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Convergence and almost convergence of certain sequences of positive linear operators

by

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1. Let $L_n(f(t); x)$, $n = 1, 2, \ldots$, be a sequence of linear operators transforming $f(t)$ to functions of x defined on $-\infty < a \leqslant x \leqslant b < \infty$. Assume that the sequence is ultimately positive, i.e. for all large n and every non-negative definite function $f(t)$

$$L_n(f(t); x) \geqslant 0 \quad (a \leqslant x \leqslant b).$$

A theorem of Korovkin ([6], p. 14) states essentially that if $L_n(1; x) \to 1$, $L_n(t; x) \to x$ and $L_n(t^2; x) \to x^2$ uniformly on $[a, b]$, then $L_n(f(t); x) \to f(x)$ uniformly on $[a, b]$ for all $f \in c[a, b]$. Recently Hsu [2] has given an extension of this theorem which applies to uniform approximation on any finite interval of functions defined and continuous on $(-\infty, \infty)$. His result asserts that certain sequences of linear operators which are positive on a particular finite interval may be modified so as to be capable of approximation to non-bounded continuous functions defined on $(-\infty, \infty)$.

It is the purpose of this note to point out that Hsu's result remains true if "convergent" is replaced by "almost convergent" in the statement of that result. In addition, the general theorems are applied to certain types of generalized Bernstein polynomials.

2. For the basic properties of almost convergent sequences, see [7]. Unless otherwise stated, it will be assumed in this section that $[a, b] = [-1, 1]$ and $L_n(f(t); x)$ is positive in $[-1, 1]$ for all large n. Modifications of the results for other finite intervals will be obvious.

THEOREM 1. *Let a_n be increasing to $+\infty$ with n and let*

(1) $\{L_n((a_n t)^k; a_n^{-1} x)\}$ *be almost convergent to x^k uniformly on every finite interval, where $k = 0, 1, 2, m, m+1, m+2$, and m is certain non-negative even integer.*

Then for every continuous function $f(x)$ defined on $(-\infty, \infty)$ and satisfying the condition $f(x) = O(|x|^m)$, $x \to \pm\infty$, it follows that $\{L_n(f(a_n t); a_n^{-1} x)\}$ is almost convergent to $f(x)$ uniformly on any finite interval.

Proof. Let

$$t_{pv}(t^i; x) = \frac{1}{p} \sum_{n=v}^{v+p-1} L_n\big((a_n t)^i; a_n^{-1}x\big),$$

$v = 0, 1, \ldots,$ $p = 1, 2, \ldots$ and $i = 0, 1, 2, m, m+1, m+2$. It follows from a result of Lorentz [7] and (1) that

(2) $$t_{pv}(t^i; x) = x^i + a^i_{pv}(x),$$

where $\lim_{p\to\infty} a^i_{pv}(x) = 0$ uniformly in $v = 0, 1, \ldots$ and uniformly on any finite interval for each $i = 0, 1, 2, m, m+1, m+2$. If $L_n\big(f(a_n t); a_n^{-1}x\big) \leqslant L_n\big(g(a_n t); a_n^{-1}x\big)$ for two functions $f(t)$ and $g(t)$, a fixed x and $n = 0, 1, 2, \ldots,$ then

(3) $$t_{pv}\big(f(t); x\big) \leqslant t_{pv}\big(g(t); x\big), \quad v = 0, 1, \ldots \text{ and } p = 1, 2, \ldots$$

Employing (2) and (3), the proof becomes a reproduction of that of Hsu's Theorem 1.

Similarly Hsu's Corollary 1, Corollary 2 and Theorem 2 remain valid if "convergent" is replaced by "almost convergent" in those results.

Various applications of the general results will now be considered.

3. Let $\{h_i(x)\}$ be a sequence of functions defined on $[0, 1]$. The generalized Lototsky or $[F, h_i] = (a_{nk})$ matrix [3] is defined by

$$a_{00} = 1, \quad a_{0k} = 0 \quad (k \neq 0),$$

$$\prod_{i=1}^{n} (h_i y + 1 - h_i) = \sum_{k=0}^{n} a_{nk} y^k.$$

For each f defined on $[0, 1]$ let

$$L_n\big(f(t); x\big) = \sum_{v=0}^{n} f(v/n)\, a_{nv}(x).$$

These operators were first investigated by King [5] and were proved to be capable of approximation to continuous functions defined on $[0, 1]$ with a suitable choice of $h_i(x)$. When $h_i(x) = x$ $(i = 1, 2, \ldots)$, $L_n\big(f(t); x\big)$ becomes the classical Bernstein polynomial [8]. Assume that

$$h_i(z) = \sum_{v=1}^{\infty} \beta_{vi} z^v$$

for $|z| \leqslant 1$ and $i = 1, 2, \ldots,$ where $\beta_{vi} \geqslant 0$ for all i, v and

$$\sum_{v=1}^{\infty} \beta_{vi} \leqslant 1 \quad \text{for} \quad i = 1, 2, \ldots$$

Let $\{s_i\}$ denote the $(c, 1)$-transform of the sequence $\{\beta_{1i}\}$.

THEOREM 2. *Let a_n be increasing to $+\infty$ with n and let $a_n^2 = o(n)$. Let $f(x)$ be defined, continuous and bounded on $[0, \infty)$. Then $\{s_i\}$ convergent (almost convergent) to 1 implies $\{L_n\big(f(a_n t); a_n^{-1}x\big)\}$ convergent (almost convergent) to $f(x)$ uniformly on any finite interval of $[0, \infty)$.*

Proof. Let $0 \leqslant a \leqslant x \leqslant b < +\infty$. Since $0 \leqslant h_i(x) \leqslant 1$ for $0 \leqslant x \leqslant 1$, $L_n\big(f(t); x\big)$ is positive on $[0, 1]$. It follows that $L_n\big(f(a_n t); a_n^{-1}x\big)$ is positive on $[a, b]$ for n large. According to the general results, it must be verified that $\{L_n\big((a_n t)^k; a_n^{-1}x\big)\}$ is convergent (almost convergent) to x^k uniformly on $[a, b]$, where $k = 0, 1, 2$. A modification of the computations of [5] shows that

(4) $$L_n(1; a_n^{-1}x) = 1,$$

(5) $$L_n(a_n t; a_n^{-1}x) = \frac{a_n}{n} \sum_{i=1}^{n} h_i(x/a_n),$$

and

(6) $$L_n\big((a_n t)^2; a_n^{-1}x\big) = (a_n/n)^2 \Big(\sum_{i=1}^{n} h_i(x/a_n) - \sum_{i=1}^{n} h_i^2(x/a_n) + \Big[\sum_{i=1}^{n} h_i(x/a_n)\Big]^2\Big).$$

For n sufficiently large, the assumptions on $h_i(z)$ imply

(7) $$\frac{a_n}{n} \sum_{i=1}^{n} h_i(x/a_n) = x \sum_{v=1}^{\infty} s_{nv} (x/a_n)^{v-1},$$

where $s_{nv} = (1/n) \sum_{i=1}^{n} \beta_{vi}$ $(v = 1, 2, \ldots)$. The result now follows from (4)-(7) and the hypotheses.

4. Let $\{g_m\}$ be a sequence of functions, each of which is analytic in the disk $|z| < R$ $(R > 1)$ and such that $\{g_m\}$ converges to a function g uniformly in some disk $|z| < r$ for some r $(1 < r < R)$. Define the generalized Boole polynomials $\{\zeta_n^{(m)}(x)\}$ by the equation

$$g_m(u)(1+u)^x = \sum_{n=0}^{\infty} \zeta_n^{(m)}(x)\, u^n.$$

With every function f defined on $[0, 1]$ associate for $m = 1, 2, \ldots$ the operator

$$L_m\big(f(t); x\big) = \frac{1}{g(x-1)} \sum_{n=0}^{m} (-1)^{m-n} \zeta_{m-n}^{(m)}(-n-1)\, x^n (1-x)^{m-n} f\Big(\frac{n}{m}\Big)$$

for $0 \leqslant x \leqslant 1$. For $g_m(u) = g(u) \equiv 1$ $(m = 1, 2, \ldots)$, $L_m\big(f(t); x\big)$ becomes the Bernstein polynomial. These operators have been studied in [4] and [10]. Properties of associated summability matrices have been investigated in [9].

THEOREM 3. *Let a_m be increasing to $+\infty$ with m and let $a_m = o(m)$. Let W denote the class of all the continuous functions satisfying the condition of the type $f(x) = O(x^N)$, $x \to +\infty$, $N = 0, 1, 2, \ldots$ Suppose*

$$(8) \qquad (-1)^{m-n}\zeta_{m-n}^{(m)}(-n-1) \geqslant 0, \qquad 0 \leqslant n \leqslant m,\ m = 0, 1, \ldots$$

Then for all $f \in W$ it follows that

$$\lim_{m\to\infty} L_m\big(f(a_m t);\, a_m^{-1}x\big) = f(x)$$

uniformly on any finite subinterval of $[0, \infty)$.

Proof. Condition (8) implies that $L_m\big(f(t); x\big)$ is positive on $[0, 1]$ for all m. Let $[a, b]$ be any finite subinterval of $[0, \infty)$. Then, for m sufficiently large, $L_m\big(f(a_m t); a_m^{-1}x\big)$ is positive on $[a, b]$.

Write

$$g_m(z) = \sum_{k=0}^{\infty} a_{mk} z^k, \qquad m = 1, 2, \ldots$$

It is easy to see that

$$(-1)^{m-n}\zeta_{m-n}^{(m)}(-n-1) = \sum_{k=0}^{m-n} \binom{m-k}{n} a_{mk}.$$

It follows that

$$L_m(1;\, a_m^{-1}x) = \frac{1}{g(x/a_m-1)} \sum_{n=0}^{m} \left(\frac{x}{a_m}\right)^n \left(1-\frac{x}{a_m}\right)^{m-n} \sum_{k=0}^{m-n} \binom{m-k}{n} a_{mk}$$
$$= \frac{1}{g(x/a_m-1)} \sum_{k=0}^{m} a_{mk}\left(1-\frac{x}{a_m}\right)^k.$$

A slight modification of the analysis given in [9], p. 37, shows that this last expression converges to 1 uniformly on $[a, b]$. Next

$$L_m(a_m t;\, a_m^{-1}x)$$
$$= \frac{1}{m} g\left(\frac{x}{a_m}-1\right) \sum_{k=0}^{m} a_{mk}\left(1-\frac{x}{a_m}\right)^k a_m \sum_{n=0}^{m-k} n\binom{m-k}{n}\left(\frac{x}{a_m}\right)^n\left(1-\frac{x}{a_m}\right)^{m-k-n}$$
$$= \frac{x}{g(x/a_m-1)} \sum_{k=0}^{m} \frac{m-k}{m} a_{mk}\left(1-\frac{x}{a_m}\right)^k.$$

Again, a modification of the analysis of [9], p. 37, shows that the last expression converges to x uniformly on $[a, b]$. Repeated computations yield

$$L_m\big((a_m t)^2;\, a_m^{-1}x\big) = \frac{x^2}{g(x/a_m-1)} \sum_{k=0}^{m} \frac{(m-k)(m-k-1)}{m^2} a_{mk}\left(1-\frac{x}{a_m}\right)^k +$$
$$+ \frac{x}{g(x/a_m-1)} \sum_{k=0}^{m} \frac{a_m}{m}\,\frac{m-k}{m} a_{mk}\left(1-\frac{x}{a_m}\right)^k,$$

$$L_m\big((a_m t)^3;\, a_m^{-1}x\big) = \frac{x^3}{g(x/a_m-1)} \sum_{k=0}^{m} \frac{(m-k)(m-k-1)(m-k-2)}{m^3} \times$$
$$\times a_{mk}\left(1-\frac{x}{a_m}\right)^k + \frac{3x^2}{g(x/a_m-1)} \sum_{k=0}^{m} \frac{a_m}{m}\,\frac{(m-k)(m-k-1)}{m^2} a_{mk}\left(1-\frac{x}{a_m}\right)^k +$$
$$+ \frac{x}{g(x/a_m-1)} \sum_{k=0}^{m} \left(\frac{a_m}{m}\right)^2 \frac{m-k}{m} a_{mk}\left(1-\frac{x}{a_m}\right)^k,$$

and, in general, for any positive integer r,

$$(9) \quad L_m\big((a_m t)^r;\, a_m^{-1}x\big)$$
$$= \frac{x^r}{g(x/a_m-1)} \sum_{k=0}^{m} \frac{(m-k)\ldots(m-k-r+1)}{m^r} a_{mk}\left(1-\frac{x}{a_m}\right)^k +$$
$$+ \ldots + \frac{x}{g(x/a_m-1)} \sum_{k=0}^{m} \left(\frac{a_m}{m}\right)^{r-1} \frac{m-k}{m} a_{mk}\left(1-\frac{x}{a_m}\right)^k.$$

The hypothesis $a_m = o(m)$, (9) and (essentially) the analysis of [9], p. 37, show that

$$\lim_{m\to\infty} L_m\big((a_m t)^r;\, a_m^{-1}x\big) = x^r$$

uniformly on $[a, b]$ for $r = 2, 3, \ldots$ An appeal to Corollary 2 of [2] completes the proof.

It should be mentioned that Theorem 3 for the case $g_m(u) = g(u) \equiv 1$ extends a result of Chlodovsky ([8], p. 36) for the Bernstein polynomials.

5. Let $\{u_n(x)\}$ be a sequence of real-valued functions defined on $[0, 1]$. Denote by $\big(h_{nk}(x)\big)$ the Hausdorff matrix generated by $\{u_n(x)\}$ ([1], chapter 11). Then

$$h_{nk}(x) = \begin{cases} \binom{n}{k} \Delta^{n-k} u_k(x), & 0 \leqslant k \leqslant n, \\ 0 & k > n, \end{cases}$$

where, for any integers $m, p \geqslant 0$,

$$\Delta^p u_m(x) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} u_{m+j}(x).$$

The sequence $\{u_n(x)\}$ is called a *moment sequence* ([8], p. 57) if there exists a function $\alpha(x, t)$, of bounded variation in t for each $x \in [0, 1]$, such that for all $x \in [0, 1]$

$$u_n(x) = \int_0^1 t^n d\alpha(x, t), \qquad n = 0, 1, 2, \ldots$$

The sequence $\{u_n(x)\}$ is said to be *totally monotone* if $\Delta^p u_n(x) \geqslant 0$ for all $x \in [0, 1]$ and all integers $n, p \geqslant 0$. Let $\{u_n(x)\}$ be a moment sequence. For all functions f defined on $[0, 1]$, define a linear operator by

$$H_n\big(f(t); x\big) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) h_{nk}(x).$$

When

$$\alpha(x, t) = \begin{cases} 0, & 0 \leqslant t < x, \\ 1, & x \leqslant t \leqslant 1, \end{cases}$$

these operators become the Bernstein polynomials.

THEOREM 4. *Let $\{u_n(x)\}$ be a totally monotone sequence. Let α_n be increasing to $+\infty$ with n and let $\alpha_n = o(n)$. Let $f(x)$ be defined, bounded and continuous in $[0, \infty)$. Assume that $\{u_0(x/\alpha_n)\}$ is convergent (almost convergent) to 1, $\{\alpha_n u_1(x/\alpha_n)\}$ is convergent (almost convergent) to x, and $\{\alpha_n^2 u_2(x/\alpha_n)\}$ is convergent (almost convergent) to x^2, uniformly on any finite interval of $[0, \infty)$. Then $\{H_n(f(\alpha_n t); \alpha_n^{-1} x)\}$ is convergent (almost convergent) to $f(x)$, uniformly on any finite interval of $[0, \infty)$.*

Proof. Since $\{u_n(x)\}$ is totally monotone, H_n is positive on $[0, 1]$. Thus $H_n\big(f(\alpha_n t); \alpha_n^{-1} x\big)$ is positive on $[a, b] \subset [0, \infty)$ for n large. The result is now an immediate consequence of the following easy computations:

$$H_n(1, \alpha_n^{-1} x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} u_k\left(\frac{x}{\alpha_n}\right) = u_0\left(\frac{x}{\alpha_n}\right),$$

$$H_n(\alpha_n t; \alpha_n^{-1} x) = \alpha_n \sum_{k=0}^{n-1} \binom{n-1}{k} \Delta^{n-1-k} u_{k+1}\left(\frac{x}{\alpha_n}\right)$$

$$= \alpha_n \int_0^1 t d\alpha\left(\frac{x}{\alpha_n}, t\right) = \alpha_n u_1(x/\alpha_n),$$

and

$$H_n\big((\alpha_n t)^2; \alpha_n^{-1} x\big) = \frac{\alpha_n^2 (n-1)}{n} u_2\left(\frac{x}{\alpha_n}\right) + \frac{\alpha_n^2}{n} u_1\left(\frac{x}{\alpha_n}\right).$$

Of course, Theorem 2 of [2] shows that Theorems 2, 3, and 4 are also valid for all functions of the Lipschitz class $\mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$.

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