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On unconditional bases in certain Banach function spaces

by

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1. It was observed by Marcinkiewicz [7] that the Haar system, $\mathscr{H} = \{h_n\colon n = 1, 2, \ldots\}$, constitutes an unconditional Schauder basis for each of the spaces $L^p[0,1]$, $p > 1$. The proof amounts to a demonstration that a certain norm on L^p defined in terms of the Haar functions is equivalent to the ordinary norm, a fact, Marcinkiewicz noted, that follows easily from a theorem of Paley [9] concerning Walsh series. The norm in question is obtained in the following manner. Let f be an element of $L^p[0,1]$, $p > 1$, let $\sum_{n=1}^{\infty} a_n h_n$ be its Haar expansion ($\mathscr{H}$ is a basis for each of the L^p-spaces [10]), and let

$$G_p(f) = \Big\|\Big[\sum_{n=1}^{\infty} a_n^2 h_n^2\Big]^{1/2}\Big\|_p.$$

From the theorem of Paley follows the existence of positive constants A_p and B_p such that $A_p G_p(\cdot) \leqslant \|\cdot\|_p \leqslant B_p G_p(\cdot)$, so that each series $\sum_{n=1}^{\infty} \varepsilon_n a_n h_n$, with $\varepsilon_n = \pm 1$, must converge, a condition equivalent to the unconditional convergence of $\sum_{n=1}^{\infty} a_n h_n$ (see, for example, [1]).

Motivated by this result Gaposhkin [2] proved that a basis $\Phi = \{\varphi_n\colon n = 1, 2, \ldots\}$ for L^p, $p > 1$, is unconditional precisely when the norm G_p defined by the relation

$$G_p(f) = \Big\|\Big[\sum_{n=1}^{\infty} a_n^2 \varphi_n^2\Big]^{1/2}\Big\|_p, \quad \text{where } f = \sum_{n=1}^{\infty} a_n \varphi_n,$$

is equivalent to the customary norm. Subsequently Gaposhkin [3] and others extended this result to include the reflexive Orlicz spaces as well.

One cannot but wonder why this peculiar norm should play such a prominent rôle. An answer, on one level, is derived from an inspection of the original work of Paley. There one finds that the Khintchine inequality (see, for example, [6]) concerning series of Rademacher functions occupies a central position in the proof of the critical theorem. The connection of the Rademacher system with the notion of unconditional convergence is furnished by the criterion, cited above, concerning the convergence of each of the series $\sum_{n=1}^{\infty} \varepsilon_n a_n \varphi_n$, where $\varepsilon_n = \pm 1$, for each n. Thus motivated, it is a not too difficult task to show that the analogue of the Marcinkiewicz-Gaposhkin result continues to hold in a very broad class of function spaces, and the present work is devoted to that end.

2. Let X be a Banach space, let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of X, and let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of continuous linear functionals on X. The sequence $\{x_n, Y_n\}_{n=1}^{\infty}$ is biorthogonal if $Y_n(x_m) = \delta_{nm}$, for all natural number pairs (n, m). A biorthogonal system $\{x_n, Y_n\}$ is a Schauder basis for X if, for every x in X, $\sum_{n=1}^{\infty} Y_n(x)x_n$ converges to x.

Let $\{u_n\}_{n=1}^{\infty}$ be an arbitrary sequence of elements of X. The series $\sum_{n=1}^{\infty} u_n$ is unconditionally convergent if, for every permutation π of the natural numbers, $\sum_{n=1}^{\infty} u_{\pi(n)}$ converges. One of several useful criteria for this mode of convergence is the following: $\sum_{n=1}^{\infty} \varepsilon_n u_n$ converges for every sequence $\varepsilon\colon N \to \{-1, 1\}$.

A Schauder basis $\{x_n, Y_n\}$ is an unconditional basis for X if every expansion $\sum_{n=1}^{\infty} Y_n(x)x_n$ converges unconditionally.

Gelbaum [4] has examined unconditional convergence of basis expansions in a Banach space X in the following manner. Let G denote the set of all sequences $\varepsilon\colon N \to \{-1, 1\}$, and, for each x in X, let $C(x)$ denote the subset of G for which $\sum_{n=1}^{\infty} \varepsilon_n Y_n(x)x_n$ converges. Let $\boldsymbol{Z}(2)$ be given the discrete topology. If G be endowed with the group structure of $(\boldsymbol{Z}(2))^{\omega}$ and with the corresponding product topology, then the normalized Haar measure μ on G is the product measure induced by the normalized Haar measure on $\boldsymbol{Z}(2)$. The sets $C(x)$, for each x, and $C = \bigcap_{x \in X} C(x)$ are shown to be measurable, and an application of the 0-1 law shows that each of these sets has Haar measure either 0 or 1. Because C proves to be a subgroup of G, it follows that either $\mu(C) = 0$ or $C = G$.

By virtue of the measure-theoretic equivalence of the Lebesgue measure space on $[0,1]$ and the measure space associated with G, the above results can be put in a form more suitable for the demonstration of the theorem that follows. Let $\{r_n\colon n = 1, 2, \ldots\}$ denote the orthonormal system of Rademacher, so modified as to make each r_n right continuous. Since to each ε in G there corresponds a unique element θ of $[0,1]$, one finds that $\{\theta\colon \sum_{n=1}^{\infty} r_n(\theta) Y_n(x)x_u \text{ converges}\}$ is, for each x, Borel measurable and has Lebesgue measure 0 or 1. Without great danger of confusion, these sets are also denoted by $C(x)$, and the intersection of all such sets is again denoted by C. Either C is a (Borel) null set or $C = [0,1]$, so that the basis is unconditional if and only if $|C|$ is positive.

Certainly it is clear that if $\{x_n, Y_n\}$ is an unconditional basis for X, then each function $T_\theta\colon X \to X$, $\theta \in [0,1]$ ($T_\varepsilon\colon X \to X$, $\varepsilon \in G$), given by the relation

$$T_\theta x = \sum_{n=1}^{\infty} r_n(\theta) Y_n(x)x_n, \quad x \in X,$$

$$\left(T_\varepsilon x = \sum_{n=1}^{\infty} \varepsilon_n Y_n(x)x_n, \quad x \in X\right),$$

is a bounded linear transformation on X. Moreover, Gelbaum (see also [5]) has observed that the family $\{T_\varepsilon\colon \varepsilon \in G\}$ ($\{T_\theta\colon \theta \in [0,1]\}$) is (uniformly) bounded, a condition that is, of course, also sufficient for the unconditionality of the basis.

3. Let X be a Banach space whose elements belong to $\mathscr{M}[0,1]$, the set of all measurable real-valued functions on $[0,1]$, where the norm is determined by another subset $\mathscr{C}$ of $\mathscr{M}[0,1]$ according to the formula

$$\|x\| = \sup\{\int_0^1 |x(t)c(t)|\,dt\colon c \in \mathscr{C}\}.$$

In order to exclude the more pathological members of this species, assume that there is an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of Lebesgue measurable sets such that $[0,1] = \bigcup_{n=1}^{\infty} E_n$ and χ_{E_n} belongs to X for all n. The Orlicz spaces (in particular, the L^p-spaces with $1 < p < \infty$), L^1, L^∞, and the Lorentz spaces are familiar examples of this type of Banach function space.

Let $\{x_n, Y_n\}$ be an unconditional basis for X and, for each x in X, let

$$G(x) = \left\|\left[\sum_{n=1}^{\infty} Y_n^2(x)x_n^2\right]^{1/2}\right\|.$$

A routine calculation reveals that G is a norm.

LEMMA (cf. Orlicz [8]). *There is a positive constant B such that $G(x) \leqslant B\|x\|$ for all x in X.*

Proof. If

$$s_n = \sum_{k=1}^{n} Y_k(x)x_k, \quad n = 1, 2, \ldots,$$

then ($\lim_n s_n = x$) $\lim_n \|s_n\| = \|x\|$, and $\lim_n G(s_n) = G(x)$. According to the Khintchine inequality for L^1, there exists a positive constant A_1 such that

$$A_1\Big[\sum_{k=1}^{n} Y_k^2(x)x_k^2(t)\Big]^{1/2} \leqslant \int_0^1 \Big|\sum_{k=1}^{n} r_k(\theta)\, Y_k(x)x_k(t)\Big|\,d\theta,$$

so that, for each c in $\mathscr{C}$,

$$A_1\int_0^1 \sum_{k=1}^{n} Y_k^2(x)x_k^2(t)]^{1/2}|c(t)|\,dt \leqslant \int_0^1\Big\{\int_0^1\Big|\sum_{k=1}^{n} r_k(\theta)\, Y_k(x)x_k(t)\Big|d\theta\Big\}|c(t)|\,dt$$

$$= \int_0^1\Big\{\int_0^1\Big|\sum_{k=1}^{n} r_k(\theta)\, Y_k(x)x_k(t)\Big||c(t)|\,dt\Big\}\,d\theta.$$

(Each of the functions $(\theta, t) \to r_k(\theta)x_k(t)$ is measurable on the square.) But

$$\int_0^1\Big|\sum_{k=1}^{n} r_k(\theta)\, Y_k(x)x_k(t)\Big||c(t)|\,dt = \int_0^1 |T_\theta s_n(t)|\,|c(t)|\,dt$$

$$\leqslant \|T_\theta s_n\| \leqslant K\|s_n\|,$$

where K is a bound for the family $\{T_\theta\colon\ \theta\in[0,1]\}$. It now follows that

$$A_1 G(s_n) \leqslant K\|s_n\|, \qquad n = 1, 2, \ldots,$$

and the desideratum is obtained by taking limits.

THEOREM. *If, in addition to the above hypotheses, X is reflexive, then there is a positive constant A such that $A\|x\| \leqslant G(x)$, for all x in X.*

Preliminary remarks. (α) Under these conditions, therefore, the norm G is equivalent to $\|\cdot\|$. (β) The reflexivity condition forces all bounded linear functionals on X to be integrals. (For this and other pertinent facts concerning Banach function spaces, see, for example, [12].) In the argument that follows, this property is used only to insure that each coefficient functional is of integral type. Accordingly, a slight modification of the proof will produce a more general result.

Proof. If, for each measurable y,

$$\|y\|' = \sup\Big\{\int_0^1 |y(t)x(t)|\,dt\colon\ \|x\| \leqslant 1\Big\},$$

then $\|\cdot\|'$ proves to be a norm on $X' = \{y\colon\ \|y\|' < +\infty\}$, and $(X', \|\cdot\|')$ is again a Banach function space, known as the *associate space* of X. It develops that $\|y\|'$ may also be calculated by taking the supremum of

$$\Big\{\Big|\int_0^1 y(t)x(t)\,dt\Big|\colon\ \|x\| \leqslant 1\Big\}$$

and that, for each x in X,

$$\|x\| = \sup\Big\{\int_0^1 |x(t)y(t)|\,dt\colon\ \|y\|' \leqslant 1\Big\}$$

$$= \sup\Big\{\Big|\int_0^1 x(t)y(t)\,dt\Big|\colon\ \|y\|' \leqslant 1\Big\}.$$

The functions $\|\cdot\|$ and $\|\cdot\|'$ are further connected by a generalized Hölder inequality; viz.,

$$\int_0^1 |x(t)y(t)|\,dt \leqslant \|x\|\cdot\|y\|',$$

for all x and y in $\mathscr{M}[0,1]$.

One consequence of the reflexivity of X is that X' coincides with X^*, the dual space of X. That is to say, for every bounded linear functional Y on X, there is a y in X' such that

$$Y(\cdot) = \int_0^1 (\cdot)(t)y(t)\,dt \quad \text{and} \quad \|y\|' = \|Y\|^*.$$

Let y_n be the element of the associate space that corresponds to the coefficient functional Y_n. Because $\{x_n, Y_n\}$ is a basis for X, $\{Y_n, x_n\}$ is a basis for Λ^*, the closed linear span of $\{Y_n\colon\ n = 1, 2, \ldots\}$ in X^*. Because the mappings T_θ^*, where

$$T_\theta^* Y = \sum_{n=1}^{\infty} r_n(\theta)\, Y(x_n)\, Y_n,$$

for Y in X^*, are the adjoints of the transformations T_θ, it follows that $\{Y_n, x_n\}$ is unconditional. For each y in Λ, the subspace of X' corresponding to Λ^*, define

$$G'(y) = \Big\|\Big[\sum_{k=1}^{n} Y^2(x_k)y_k^2\Big]^{1/2}\Big\|',$$

where Y is the linear functional determined by y. By virtue of the lemma, there is a positive constant B' such that $G'(\cdot) \leqslant B'\|\cdot\|'$.

Given Z in X^* and x in X, let

$$Z_m(x) = \sum_{k=1}^{m} Z(x_k)\, Y_k(x).$$

It is clear that $\lim_m Z_m(x) = Z(x)$. Moreover, the operators

$$Z \to Z_m = \sum_{k=1}^{m} Z(x_k)\, Y_k$$

are the adjoints of the operators

$$x \to \sum_{k=1}^{m} Y_k(x)x_k,$$

and are, consequently, uniformly bounded. Let the positive constant K be chosen so that

$$\|Z_m\|^* \leqslant K\|Z\|^*, \quad m = 1, 2, \ldots$$

Let x be an element of X, let

$$s_n = \sum_{k=1}^{n} Y_k(x)x_k,$$

and let the positive number ε be given. Choose z in X' so that $\|z\|' \leqslant 1$, and

$$\|s_n\| \leqslant \Big|\int_0^1 s_n(t)z(t)\,dt\Big| + \varepsilon/2.$$

Let Z be the element of X^* to which z corresponds, let m be a natural number exceeding n such that

$$|Z_m(s_n) - Z(s_n)| < \varepsilon/2,$$

and let z_m be the correspondent of Z_m in X'. Then

$$\|s_n\| \leqslant \Big|\int_0^1 s_n(t)z_m(t)\,dt\Big| + \varepsilon.$$

Because $Z_m = \sum_{i=1}^{m} Z(x_i)\,Y_i$, the integral standing in the right member reduces to

$$\sum_{k=1}^{n} Y_k(x)Z(x_k) = \int_0^1 \sum_{k=1}^{n} Y_k(x)x_k(t)Z(x_k)y_k(t)\,dt;$$

thus,

$$\begin{aligned}
\|s_n\| &\leqslant \Big|\int_0^1 \sum_{k=1}^{n} Y_k(x)x_k(t)Z(x_k)y_k(t)\,dt\Big| + \varepsilon \\
&\leqslant \int_0^1 \sum_{k=1}^{n} |Y_k(x)x_k(t)Z(x_k)y_k(t)|\,dt + \varepsilon \\
&\leqslant \int_0^1 \Big[\sum_{k=1}^{n} Y_k^2(x)x_k^2(t)\Big]^{1/2}\Big[\sum_{k=1}^{n} Z^2(x_k)y_k^2(t)\Big]^{1/2} dt + \varepsilon \\
&\leqslant \Big\|\Big[\sum_{k=1}^{n} Y_k^2(x)x_k^2\Big]^{1/2}\Big\| \cdot \Big\|\Big[\sum_{k=1}^{n} Z^2(x_k)y_k^2\Big]^{1/2}\Big\|' + \varepsilon \\
&= G(s_n)\cdot G'\Big(\sum_{k=1}^{n} Z(x_k)y_k\Big) + \varepsilon \\
&\leqslant G(s_n)\cdot B'\|Z_n\|^* + \varepsilon \\
&\leqslant B'KG(s_n) + \varepsilon,
\end{aligned}$$

from which follows

$$A\|s_n\| \leqslant G(s_n)$$

with $A = 1/B'K$. Again, the required inequality is obtained by passing to the limit.

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