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Perfect sets in some groups

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Let G be a compact, metric, totally disconnected abelian group, and $G_1 \supset \ldots \supset G_n \supset G_{n+1} \supset \ldots$ a decreasing sequence of open subgroups meeting in $\{0\}$. Let $(H_n)_1^\infty$ be a sequence of positive numbers; a closed subset X of G is said to have *positive H-capacity* if X supports a Borel probability measure μ with the property

$$\mu(b+G_n) \leqslant KH_n, \quad 1 \leqslant n < \infty, b \in G.$$

In the first paragraph we prove an abstract lemma relating "economical coverings" of a set with additive set functions; it follows that capacity and metric covering properties are connected much as in a Euclidean space.

Next we specialize to the group of p-adic integers, as the multiplication in this ring yields an abundance of continuous endomorphisms. An analogue of C^1 mappings is introduced, in terms of which a p-adic analogue of the construction in [4] is accomplished.

I. Let S be a set and A a collection of subsets with this property:

(1) For each choice $\{T_1, \ldots, T_r\}$ from A of a covering of S (namely $S = \bigcup_{i=1}^{r} T_i$) there is a choice $\{T_1', \ldots, T_t'\} \subseteq \{T_1, \ldots, T_r\}$ of pairwise disjoint subsets such that $S = \bigcup_{j=1}^{t} T_j'$.

Moreover, let $h \geqslant 0$ be a real function on A such that

(2) $\sum_{i=1}^{r} h(T_i) \geqslant 1$ whenever each T_i is in A and $S = \bigcup_{i=1}^{r} T_r$.

LEMMA. *There is a non-negative finitely additive set function σ, so defined on all the subsets of S that $\sigma(S) = 1$, and $\sigma(T) \leqslant h(T)$ for each T in A.*

Proof. The argument is based on [3]. First, the covering property (2) is valid for multiple coverings: writing I_T for the characteristic function of T,

$(2')$ $\sum_{i=1}^{r} I_{T_i}(x) \geqslant m$, for all x in S, implies $\sum_{i=1}^{r} h(T_i) \geqslant m$ $(m = 1, 2, \ldots)$.

In fact, from the sets $T_1, \ldots, T_r$ a disjoint collection, say $T_1, \ldots, T_t$, can be chosen so that $\sum_{i=1}^{t} I_{T_i} = 1$, whence $\sum_{i=1}^{t} h(T_i) \geqslant 1$. In case $m > 1$, $\sum_{i>t} I_{T_i} \geqslant m-1$, and $(2')$ is obtained by exhaustion, using (1) at each step.

Let now Z be the group of bounded integer-valued functions on S, and L a functional on Z defined by $L(1-I_T) = 1-h(T)$ for T in A, and $L = -\infty$ for other functions. We shall find a functional ξ on Z so that

(3) $\xi(z_1)+\xi(z_2) = \xi(z_1+z_2)$, $z_i \in Z$;

(4) $\sup z \geqslant \xi(z)$;

(5) $\xi(z) \geqslant L(z)$.

If such a functional ξ exists we need only define $\sigma(R) = \xi(I_R)$ for $R \subseteq S$; for σ is additive by (3), $\sigma(S) = 1$ by (4), $\sigma(T) \leqslant h(T)$ by (5).

A functional ξ exists, after [3], if $\sup(z_1+\ldots+z_r) \geqslant L(z_1)+\ldots+L(z_r)$ whenever all z_i are in Z. This is trivially true unless $z_i = 1-I_{T_i}$ with each T_i in A, and it then becomes

$$\sup \sum_{i=1}^{r} (1-I_{T_i}) \geqslant \sum_{i=1}^{r} \big(1-h(T_i)\big),$$

or

$$\sum_{i=1}^{r} h(T_i) \geqslant \inf \sum_{i=1}^{r} I_{T_i}.$$

But this is just $(2')$ and the lemma is proved.

Corollary. *Each of the following properties of a closed set X in G implies the other*:

(1) *X supports a Borel probability measure μ for which $\mu(b+G_n) \leqslant KH_n$, $1 \leqslant n < \infty$, $b \in G$.*

(2) $X \subseteq \bigcup_{j=1}^{t} (b_j+G_{n_j}) \Rightarrow \sum_{j=1}^{t} KH_{n_j} \geqslant 1$.

Proof. It is plain that (1) is stronger than (2) because $\sigma(X) = 1$. If (2) holds we take for A, in the lemma, all intersections $X \cap (b+G_n)$, so that (1) is an easy consequence of the fact that distinct cosets of a fixed subgroup are disjoint. We choose $h\big(X \cap (b+G_n)\big) = KH_n$ and let σ be the set function obtained in the lemma. Restricting σ to the Boolean algebra of open-closed subsets of X, we obtain a measure μ that is trivially countably additive, and so can be extended to a measure on the Borel sets ([1], §§ 7,13).

We indicate a property of H-capacity involving product measures, related to concepts in classical potential theory.

Theorem. *Suppose X supports a probability measure μ, such that $H_n \geqslant (\mu\times\mu)\{(x, y)\colon x \equiv y \text{ modulo } G_n\}$. Then, for any sequence $r_n \geqslant 0$, $\sum r_n < \infty$, X has positive capacity for the sequence $(r_n^{-1}H_n)_1^\infty$.*

Proof. Set $\psi_n(x) = \mu(x+G_n)H_n^{-1}$, $1 \leqslant n < \infty$, $x \in G$. Then $\int \psi_n(x)\mu(dx) = H_n^{-1}(\mu\times\mu)\{x \equiv y \text{ modulo } G_n\} \leqslant 1$, so $\sum_{n=1}^{\infty} r_n\psi_n$ is integrable with respect to μ. Thus there is a closed subset Y of X, with $\mu(Y) > 0$ and $r_n\psi_n \leqslant C$ on Y $(1 \leqslant n < \infty)$. In particular, $\mu\big(Y \cap (x+G_n)\big) \leqslant Cr_n^{-1}H_n$ for x in Y, and we can prove the theorem by showing that $\mu\big(Y \cap (b+G_n)\big) \leqslant Cr_n^{-1}H_n$ for each b in G. But this is plain unless $(b+G_n) \cap Y \neq \emptyset$, in which case $b+G_n = y+G_n$ with y in Y.

II. In this section G is the ring of p-adic integers for a prime p ([2], § 10). Elements of G are sequences of integers

$$x = (x_1, x_2, \ldots, x_n, \ldots), \qquad 0 \leqslant x_n < p;$$

the subgroup $G_n = p^nG$, or, which is the same,

$$x \in G_n \Leftrightarrow x_1 = x_2 = \ldots = x_n = 0 \Leftrightarrow |x|_p \leqslant p^{-n}.$$

Let φ and $D\varphi$ be continuous mappings of G into itself. We say that φ is C^1 *with derivative* $D\varphi$ provided

$$|\varphi(x+y)-\varphi(x)-yD\varphi(x)|_p = o(|y|_p) \quad \text{as } |y|_p \to 0,$$

uniformly for all x in G. The C^1-norm of φ is defined as

$$\|\varphi\| = \sup|\varphi|_p + \sup|D\varphi|_p + \sup_{x,y} \frac{|\varphi(x+y)-\varphi(x)-yD\varphi(x)|_p}{|y|_p}.$$

By an argument familiar from Banach spaces, we find that C^1 is a complete metric group with regard to the norm $\|\varphi\|$.

Now let φ be C^1 with a derivative which at each point is different from zero; we call φ "non-singular". There is an integer N such that, whenever $|x-\bar{x}|_p \leqslant p^{-N}$ in G, then $|\varphi(x)-\varphi(\bar{x})| \geqslant p^{-N}|x-\bar{x}|_p$. Thus, for each coset $b+G_n$, $\varphi^{-1}(b+G_n)$ intersects at most p^{2N} cosets of G_n $(1 \leqslant n < \infty)$. If, then, μ is a probability measure on G, and

$$H_n \geqslant \sup\mu(b+G_n) \qquad (1 \leqslant n < \infty, b \in G),$$

then

$$\sup\mu\big(\varphi^{-1}(b+G_n)\big) \leqslant p^{2N}H_n.$$

Thus φ maps sets of positive H-capacity onto sets with the same property.

Recall that a closed set E in G is a *Kronecker set* ([1], 5.2; 6) if each continuous unimodular function on E admits uniform approximation by characters of G.

THEOREM 1. *For every modulus function H such that $p^n H_n \to \infty$, G contains a Kronecker set of positive H-capacity.*

The method of proof is to construct a subset of positive H-capacity (in the most obvious way) and then produce a non-singular C^1 mapping φ of this set onto a Kronecker set. For any set M of positive integers we define

$$G[M] = \{x \in G: x_i = 0 \text{ for all } i \notin M\}.$$

Let m be the counting function of M:

$$m(n) = \sum_{\substack{i \leqslant n \\ i \in M}} 1 \quad (n = 1, 2, \ldots).$$

LEMMA. *$G[M]$ has positive capacity for the function $H_n = p^{-m(n)}$.*

Proof. Regarding $G[M]$ as a product $\prod_M \{0, 1, \ldots, p-1\}$ we provide $G[M]$ with a probability measure μ rendering the co-ordinates mutually independent, and equally distributed upon $\{0, 1, \ldots, p-1\}$. Then a coset of G_n has μ-measure H_n if it meets $G[M]$, and 0 otherwise.

THEOREM 2. *Let M be a set of positive integers whose complement contains segments of unbounded lengths.*

Then each mapping φ in the space C^1, excepting only a set of Baire's category 1, determines a homeomorphism of $G[M]$ onto a Kronecker set.

Proof. Suppose that $\{F_j\}_1^\infty$ is a sequence of continuous unimodular functions on $G[M]$, uniformly dense in the metric space of all these functions. Let $V_{j,k}$ be the set of C^1 defined thus:

For some character γ of G, $|\gamma(\varphi(y)) - F_j(y)| < k^{-1}$ for all y in $G[M]$. Then $V_{j,k}$ is open, the intersection $\bigcap V_{j,k}$ is exactly the set described in the theorem, and we proceed to show that each $V_{j,k}$ is dense in C^1.

Let $\psi \in C^1$, $\varepsilon > 0$, and F be a unimodular function on $G[M]$. There is a constant B so that $|\psi(x) - \psi(\bar{x})|_p \leqslant B|x - \bar{x}|_p$ for any pair $x, \bar{x}$ in G. Let $[u, v]$ be a segment of positive integers so that $[u, v] \cap M = \emptyset$, and let w and w_1 be integers so chosen that

$$\left|w - \frac{2}{3}v - \frac{1}{3}u\right| \leqslant 1, \qquad \left|w_1 - \frac{2}{3}u - \frac{1}{3}v\right| \leqslant 1.$$

For γ we choose a certain character of order p^w ([2], § 25.2) (later we shall give the exact form of γ).

We are going to displace ψ by a function α, constant on cosets of G_u, while $|\alpha|_p \leqslant p^{-w_1}$. Thus $D_\alpha = 0$, and $|\alpha(x+y) - \alpha(x)| \leqslant p^{u-w_1}|y|_p$ because $\alpha(x+y) = \alpha(x)$ unless $y \notin G_u$. As $u - w_1 \geqslant \frac{1}{3}(v-u) - 1$, $\|\alpha\|$ can be made as small as we please. We note also that if y_1, y_2 belong to $G[M]$ and $y_1 \equiv y_2$ modulo G_u, then $y_1 \equiv y_2$ modulo G_v. Then $|\psi(y_1) - \psi(y_2)| \leqslant B_p^{-v}$; if $v - u$ is sufficiently large, then $B_p^{-v} < p^{-w}$, so that $\gamma(\psi(y_1)) = \gamma(\psi(y_2))$. Similarly, $\gamma(\alpha(y_1)) = \gamma(\alpha(y_2))$ as soon as $v > w$.

The character γ has the formula

$$\gamma(x_1, \ldots, x_w, x_{w+1}, \ldots) = \exp 2\pi i q(p^{-w}x_1 + \ldots + p^{-1}x_w)$$

for an integer q with $(q, p) = 1$, for example $q = 1$. The range of γ on G_{w_1} is the group of p^{w-w_1} roots of 1, and since $w - w_1 \geqslant \frac{1}{3}(v-u) - 2$, the number $\delta = |1 - \exp 2\pi i p^{w_1 - w}|$ can be made arbitrarily small.

From each coset of G_u that meets $G[M]$, we choose an element y_0 and specify $\alpha(y_0)$ in G_{w_1} so that

$$\left|\gamma(\alpha(y_0)) - \overline{\gamma(\psi(y_0))}\, F(y_0)\right| < \delta,$$

or

$$\left|\gamma(\alpha(y_0)) + \psi(y_0)) - F(y_0)\right| < \delta.$$

Because the function $\gamma(\alpha + \psi)$ is constant on cosets of G_u, the error $\|\gamma(\alpha+\psi) - F\|_\infty$ is easily estimated by means of δ and the degree of continuity of F. Making u and $v - u$ increase without bound we obtain element $\alpha + \psi$, arbitrarily close to ψ, with $\|\gamma(\alpha+\psi) - F\|_\infty$ arbitrarily small.

We note that the non-singular mappings in C^1 form a neighborhood of the identity mapping of G, so that mappings of this kind are determined in our theorem. To verify Theorem 1, a choice of M is the final step; the complement of M must contain segments of unbounded length, while $p^{-m(n)} = O(H_n)$. We may assume that $p^{-n} \leqslant H_n$ for all n. Suppose we have found sets $P_1 \supseteq \ldots \supseteq P_j$ of natural numbers whose complements are finite, and the counting number π_j of P_j fulfills $p^{-\pi_j(n)} \leqslant H(n)$ for all n. Then we can remove j consecutive integers from P_j, so as to leave an acceptable set P_{j+1}. Plainly $M = \bigcap P_j$ will serve for Theorems 1 and 2.

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