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On topological conjugation in linear groups

by

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1. Introduction. If X is a Banach space, then by $GL(X)$ we denote the group of linear, bounded and invertible operators in X.

The identity operator in X will be denoted by I_X or simply by I.

By R^n and C^n ($n = 1, 2, \ldots$) we will denote the n-dimensional real and complex vector space respectively.

We recall that maps ($=$ continuous transformations) $\alpha\colon X \to X$ and $\beta\colon Y \to Y$ are *topologically conjugated* if there exists a homeomorphism $\varphi\colon X \xrightarrow[\text{onto}]{} Y$ such that $\varphi\alpha\varphi^{-1} = \beta$.

Let O_X and O_Y be neighborhoods of zero of topological vector spaces X and Y respectively. Maps $\alpha\colon O_X \to X$ and $\beta\colon O_Y \to Y$ are said to be *locally topologically conjugated* if there exist neighborhoods Z_X and Z_Y of zeros in X and Y respectivelly, and a homeomorphism $\varphi\colon Z_X \xrightarrow[\text{onto}]{} Z_Y$ such that $(\varphi\alpha\varphi^{-1})(y) = \beta(y)$ for each y in a certain neighborhood of zero in Y.

For arbitrary Banach space X (real or complex), by $GL_U(X)$ we denote the subset of $GL(X)$ consisting of U-operators. An operator $T \in GL(X)$ is said to be an *U-operator* if its symmetric spectrum is disjoint with the unit circle $S^1 = \{z \in C^1; |z| = 1\}$.

If X is a complex Banach space, then the symmetric spectrum of an operator $T\colon X \to X$ is the smallest set containing the spectrum of T and symmetric with respect the real axis (in the general case, see definition in section 2).

We use here the following notation. For $T \in GL_U(E)$, where E is a Banach space, we put:

$$E_c(T) = \{e \in E; \lim_{n\to\infty} \|T^n e\| = 0\},$$

$$E_d(T) = \{e \in E; \lim_{n\to\infty} \|T^{-n} e\| = 0\}.$$

It is easy to see that $E_c(T)$ and $E_d(T)$ are invariant subspaces of T. We shall denote by T_c and T_d the operator T restricted to E_c and E_d, respectively.

The name "U-operators" is justified by the Anosov's theory of U-diffeomorphism (see [1]).

In the present paper we study the problem of the classification of U-operators in separable Banach spaces with respect to the relation of (local) topological conjugacy.

THEOREM I. *Let X and Y be separable Banach spaces and let $A \in GL_U(X)$ and $B \in GL_U(Y)$. The operators A and B are locally topologically conjugated iff they are topologically conjugated.*

THEOREM II. *Let X and Y be separable Banach spaces. Let $A \in GL_U(X)$ and $B \in GL_U(Y)$.*

1. *Let X and Y be real Banach spaces. Then, the conjunctions of the following three conditions is necessary and sufficient in order that operators A and B will be topologically conjugated*:

(a) $\dim X_c(A) = \dim Y_c(B)$ *and* $\dim X_d(A) = \dim Y_d(A)$;

(b) *if* $\dim X_c(A) < \infty$, *then* $\det A_c \cdot \det B_c > 0$;

(c) *if* $\dim X_d(A) < \infty$, *then* $\det A_d \cdot \det B_d > 0$.

2. *If X and Y are complex spaces, then A and B are topologically conjugated iff* $\dim X_c(A) = \dim Y_c(B)$ *and* $\dim X_d(A) = \dim Y_d(B)$.

In the case of finite-dimensional spaces, the second assertion of Theorem II can be derived from a result of Vajsbord (see [18] and [4]). For the finite-dimensional space the first assertion of Theorem II, where "topological conjugation" is replaced by "local topological conjugation", has been announced by Smale in [17], p. 753.

To derive some corollaries, we shall need the following result due to Hartman and Grobman (see [4] and [8]):

Let φ be a diffeomorphism of a neighborhood of zero in R^n onto another neighborhood of zero in R^n and let $\varphi \in C^1$ and $\varphi(0) = 0$. If the first derivative φ' of φ at 0 is an U-operator, then φ and φ' are locally topologically conjugated.

By M_n^R [M_n^C] we denote the set all diffeomorphisms of a neighborhood of zero in R^n [in C^n] onto another neighborhood of zero in R^n [in C^n] which satisfy the assumption of the Hartman-Grobman theorem.

Combining this theorem with Theorems I and II we get the following corollary:

COROLLARY 1. *The set of equivalence classes in M_n^R [in M_n^C] with respect to the relation of local topological conjugacy contains exactly $4n$ elements [$n+1$ elements].*

The present paper consists of four sections. In section 2 we study some simple facts on U-operators and on Banach spaces. The proofs of Theorems I and II are presented in section 4. These proofs are based on topological facts summarized in section 3.

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2. U-operators. If X is a real Banach space, then by $\tilde{X}$ we denote the *complexification* of X, i.e. the complex Banach space of ordered pairs $\{(x, y); x, y \in X\}$ in which $(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2)$, $(\alpha+i\beta)\times(x, y) = (\alpha x-\beta y, \alpha y+\beta x)$ for $\alpha, \beta \in R^1$ and $\|(x, y)\| = (\|x\|^2+\|y\|^2)^{1/2}$. In the sequel we will denote the pair (x, y) by $x \oplus iy$.

By a *symmetric spectrum* of a linear operator T defined in a real Banach space X we shall understand the spectrum of the operator $\tilde{T}$ in $\tilde{X}$ given by the formula $\tilde{T}(x \oplus iy) = Tx \oplus iTy$ for $x, y \in X$.

We recall that an operator $T \in GL(X)$ is said to be an U-operator if its symmetric spectrum is disjoint with S^1.

Let now X be a complex Banach space and let $T\colon X \to X$ be a linear operator. Denote by X_R the space X regarded as the real Banach space and by T_R the operator T regarded as the operator in X_R (i.e. $T_R x = Tx$). We define the symmetric spectrum of T as the symmetric spectrum of T_R.

PROPOSITION 1. *If T is an operator in a complex Banach space X, then* $\sigma(T) \cup \overline{\sigma(T)} = \sigma(\tilde{T})$, *where* $\overline{\sigma(T)} = \{z \in C^1; \bar{z} \in \sigma(T)\}$.

We omit the proof of Proposition 1. It is based on the Interior Mapping Principle (see [6], p. 57, Theorem 2).

By Proposition 1, an operator T in a complex Banach space is an U-operator iff $\sigma(T)$ is disjoint with the unit circle.

Before stating the main result of this section we recall the following fact, implicitly formulated in [8] and [16]:

LEMMA 1. *Let X be a real or complex Banach space and let $\|\cdot\|$ denote the norm on X. Then for every $A \in GL(X)$ the following conditions are equivalent*:

a. $\|A^n\| < 1$ *for some natural number* n.

b. *There exists on X a norm $|||\cdot|||$ such that $|||A||| < 1$ and the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent, i.e.*

$$\lim_{n\to\infty}\|x_n - x_0\| = 0 \quad \textit{iff} \quad \lim_{n\to\infty}|||x_n - x_0||| = 0$$

for every sequence $(x_n)_{n=0}^{\infty}$ in X.

c. $\tilde{\sigma}(A) \subset K(0, 1) = \{z \in C^1; |z| < 1\}$.

d. $\lim_{n\to\infty}\|A^n\| = 0$.

Proof. a $\Rightarrow$ b. If $\|A^n\| < 1$, then the norm

$$|||x||| = \frac{\|x\| + \ldots + \|A^{n-1}x\|}{n}$$

satisfies b.

The implications $b \Rightarrow c$ and $c \Rightarrow d$ are immediate consequences of the formula for spectral radius (see [7], p. 864). The implication $d \Rightarrow a$ is obvious.

Proposition 2. *Let X be a real or a complex Banach space. Then for every $A \in GL(X)$ the following conditions are equivalent:*

a. *A is a U-operator.*

b. *$X_c(A)$ and $X_d(A)$ are closed subspaces of X and $X = X_c(A) \oplus X_d(A)$.*

Proof. 1. First let us assume that X is a complex space.

$a \Rightarrow b$. By Proposition 1, A is a U-operator iff $\sigma(A)$ is disjoint with S^1. We write $\sigma_c(A) = \sigma(A) \cap K(0,1)$ and $\sigma_d(A) = \sigma(A) \setminus \sigma_c(A)$. By F. Riesz decomposition theorem (see [14], chapter XI, §§ 147, 148) it follows that $X = X_1 \oplus X_2$, where X_1 and X_2 are closed invariant subspaces of A and $\sigma(A|_{X_1}) = \sigma_c(A)$, $(A|_{X_2}) = \sigma_d(A)$. By $A|_Y$ we denote the restriction of A to the subspace Y of X. Obviously, $\sigma\big((A|_{X_2})^{-1}\big) \subset K(0,1)$. It follows from Lemma 1 that $X_1 \subset X_c(A)$ and $X_2 \subset X_d(A)$.

We shall prove that $X_1 = X_c(A)$ and $X_2 = X_d(A)$. Let $x \in X_c(A)$. Then $\lim\limits_{n\to\infty} \|A^n x\| = 0$. Let $x = u+v$, where $u \in X_1$ and $v \in X_2$. By Lemma 1, $\lim\limits_{n\to\infty} \|A^n u\| = 0$. Therefore $\lim\limits_{n\to\infty} \|A^n v\| = 0$. Since $v \in X_2$, we have

$$\|v\| \leqslant \|(A|_{X_2})^{-n}\| \, \|A^n v\| \quad \text{for } n = 1, 2, \ldots$$

It follows from Lemma 1 applied to the operator $(A|_{X_2})^{-1}$ that $\lim\limits_{n\to\infty} \|(A|_{X_2})^{-n}\| = 0$. Hence $v = 0$ and $X_1 = X_c(A)$. The proof that $X_2 = X_d(A)$ is similar.

$b \Rightarrow a$. This implication is obvious.

2. Now let us assume that X is a real Banach space. For $M \subset X$ let $\operatorname{Re} M = \{u \in X;$ there exists $v \in X$ such that $u \oplus iv \in M\}$.

$a \Rightarrow b$. It follows from part 1 of this proof that $\tilde{X}_c(\tilde{A})$ and $\tilde{X}_d(\tilde{A})$ are closed invariant subspaces of $\tilde{A}$, $\tilde{X} = \tilde{X}_c(\tilde{A}) \oplus \tilde{X}_d(\tilde{A})$. Observe that $x \oplus iy \in \tilde{X}_c(\tilde{A})$ iff $x \in X_c(A)$ and $y \in X_c(A)$. Hence $\operatorname{Re} X_c(\tilde{A}) = X_c(A)$. Analogically, $\operatorname{Re} \tilde{X}_d(\tilde{A}) = X_d(A)$. Obviously, $X_c(A)$ and $X_d(A)$ are invariant subspaces of A. Let $x_n \in X_c(A)$, $n = 1, 2, \ldots$, and $\lim\limits_{n\to\infty} x_n = x$. Then $x_n \oplus i0 \in \tilde{X}_c(\tilde{A})$ for $n = 1, 2, \ldots$ and $\lim (x_n \oplus i0) = x \oplus i0 \in \tilde{X}_c(\tilde{A})$. Thus $x \in X_c(A)$. The proof that $X_d(A)$ is a closed subspace is similar. Hence $X_c(A)$ and $X_d(A)$ are closed subspaces of X.

Let us note that if $x \in X$, then $x = \operatorname{Re} P_1(x \oplus i0) + \operatorname{Re} P_2(x \oplus i0)$, where P_1 and P_2 denote the projections of $\tilde{X}$ onto $\tilde{X}_c(\tilde{A})$ and $\tilde{X}_d(\tilde{A})$, respectively, such that $P_1P_2 = P_2P_1 = 0$. This representation of x as the sum of elements $X_c(A)$ and $X_d(A)$ is unique. Indeed, let $x = a+b = c+d$, where $a, c \in X_c(A)$ and $b, d \in X_d(A)$; then $0 = u+v$ for $u = a - c \in X_c(A)$ and $v = b - d \in X_d(A)$. Using the similar arguments as in the proof of point 1, we get $u = 0$ and $v = 0$. Hence $a = c$ and $b = d$.

The projections $x \to \operatorname{Re} P_1(x \oplus i0)$ and $x \to \operatorname{Re} P_2(x \oplus i0)$ are bounded, because P_1 and P_2 are bounded. This completes the proof of the implication $a \Rightarrow b$.

$b \Rightarrow a$. $X_c(A)$ and $X_d(A)$ are naturally embedded in X as closed invariant subspaces of A.

It is easy to see that $\tilde{X} = \widetilde{X_c(A)} \oplus \widetilde{X_d(A)}$. If $u \oplus iv \in \widetilde{X_c(A)}$, then

$$\lim_{n\to\infty} \|A^n(u \oplus iv)\| = 0$$

and if $u \oplus iv \in \widetilde{X_d(A)}$, then

$$\lim_{n\to\infty} \|A^{-n}(u \oplus iv)\| = 0.$$

Thus our assertion follows immediately from Lemma 1.

Let X be a real or complex Banach space with the norm $\|\cdot\|$. An operator $A \in GL(X)$ is said to be a *strong contraction* if there exists on X a norm $|||\cdot|||$ which is equivalent to $\|\cdot\|$ and such that $|||A||| < 1$.

Obviously, strong contractions and their inverse are U-operators. If A is a U-operator, then operators A_c and A_d^{-1}, considered in the spaces $X_c(A)$ and $X_d(A)$ respectively, are strong contractions.

We call an operator $V \in GL(X)$ a *pre-isometry* if there exists a constant M such that $\|V^n\| < M$ for every integer n.

The formula for spectral radius immediately implies that $\sigma(\tilde{V}) \subset S^1$. It is easy to see that the notion of pre-isometry does not depend on the particular choice of the equivalent norm in the space.

Lemma 2. *Let X be a real or complex Banach space, let V be a pre-isometry in X. Then there exists a norm $|||\cdot|||$, equivalent to the original norm on X, such that:*

$$|||Vx||| = |||x||| \quad \text{for } x \in X. \tag{1}$$

Proof. It is easy to see that the norm defined as follows

$$|||x||| = \sup_{-\infty<n<+\infty} \|V^n x\|_1$$

satisfies (1)

3. Some topological theorems.

Lemma 3. *Let H be a linear transformation of R^n, given in the unit vector basis by the matrix*

$$\begin{bmatrix} -1 & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{bmatrix}.$$

Then the transformation $A = \frac{1}{2}I$ and $B = \frac{1}{2}H$ are not locally topologically conjugated.

Proof. Let us assume that A and B are locally topologically conjugated. Then there exists a homeomorphism φ of a neighborhood of zero in R^n into the neighborhood of zero in R^n and an $\varepsilon > 0$ such that $\varphi^{-1}A\varphi(x) = B(x)$ for $x \in K(0, \varepsilon) = \{x\colon \|x\| < \varepsilon\}$. Thus $\varphi(x) = 2\varphi(Bx)$ for $x \in K(0, \varepsilon)$. Obviously, there exists a $\delta > 0$ such that $\varphi(K(0, \varepsilon)) \supset K(0, \delta)$.

We define a homeomorphism $F\colon R^n \xrightarrow[\text{onto}]{} R^n$ such that $F(x) = 2F(Bx)$ for $x \in R^n$ as follows. For $x \in K(0, \varepsilon)$ we define $F_1(x) = \varphi(x)$. On $K(0, 2\varepsilon)$ we define $F_2(x)$ by $F_2(x) = 2F_1(Bx)$. It is easy to see that F_2 is a homeomorphism on $K(0, 2\varepsilon)$ and that $F_2(K(0, 2\varepsilon)) \supset K(0, 2\delta)$. Obviously, F_2 is an extension of $F_1 = \varphi$. Proceeding analogously, we define $F_3(x)$ on $K(0, 4\varepsilon)$ by the formula $F_3(x) = 2F_2(Bx)$ etc. Let $F(x) = F_{n+1}(x)$ for $\|x\| \leqslant 2^n\varepsilon$. F is a well defined homeomorphism of R^n onto R^n and F satisfies the equation $F(x) = 2F(Bx)$ for $x \in R^n$. Hence if A and B are locally topologically conjugated, then A and B are topologically conjugated in R^n.

Regard the n-dimensional unite sphere S^n as $R^n \cup \{\infty\}$, the one-point compactification of R^n. We extend A to the homeomorphism $\bar{A}$ of S^n onto S^n by the following formula:

$$\bar{A}(x) = \begin{cases} A(x) & \text{if } x \neq \infty, \\ \infty & \text{if } x = \infty. \end{cases}$$

Analogously we define the homeomorphisms $\bar{B}$ and $\bar{F}$. Obviously $\bar{A}\bar{F} = \bar{F}\bar{B}$ on S^n.

If f is a homeomorphism of S^n onto itself, then by $\deg f$ we denote its degree (see [10], chapter 2, or [9]) We have:

$$\deg f = \pm 1 \quad \text{and} \quad \deg(fg) = \deg f \cdot \deg g.$$

Therefore $\deg \bar{A} = \deg \bar{B}$. But from the last problem of Section D of chapter 2 in [10] we easily obtain that $\deg \bar{A} = +1$ and $\deg \bar{B} = -1$. This contradiction completes the proof.

LEMMA 4. *Let X and Y be Banach spaces and let $A \in GL_U(X)$ and $B \in GL_U(Y)$. The following conditions are equivalent:*

a. *A and B are locally topologically conjugated.*

b. *A_c and B_c are locally topologically conjugated and A_d and B_d are locally topologically conjugated.*

In particular, if A and B are locally topologically conjugated, then

$$\dim X_c(A) = \dim Y_c(B) \quad \text{and} \quad \dim X_d(A) = \dim Y_d(B).$$

Proof. a $\Rightarrow$ b. Let L be a homeomorphism of a neighborhood of zero in X onto a neighborhood of zero in Y such that $LAL^{-1}x = Bx$ for x belonging to a certain neighborhood of zero. It is enough to show that for every $\varepsilon_i > 0$ there exists $\delta_i > 0$ for $i = 1, 2, 3, 4$ such that:

(1) $$L(X_c(A) \cap K(0, \delta_1)) \subset Y_c(B) \cap K(0, \varepsilon_1),$$

(2) $$L(X_d(A) \cap K(0, \delta_2)) \subset Y_d(B) \cap K(0, \varepsilon_2),$$

(3) $$L^{-1}(Y_d(B) \cap K(0, \delta_3)) \subset X_d(A) \cap K(0, \varepsilon_3),$$

(4) $$L^{-1}(Y_d(B) \cap K(0, \delta_4)) \subset X_d(A) \cap K(0, \varepsilon_4).$$

If $L(Ax) = B(Lx)$ in a neighborhood of zero, then the equality $L^{-1}BL = A$ holds in a neighborhood of zero. Then (1) and (2) imply (3) and (4). Since $X_c(A) = X_d(A^{-1})$ and $X_d(A) = X_c(A^{-1})$, it follows from the identity $LA^{-1}L^{-1} = B^{-1}$ (which holds in a neighborhood of zero) that (1) implies (2). Therefore it is sufficient to prove (1).

In view of Lemma 1, we can assume without loss of generality that $\|A_c\| < 1$. We know that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that:

1. L and L^{-1} are homeomorphisms on $K(0, \delta)$;
2. $L(K(0, \delta)) \subset K(0, \varepsilon)$;
3. $L^{-1}BL(x) = A(x)$ for $x \in K(0, \delta)$.

If $x \in X_c(A)$, then $\|A^n x\|$ converges monotonically to zero for $n \to \infty$. Hence for $x \in X_c(A) \cap K(0, \delta)$ we have $L^{-1}B^nL(x) = A^n(x)$. Next observe that if $BL(x) = L(Ax)$, then $BL(0) = L(0)$. Since $B \in GL_U(Y)$, we have $L(0) = 0$. Since L is a homeomorphism, we have $L(0) = L^{-1}(0) = 0$. Hence

$$\lim_{n\to\infty} \|B^n L(x)\| = 0.$$

Consequently, $L(x) \in Y_c(B) \cap K(0, \varepsilon)$. This proves (1).

b $\Rightarrow$ a. This implication is obvious.

By Lemma 4 the problem of classification of elements of GL_U with respect to the relation of the local topological conjugancy is reduced to the case of strong contractions.

The proof in this case will employ the following results:

THEOREM OF KADEC. *All infinite-dimensional separable Banach spaces are homeomorphic* (see [11] and [3]).

THEOREM OF WONG. *If X is an infinite-dimensional separable Hilbert space, then every homeomorphism of X onto itself is isotopic with I_x* (see [19]).

THEOREM OF KLEE. *In every infinite-dimensional Banach space X, the unit sphere S_X is homeomorphic to X.*

Proof. The proof of the last theorem is based on the following fact formulated by Klee in [12]. If X is an arbitrary infinite-dimensional normed linear space, then S_X is homeomorphic to a closed hyperplane of X (i.e. closed linear subspace of codimension 1).

Now, let S be homeomorphic to a hyperplane Y. Let Z be an infinite-dimensional separable and closed subspace of Y. Then it follows from the Bartle-Graves theorem (see [2]) that Y is homeomorphic to $Z\times Y/Z$. From theorem of Kadec we infer that $R^1\times Z$ is homeomorphic to Z. Obviously, $R^1\times Z\times Y/Z$ is homeomorphic to X, so Y is homeomorphic to X. (This proof was communicated to the author by Professor C. Bessaga.)

In the sequel we shall essentially employ the following

COROLLARY 2. *If X and Y are separable infinite-dimensional Banach spaces and f and g are homeomorphisms of S_X onto S_Y, then f and g are isotopic (in the class of homeomorphisms of S_X onto S_Y).*

Proof. It follows by theorems of Kadec and Wong that if X and Y are separable infinite-dimensional Banach spaces and f and g are homeomorphisms of X onto Y, then f and g are isotopic. Hence, by theorem of Klee, it follows that if f and g are homeomorphisms of S_X onto S_Y, then f and g are isotopic.

4. Proofs of Theorems I and II. We begin with the following proposition:

PROPOSITION 3. *Let X and Y be separable, infinite-dimensional Banach spaces. If A and B are strong contractions in X and Y respectively, then A and B are topologically conjugated.*

Proof. Let Z be a Banach space. If $x\in Z$ and $A\in GL(Z)$, then we write

$$\overrightarrow{Ox} = \{z\in Z;\ z = tx \text{ for } t\geqslant 0\} \quad\text{and}\quad C_A(x) = \overrightarrow{Ox}\cap A(S_Z).$$

a. It is sufficient to show that if A is a strong contraction in X, then A is topologically conjugated with $B = \frac{1}{2}I_Y$. In view of Lemma 1 we can assume, without loss of generality, that $\|A\|<1$ and, therefore, $\|C_A(x)\|<1$ for $x\in X$. We write

$$M = \bigcup_{0\leqslant t\leqslant 1} a_t,$$

where $a_t = \{z\in X;\ z = x+t(C_A(x)-x),\ x\in S_X\}$.

Obviously, for every point $x\in M$, there exists exactly one number $t(x)$, $0\leqslant t(x)\leqslant 1$, such that $x\in a_{t(x)}$.

We put

$$M_1 = \{y\in Y;\ \tfrac{1}{2}\leqslant \|y\|\leqslant 1\},\qquad M_1 = \bigcup_{0\leqslant t\leqslant 1}\gamma_t,$$

where $\gamma_t = \{y\in Y;\ \|y\| = -\frac{1}{2}t+1\}$.

For $y\in Y$ we write $\psi_t(y) = \overrightarrow{O_y}\cap\gamma_t$, $0\leqslant t\leqslant 1$.

b. By the theorems of Kadec and Klee there exists a homeomorphism $F\colon S_X \xrightarrow[\text{onto}]{} S_Y$. We will define here a new homeomorphism $G\colon S_X\xrightarrow[\text{onto}]{} S_Y$ by the following formula:

$$\text{if } x\in S_X, \text{ then } G\left(\frac{Ax}{\|Ax\|}\right) = F(x).$$

Since every two homeomorphisms of S_X onto S_Y are isotopic, there exits an isotopy $\{f_t\}$, $0\leqslant t\leqslant 1$, such that $f_0 = F$ and $f_1 = G$. We define a homeomorphism $L\colon M\xrightarrow[\text{onto}]{} M_1$ by

$$L(x) = \psi_{t(x)}\left(f_{t(x)}\left(\frac{x}{\|x\|}\right)\right)\quad \text{for } x\in M.$$

It is easy to see that if $x\in S_X$, then $L(Ax) = \frac{1}{2}L(x)$.

We extend L to the homeomorphism $\bar{L}\colon X\xrightarrow[\text{onto}]{} Y$ by the following formula:

$$\bar{L}(x) = \begin{cases} \dfrac{1}{2^k}L(A^{-k}x) & \text{for } x\in A^k(N),\ -\infty<k<\infty,\\ 0 & \text{for } x = 0,\end{cases}$$

where $N = M\setminus A(S_X)$. (Note that $X = \bigcup_{-\infty}^{\infty}A^k(N)\cup\{0\}$.)

It is obvious that $\bar{L}$ is a homeomorphism of X onto Y and that $\bar{L}(Ax) = \frac{1}{2}\bar{L}(x)$ for $x\in X$.

Let A be a strong contraction defined on the Banach space $(X, \|\cdot\|)$, let V be a pre-isometry defined on X. We say that the pair (A, X) satisfies the *condition* (t) if

(t) *there exists a norm $|||\cdot|||$, equivalent to the norm $\|\cdot\|$, such that*

(a) $|||A|||<1$,

(b) $|||Vx||| = |||x|||$ *for* $x\in X$.

PROPOSITION 4. *Let X be an arbitrary Banach space and let $GL_I(X)$ denote the component of identity in $GL(X)$. Let A and B be strong contractions belonging to $GL_I(X)$, and let V be a pre-isometry in X. If the pairs (A, V) and (B, V) satisfy condition* (t), *then the operators VA and VB are topologically conjugated.*

Proof. It is sufficient to prove that the operators VA and $\frac{1}{2}VI_X$ are topologically conjugated in X. Since the pair (A, V) satisfies condition (t), we can assume that $\|A\|<1$ and $\|Vx\| = \|x\|$ for $x\in X$.

Let $\{A_t\}$, $0\leqslant t\leqslant 1$, be a curve in $GL_I(X)$ which links together I_X with A; $A_0 = I_X$, $A_1 = A$. The curve $B_t = A_t^{-1}V^{-1}$ joints V^{-1} with $A^{-1}V^{-1}$.

Analogically as in the proof of Proposition 3 we introduce the following sets and functions:

$$c(x) = \overrightarrow{Ox} \cap VA(S_X);$$

$$M = \bigcup_{0 \leqslant t \leqslant 1} a_t,$$

where $a_t = \{z \in X;\ z = x + t(c(x) - x),\ x \in S_X\}$;

$$M_1 = \{x \in X;\ \tfrac{1}{2} \leqslant \|x\| \leqslant 1\} = \bigcup_{0 \leqslant t \leqslant 1} \gamma_t,$$

where $\gamma_t = \{y \in X;\ \|y\| = -\tfrac{1}{2}t + 1\}$, $\psi_t(x) = V(\overrightarrow{Ox} \cap \gamma_t)$.

It is easy to see that the mapping L given by the formula

$$L(x) = \psi_{t(x)}\Big(B_{t(x)}\frac{x}{\|x\|}\Big) \quad \text{for } x \in a_t$$

is a homeomorphism of M onto M_1.

Let $x \in S_X$; then $x \in a_0$. Hence

$$L(x) = \psi_0\Big(B_0\frac{x}{\|x\|}\Big) = V(V^{-1}x) = x$$

and

$$L(VAx) = \psi_1\Big(B_1\frac{VAx}{\|VAx\|}\Big) = \psi_1\Big(A^{-1}V^{-1}\frac{VAx}{\|VAx\|}\Big) = \psi_1\Big(\frac{x}{\|Ax\|}\Big)$$
$$= \tfrac{1}{2}V(x) = \tfrac{1}{2}V(Lx).$$

Let us set

$$\bar{L}(x) = \begin{cases} \dfrac{1}{2^n}V^n L\big((VA)^{-n}x\big) & \text{for } x \in (VA)^n\big(M \setminus VA(S_X)\big), \\ 0 & \text{for } x = 0. \end{cases}$$

Clearly, $\bar{L}$ is a homeomorphism of X onto X and $\bar{L}$ is an extension of L. Moreover, $\bar{L}(VAx) = \frac{1}{2}V\bar{L}(x)$ for $x \in X$.

It is well known that the group $GL(R^n)$ has two components which are arcwise connected: the operators with a positive determinant and the operators with a negative determinant. The group $GL(C^n)$ is arcwise connected.

COROLLARY 3. *Every two strong contractions in the space R^n, the determinats of which have the same sign, are topologically conjugated.*

Proof. Let K and L be two strong contractions in R^n.

(a) If $\det K > 0$, $\det L > 0$, then the pairs (K, I) and (L, I) satisfy condition (t), so that K and L are topologically conjugated.

(b) Let $\det K < 0$ and $\det L < 0$. It is sufficient to show that K is topologically conjugated to $\frac{1}{2}H$, where H is the pre-isometry of Lemma 3. Let $\|\cdot\|$ denote the Euclidean norm in R^n, let c be a positive real number such that $\|cHK\| < 1$ and $0 < c < 1$. Applying Proposition 4 to $X = R^n$, $V = H$, $A = cHK$, and $B = \frac{1}{2}I$ we conclude that cK is topologically conjugated to $\frac{1}{2}H$. It can easily be checked that the operators K and cK for $0 < c < 1$ are topologically conjugated, so are the operators K and $\frac{1}{2}H$.

Using Corollary 3 and Lemma 3 we obtain the following

COROLLARY 4. *Let A and B be strong contractions in R^n, and let $\det A > 0$ and $\det B < 0$. Then A and B are not locally topologically conjugated.*

Applying Proposition 4 to the case where $X = C^n$ and $V = I_{C^n}$ we get

COROLLARY 5. *In the space C^n every two strong contractions are topologically conjugated.*

Note that the statements of Proposition 3 and Corollaries 3, 4 and 5 are also true for the operators which are inverse to strong contractions.

Proof of Theorems I and II. If A and B or their inverse are strong contractions, then Theorems I and II are consequences of Proposition 3 and Corollaries 3, 4 and 5. In a general case, these theorems follows from the preceding remarks and Lemma 4.

We note that, in fact, the assertion 2 of Theorem II is a consequence of assertion 1, but its direct proof is essentialy easier than the proof of assertion 1.

It is well known that in every infinite-dimensional (not necessery separable) real or complex Hilbert space H the group $GL(H)$ is arcwise connected (see [13]). Therefore, in view of Proposition 4, we conclude that Theorems I and II also hold in an arbitrary infinite-dimensional Hilbert space.

It is quite probable that the following generalization of Wong's theorem is true. If X is an infinite-dimensional Banach space, then every homeomorphism of X onto X is isotopic with I_X.

If the answer is affirmative, then in infinite-dimensional spaces Proposition 4 will be a consequence of Proposition 3. Hence the only nontrivial application of Proposition 4 will be the cases of $X = R^n$ and $X = C^n$. On the other hand, in contrast with the Proposition 3, Proposition 4 has an elementary character.

It is interesting to notice that S. Rolewicz in [15] gave an example of operator in l^p, $1 \leqslant p < \infty$, which has a dense orbit. Obviously, neither the operator of Rolewicz, nor its inverse is a U-operator.

Finally, we remark that A. Douady in [5] proved that $GL(L^2 \times c_0)$ has infinitely many components.

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Analytic functions in Banach spaces

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There are not many papers devoted to the theory of analytic mappings in Banach spaces, especially in real Banach spaces; recently, however, a number of papers concerning that branch have been published (Cartan, Douady, Lelong, Ramis and others). N. Bourbaki in his forthcoming books [5] ("Fascicule de résultats" is already available) will give the systematic theory of such mappings.

This paper gives some results on analytic mappings, with values in a Banach space, defined on open subsets of Banach spaces (real or complex). In particular, we prove a natural criterion of the analyticity of mappings (Theorem 6). Some versions of that criterion were given by Alexiewicz and Orlicz [1] and Siciak [17]; the author of the present paper has been inspired by some ideas of [1], and [9].

The plan of the article is as follows. We start in Section I by proving some results on formal series in Banach spaces. In Section II we consider Gateaux-differentiable mappings. In Section III we state and prove some criteria of the holomorphicity (Theorem 4, complex case) and analyticity (Theorem 6, real case) of the mappings. Finally, in Section IV, we give the applications of the preceding results, namely: a proof of the Weierstrass preparation theorem for analytic functions in Banach spaces (another proof of that theorem was given by Ramis [14]), the generalization of a theorem of Malgrange, and some other theorems.

Theorem 4 is stated in [5] without proof (see also [9], [21]). The proof of case I of Theorem C has been communicated to me by Professor S. Łojasiewicz.

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I. FORMAL SERIES IN BANACH SPACES

Let E and F be real or complex Banach spaces. Denote by $\mathrm{Hom}^k(E, F)$ (resp. $L^k(E, F)$) the space of k-linear, symmetric (resp. k-linear, symmetric