fact, assume $\varrho(x) \leqslant 1$, $x \neq 0$. Then $x = t\bar{x}$, $\varrho(\bar{x}) = 1$, $0 \leqslant t \leqslant 1$, and so we have

$$\varrho(x) = \varrho(t\bar{x}) \geqslant t\varrho(\bar{x}) = \frac{|x|}{|\bar{x}|}.$$

Hence one can take $c = \sup_{\varrho(z)=1} |z|$ and obtain $|x| \leqslant c\varrho(x)$. We prove next the convergence of $K_{\varepsilon\eta}f$ in $L^p$ for $f \in \mathscr{C}_0^1(\boldsymbol{R}^n)$. We have

$$|f(x)-f(y)| \leqslant c_f|x-y| \leqslant c_f\,\varrho(x-y)$$

with $c_f$ depending on $f$, but not on $x$, $y$. If $\varepsilon < 1 < \eta$, we can write

$$K_{\varepsilon\eta}f(x) = \int_{\varrho(x-y)>\varepsilon} k(x-y)f(y)\,dy - \int_{\varrho(x-y)\geqslant\eta} k(x-y)f(y)\,dy.$$

The last convolution clearly converges in $L^p(\boldsymbol{R}^n)$. For the first term of the second member we have

$$K_\varepsilon f(x) = \int_{\varrho(x-y)>\varepsilon} k(x-y)f(y)\,dy$$
$$= \int_{1>\varrho(x-y)>\varepsilon} k(x-y)[f(y)-f(x)]\,dy + f(x)\int_{1>\varrho(y)>\varepsilon} k(y)\,dy +$$
$$+ \int_{\varrho(x-y)>1} k(x-y)f(y)\,dy.$$

According to condition (b) (1), $\int_{1>\varrho(y)>\varepsilon} k(y)\,dy$ converges as $\varepsilon \to 0$. We also have

$$\left|\int_{\delta>\varrho(x-y)>\varepsilon} k(x-y)[f(y)-f(x)]\,dy\right| \leqslant c\int_{\delta>\varrho(x-y)>\varepsilon} \varrho(x-y)\,|k(x-y)|\,dy \to 0$$
$$\text{as} \quad \varepsilon, \delta \to 0.$$

On the other hand, for big $|x|$, $K_\varepsilon f(x)$ is independent of $\varepsilon$. Thus $K_\varepsilon f(x)$ converges uniformly in $\varepsilon$ as $\varepsilon \to 0$. Hence $K_\varepsilon f$ converges in $L^p(\boldsymbol{R}^n)$. For any other $f \in L_0^\infty(\boldsymbol{R}^n)$ the result is obtained by approximation.

**3.2.** Remarks. (a) Note that we obtain easily from the theorem the same results for $f \in L^p(\boldsymbol{R}^n)$, $1 \leqslant p < \infty$, and so the restriction $f \in L_0^\infty(\boldsymbol{R}^n)$ is irrelevant. If $g \in L^1(\boldsymbol{R}^n)$, take $h_k \in L_0^\infty(\boldsymbol{R}^n)$, $h_k \to g$ in $L^1$. Then $K_{\varepsilon\eta}h_k \to K_{\varepsilon\eta}g$ in $L^1$ and so also in measure. Thus for $t > 0$

$$|\{x: |K_{\varepsilon\eta}g(x)| > t\}| = \lim_{k\to\infty} |\{x: |K_{\varepsilon\eta}h_k(x)| > t\}| \leqslant c\frac{\|g\|_1}{t}.$$

Also it is obvious $\|K_{\varepsilon\eta}g\|_2 \leqslant c\|g\|_2$ if $g \in L^2(\boldsymbol{R}^n)$.

(b) Metrics $\varrho$ and kernels $k$ satisfying the conditions of Theorem 3.1 appear in a natural way when one studies singular integrals with generalized homogeneity which are associated with differential equations of parabolic type. Consider a real $n\times n$ matrix $P$ such that $P^*P = PP^*$ ($P^*$ is the adjoint of $P$) and with eigenvalues with real part $\geqslant 2$. Then one can define $\varrho(x)$ for $x \neq 0$ as the unique solution of $|[\exp(-P\log\varrho)]x| = 1$. The function so obtained yields a metric satisfying all properties of Theorem 3.1. A function $k$ is said to be *$P$-homogeneous of order $m$ if*

$$k\big([\exp(P\log\lambda)]x\big) = \lambda^m k(x) \quad \text{for} \quad x\in\boldsymbol{R}^n,\ x\neq 0,\ \lambda > 0.$$

It is easy to construct $P$-homogeneous kernels satisfying all conditions of Theorem 3.1 and they appear in the problems mentioned above. The theory of the singular integrals so obtained runs parallel to the one obtained for the classical singular integral operators of Calderón and Zygmund (cf. [4]). We refer to [8]–[13] for a more detailed study of such operators.

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