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An extension of Choquet boundary theory to certain partially ordered compact convex sets

by

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1. Introduction. In Choquet boundary theory [6, 8, 18, 21] one studies the Choquet simplex $\mathscr{P}(\Omega)$ of all probability Radon measures on a compact Hausdorff space Ω, together with a wedge $\mathscr{G}$ of continuous real functions on Ω. Under suitable hypotheses $\mathscr{P}(\Omega)$ can be partially ordered by writing $\mu \succ \nu$ whenever $\int g d\mu \geqslant \int g d\nu$ for all $g \in \mathscr{G}$, and the theory then has much to say about the maximal elements of $\mathscr{P}(\Omega)$ for this ordering, and about semicontinuous or continuous functions on Ω that are *$\mathscr{G}$-convex*, in the sense that $\int f d\mu \geqslant f(x)$ whenever $\mu \succ \varepsilon_x$. For all this it is enough to assume that $\mathscr{G}$ separates the points of Ω, contains the constant functions, and is such that $\max(f, g) \in \mathscr{G}$ whenever $f, g \in \mathscr{G}$. Then, by the Weierstrass-Stone theorem, $\mathscr{G} - \mathscr{G}$ is dense in the space $\mathscr{C}(\Omega)$ of real continuous functions on Ω, and so *inter alia* $\succ$ is a partial ordering for $\mathscr{P}(\Omega)$.

Most if not all of the results of the theory can be reformulated as statements about $\mathscr{P}(\Omega)$ and the affine extended-real functions on $\mathscr{P}(\Omega)$. Adopting this kind of setting for the theory, I show in the present paper that much of it can be established under much weaker hypotheses: in the revised theory the pair $(\mathscr{P}(\Omega), \mathscr{G})$ is replaced by $(X, \mathscr{E})$, where X is a compact convex set and $\mathscr{E}$ is a wedge of affine real continuous functions on X that contains the constant functions, separates the extreme points of X, and is such that the family $\{g \in \mathscr{E}\colon g < f\}$ is upward filtering whenever f is affine real and continuous on X. What makes this modification possible is the use of the generalized Weierstrass-Stone theorem of [15, 16] to replace the ordinary Weierstrass-Stone theorem.

This more general formulation of Choquet theory has some advantages: (i) it is in some ways easier to work with — it has in fact led to new results for the classical case (see e.g. Theorem 14); (ii) decomposition of maximal elements into extreme maximal elements can be given a simple treatment in this setting (see §§ 7, 8; for treatments of a special case see [23, 1]); (iii) the basic hypotheses survive restriction to suitable subsets of X (for particulars, and an application, see § 8); (iv) the theory applies directly to the natural ordering on caps (§ 9).

Partially ordered compact convex sets have been studied by Lumer [17], Rogalski [22], and Alfsen and Skau [1]. What distinguishes the present paper from these works, where there is overlap, is the systematic use here of the filtering condition on $\mathscr{E}$.

I have not attempted to include Choquet's theory of conical measures in the present work.

2. A maximum theorem. Throughout this paper X will denote a non-empty compact convex subset of a locally convex Hausdorff topological vector space V over the real field, and $\mathscr{E}$ will denote a non-empty family of upper semicontinuous affine maps of X into $[-\infty, \infty)$. We associate with $\mathscr{E}$ the *quasi-ordering* $\succsim$ of X defined by writing $y \succsim x$ (or $x \precsim y$) whenever $g(y) \geqslant g(x)$ for all $g \in \mathscr{E}$. For each $x \in X$ we define

$$R_x \equiv R_x(\mathscr{E}) = \{y \in X:\ y \succsim x\},$$

and

$$[x] \equiv [x]_{\mathscr{E}} = \{y \in X:\ y \succsim x \text{ and } x \succsim y\}.$$

An element x of X is called *maximal* for the above quasi-ordering if $R_x = [x]$. The set of all maximal elements of X will be denoted by $Z_{\mathscr{E}}(X)$ or simply by Z. Following Lumer [17] we define the *$\mathscr{E}$-boundary* of X to be the set $\partial_{\mathscr{E}} X = X_e \cap Z$, where, for any convex set $K \subseteq V$, K_e denotes the set of extreme points of K.

THEOREM 1. *Each function in $\mathscr{E}$ attains its X-maximum on $\partial_{\mathscr{E}} X$.*

This is essentially Lumer's [17] extension of Bauer's [4] maximum theorem, and the proof is similar [17, 22]. Since Lumer's proof was barely indicated in [17], and since Rogalski [22] treats a special case, it seems desirable to prove Theorem 1 here.

By an *$\mathscr{E}$-stable* subset of X we shall mean any subset Y such that $R_x \subseteq Y$ for all $x \in Y$. By an *$\mathscr{E}$-face* we shall mean any non-empty closed face of X that is $\mathscr{E}$-stable. By Zorn's lemma every $\mathscr{E}$-face of X covers an $\mathscr{E}$-face that is minimal for the partial ordering of set inclusion. If A is an $\mathscr{E}$-face and $g \in \mathscr{E}$, $\beta = \max\{g(x):\ x \in A\}$, then

$$B = \{x \in A:\ g(x) = \beta\}$$

is an $\mathscr{E}$-face of X. Consequently, if A is in fact minimal, then $B = A$; it follows from this that, when A is a minimal $\mathscr{E}$-face, $A = [x]$ for all $x \in A$. Since each closed face $K \neq \emptyset$ of X meets X_e we deduce that the minimal $\mathscr{E}$-faces are precisely the sets $[x]$ with $x \in \partial_{\mathscr{E}} X$.

Now let f be any element of $\mathscr{E}$, let $a = \max\{f(x):\ x \in X\}$, and consider

$$F = \{x \in X:\ f(x) = a\}.$$

This is an $\mathscr{E}$-face, and so it covers some minimal $\mathscr{E}$-face, and therefore meets $\partial_{\mathscr{E}} X$.

COROLLARY 2. $\partial_{\mathscr{E}} X \neq \emptyset, Z_{\mathscr{E}}(X) \neq \emptyset$.

A quite different method for proving Corollary 2, based on a sharpening of the Hahn-Banach theorem, has been discovered recently by Vincent-Smith [23] and Andenaes [2].

COROLLARY 3. *If Y is a non-empty compact convex $\mathscr{E}$-stable subset of X, then $Z_{\mathscr{E}}(X) \cap Y_e \neq \emptyset$. In particular, $Z_{\mathscr{E}}(X) \cap (R_x)_e \neq \emptyset$ for all $x \in X$.*

The first part of Corollary 3 merely says that $\partial_{\mathscr{E}_1} Y \neq \emptyset$, where $\mathscr{E}_1$ is the set of restrictions $\{f | Y:\ f \in \mathscr{E}\}$. The second part, viz. the special case $Y = R_x$, is interesting in that it shows that, among the maximal elements of X that majorize a given element x, extreme points exist; we shall return to this fact in § 8.

One of Bauer's theorems is a special case of Theorem 1:

COROLLARY 4. *Every upper semicontinuous affine map $f\colon X \to [-\infty, \infty)$ attains its X-maximum on X_e.*

For proof, one takes $\mathscr{E}$ in Theorem 1 to be the set of all such upper semicontinuous affine maps.

3. A class of partial orderings for X. We consider here some circumstances in which the quasi-ordering of § 2 is a partial ordering. Until further notice $\mathscr{E}$ will be a wedge, that contains the constant functions, in the space $\mathscr{A}(X)$ of all real continuous affine functions on X. We shall say that $\mathscr{E}$ satisfies the *filtering condition* if for each $f \in \mathscr{A}(X)$ the family $\{g \in \mathscr{E}:\ g < f\}$ is upward filtering.

When $\mathscr{E}$ satisfies the filtering condition so does $\mathscr{E} - \mathscr{E}$. To see this suppose that $u, v, u_1, v_1 \in \mathscr{E}, f \in \mathscr{A}(X)$ and

$$(u-v) \vee (u_1-v_1) < f.$$

Then

$$(u+v_1) \vee (u_1+v) < f+v+v_1,$$

and so there exists a $w \in \mathscr{E}$ such that

$$(u+v_1) \vee (u_1+v) < w < f+v+v_1,$$

whence

$$(u-v) \vee (u_1-v_1) < w-(v+v_1) < f.$$

Since we know [15, 16] that a linear subspace $\mathscr{L}$ of $\mathscr{A}(X)$ that contains the constant functions is dense in $\mathscr{A}(X)$ if and only if (a) $\{l \in \mathscr{L}:\ l < f\}$ is upward filtering for each $f \in \mathscr{A}(X)$, and (b) $\mathscr{L}$ separates the points of X_e, we are led to the following result:

THEOREM 5. *If $\mathscr{E}$ satisfies the filtering condition, then the following assertions are equivalent*:

(i) *$\mathscr{E}$ separates the points of X_e*;

(ii) *$\mathscr{E}$ separates the points of X*;

(iii) *$\mathscr{E}-\mathscr{E}$ is dense in $\mathscr{A}(X)$*;

(iv) *the quasi-ordering on X induced by $\mathscr{E}$ is a partial ordering.*

The implication (i) $\Rightarrow$ (iii) has just been discussed. The implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are trivial. Finally, (iii) $\Rightarrow$ (ii) requires only the (Hahn-Banach) fact that $\mathscr{A}(X)$ separates the points of X.

Until further notice we shall suppose that $\mathscr{E}$ satisfies all the conditions of Theorem 5. We shall study the associated partial order on X, and various classes of monotone functions on X.

4. Affine decreasing functions. We characterize here, among the continuous or the semicontinuous affine extended-real-valued functions on X, those that are decreasing Such characterizations generalize similar theorems about the $\mathscr{G}$-convex functions (see § 1) of ordinary Choquet theory [5, 7, 9, 13, 19, 22].

For each upper bounded map $f\colon X\to[-\infty,\infty)$ and each $x\in X$ we define

$$\hat{f}(x) = \inf\{g(x)\colon\ g\in-\mathscr{E},\ g>f\},$$

so that $\hat{f}$ is an upper bounded upper semicontinuous decreasing function, concave in general but affine if $f\in\mathscr{A}(X)$. If $f\in-\mathscr{E}$, then $f=f$. When $x\in X$ and $f\in\mathscr{A}(X)$, we shall write $\hat{x}(f)=\hat{f}(x)$. It is easy to see that $\hat{x}$ is a real-valued sublinear functional on $\mathscr{A}(X)$.

By a positive functional on $\mathscr{A}(X)$ we mean a functional Φ such that $\Phi(f)\geqslant 0$ whenever $f\geqslant 0$. A positive linear functional Φ on $\mathscr{A}(X)$ such that $\Phi(1)=1$ is called a *state* of $\mathscr{A}(X)$. If $x\in X$, then the functional ε_x defined on $\mathscr{A}(X)$ by $\varepsilon_x(f)=f(x)$ is a state, and the map $x\to\varepsilon_x$ is well known to be a bijection of X onto the set of all states of $\mathscr{A}(X)$. If Φ,Ψ are two functionals on $\mathscr{A}(X)$, we shall write $\Phi\leqslant\Psi$ to mean that $\Phi(f)\leqslant\Psi(f)$ for all $f\in\mathscr{A}(X)$. For instance, $\varepsilon_x\leqslant\hat{x}$ for all $x\in X$.

PROPOSITION 6. *Let $x\in X$ and let Φ be a linear functional on $\mathscr{A}(X)$. Then $\Phi\leqslant\hat{x}$ if and only if $\Phi=\varepsilon_y$ for some y in X such that $y\succeq x$.*

Suppose that $y\succeq x$. Then for all $g\in-\mathscr{E}$ we have $g(x)\geqslant g(y)$. Hence $\hat{x}\geqslant\hat{y}\geqslant\varepsilon_y$.

Conversely, let Φ be linear and such that $\Phi\leqslant\hat{x}$. By considering the action of Φ on $\{h\in\mathscr{A}(X)\colon h\leqslant 0\}$ and on the constant functions ± 1 one sees that Φ is a state ε_y of $\mathscr{A}(X)$. Whenever $g\in-\mathscr{E}$, we have $g(y)=\Phi(g)\leqslant\hat{x}(g)=g(x)$, which shows that $y\succeq x$.

COROLLARY 7. *If $f\in\mathscr{A}(X)$, then, for each $x\in X$,*

$$\hat{f}(x)=\max\{f(y)\colon\ y\succeq x\}. \tag{1}$$

By the Hahn-Banach theorem we can find a linear form Φ on $\mathscr{A}(X)$ such that $\Phi\leqslant\hat{x}$, $\Phi(f)=\hat{x}(f)$. By proposition 6, $\Phi=\varepsilon_y$ for some $y\succeq x$, so that $f(y)=\Phi(f)=\hat{x}(f)=\hat{f}(x)$, and hence

$$\hat{f}(x)\leqslant\max\{f(y)\colon\ y\succeq x\}.$$

Since the converse inequality is obvious, the proof is complete.

We shall denote by $\mathscr{U}(X)$ the space of all upper semicontinuous affine maps of X into $[-\infty,\infty)$. Corollary 7 can be extended to $\mathscr{U}(X)$ as follows:

PROPOSITION 8. *If $f\in\mathscr{U}(X)$, then $\{g\in-\mathscr{E}\colon g>f\}$ is a downward filtering family, $\hat{f}$ is affine decreasing and formula* (1) *remains true for all $x\in X$.*

By a theorem of Mokobodzki [19] the set $\{h\in\mathscr{A}(X)\colon h>f\}$ is downward filtering. Given $g_1,g_2\in-\mathscr{E}$ with $g_1\wedge g_2>f$ we can therefore choose $h\in\mathscr{A}(X)$ so that $g_1\wedge g_2>h>f$ and hence $g\in-\mathscr{E}$, so that $g_1\wedge g_2>g>h$.

It remains only to prove the formula for $\hat{f}(x)$. For this we use a variant of the Dini-Cartan theorem. Let Ω be a compact Hausdorff space and let $\mathscr{S}$ be the set of all upper semicontinuous maps of Ω into $[-\infty,\infty)$.

LEMMA 9. *Suppose that $u\in\mathscr{S}$ is the infimum of a downward filtering family $\mathscr{H}\subseteq\mathscr{S}$ and that F is a non-empty closed subset of Ω. Then*

$$\inf_{h\in\mathscr{H}}\max_{\omega\in F}h(\omega)=\max_{\omega\in F}u(\omega).$$

The proof is an obvious modification of that for Dini's theorem.

To apply Lemma 9 note first that whenever $g\in-\mathscr{E}$, $h\in\mathscr{A}(X)$, and $g>h>f$, we have

$$g(x)=\hat{g}(x)\geqslant\hat{h}(x)\geqslant\hat{f}(x).$$

This, with Mokobodzki's theorem, shows that

$$\hat{f}(x)=\inf\{\hat{h}(x)\colon\ h\in\mathscr{H}\},$$

where $\mathscr{H}=\{h\in\mathscr{A}(X)\colon h>f\}$. Taking $\Omega=X$, $u=f$, $F=R_x$, and $\mathscr{H}$ as just defined we have, by the lemma,

$$\begin{aligned}\hat{f}(x)&=\inf\{\hat{h}(x)\colon\ h\in\mathscr{H}\}\\&=\inf_{h\in\mathscr{H}}\max\{h(y)\colon\ y\succeq x\}\\&=\max\{f(y)\colon\ y\succeq x\},\end{aligned}$$

as desired.

PROPOSITION 10. *For each $f \in \mathscr{U}(X)$ the following assertions are equivalent:*

(i) $f = \hat{f}$;

(ii) f *is a decreasing function*;

(iii) *if* $x \in X_e$ *and* $y \succ x$, *then* $f(y) \leqslant f(x)$.

If, in fact, $f \in \mathscr{A}(X)$, *then these statements are equivalent to*

(iv) $f \in -\overline{\mathscr{E}}$.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. If f satisfies (iii), then by proposition 8 we have $\hat{f}(x) = f(x)$ for all $x \in X_e$. Since f and $\hat{f}$ are both in $\mathscr{U}(X)$, it now follows from Corollary 4 that, if $h \in \mathscr{A}(X)$, then $h > f$ if and only if $h > \hat{f}$. By Mokobodzki's theorem this in turn implies that $f = \hat{f}$, and so f satisfies (i).

Condition (iv) obviously implies (ii). On the other hand, if f in $\mathscr{A}(X)$ satisfies (i), then, by Dini's theorem and the filtering property, f satisfies (iv).

PROPOSITION 11. *If* $f\colon X \to (-\infty, \infty]$ *is a lower semicontinuous decreasing affine function, then the family*

$$\{g \in -\mathscr{E}\colon g < f\} \tag{2}$$

is upward filtering, with pointwise limit f.

Suppose that $h \in \mathscr{A}(X)$ and $h < f$, so that $h+\varepsilon < f$ for some $\varepsilon > 0$. Then for all $y \succ x$ we have

$$h(y)+\varepsilon \leqslant f(y) \leqslant f(x),$$

whence, by Corollary 7, $\hat{h}(x)+\varepsilon \leqslant f(x)$. Since $\{g \in -\mathscr{E}\colon g > \hat{h}\}$ is downward filtering to the limit $h < f$, we can find $g \in -\mathscr{E}$ such that $\hat{h} \leqslant h < g < f$.

Since $\{h \in \mathscr{A}(X)\colon h < f\}$ is, by Mokobodzki's theorem, upward filtering to the limit f, it follows that so is the family (2).

5. The maximal elements of X. Choquet and Meyer's characterizations of maximal measures [9] adapt easily to the present situation:

THEOREM 12. *For each $x \in X$ the following assertions are equivalent:*

(i) $x \in Z$;

(ii) $\hat{x}$ *is a linear functional on* $\mathscr{A}(X)$;

(iii) $\hat{x} = \varepsilon_x$ *on* $\mathscr{A}(X)$;

(iv) $\hat{x} = \varepsilon_x$ *on* $\mathscr{E}$.

Suppose that (i) is true. By Proposition 6 and the Hahn-Banach theorem we deduce (iii) and hence (iv).

Now suppose that (iv) is true. Then $\hat{x}$ and ε_x agree on $\mathscr{E} \cup (-\mathscr{E})$. Therefore, if $u, v \in \mathscr{E}$,

$$x(u)+\hat{x}(-v) = u(x)-v(x) = \varepsilon_x(u-v)$$
$$\leqslant \hat{x}(u-v) \leqslant \hat{x}(u)+\hat{x}(-v),$$

and hence $\hat{x}(u-v) = u(x)-v(x)$. It easily follows that $\hat{x}$ is linear on $\mathscr{E}-\mathscr{E}$. Next, if $f \in \mathscr{A}(X)$ and $\varepsilon > 0$ we can choose $h \in \mathscr{E}-\mathscr{E}$ such that $h-\varepsilon \leqslant f \leqslant h+\varepsilon$. Hence we have

$$\hat{x}(f) \leqslant \hat{x}(h+\varepsilon) \leqslant \hat{x}(h)+\varepsilon\hat{x}(1)$$
$$= h(x)+\varepsilon \leqslant f(x)+2\varepsilon.$$

Therefore on $\mathscr{A}(X)$ we have $\hat{x} \leqslant \varepsilon_x \leqslant \hat{x}$. That is, we have shown that (iv) implies (iii).

Obviously (iii) implies (ii). To see that (ii) implies (i) note that (ii) rivially implies that ε_x is the unique linear form on $\mathscr{A}(X)$ that is matjorized by $\hat{x}$. By Proposition 6 this means that $x \in Z$. This completes the proof.

If $f \in \mathscr{A}(X)$, then $\hat{f}-f$ is affine, upper semicontinuous, and nonnegative. It follows that

$$B_f = \{x \in X\colon \hat{f}(x) = f(x)\}$$

is a G_δ set, and a face of X. By Theorem 12 we now have

PROPOSITION 13. $Z = \bigcap \{B_f\colon f \in \mathscr{A}(X)\}$ *and, consequently,* Z *is a face of* X. *If* $\mathscr{A}(X)$ *contains a strictly increasing function* h, *then* $Z = B_h$.

Concerning the last part of Proposition 13 note that if X is metrizable, then $\mathscr{E}$ contains a countable dense set $\{g_n\}$ of non-zero elements, and hence $\mathscr{A}(X)$ contains a strictly increasing function, e.g. $\sum_{n=1}^{\infty} g_n(2^n\|g_n\|)^{-1}$.

We know (see [10], appendix B14, and [14]) that the Choquet boundary is a Baire space for the relative topology. It does not seem to have been observed before that the same thing is true of the set of maximal elements:

THEOREM 14. *For the relative topology from X the set Z of all maximal elements is a Baire space.*

The proof is a mild complication of the argument of [14]. We shall make use of $\mathscr{K}$, the space of all functions of the form $f_1 \vee f_2 \vee \ldots \vee f_n$, where n is an arbitrary natural number and the f_r are in $\mathscr{A}(X)$. The space $\mathscr{W}$ will denote the set of all functions formed in the same way from elements f_r of $-\mathscr{E}$. For each $x \in X$, $\mathscr{D}(x)$ will be the set $\{f \in \mathscr{W}\colon f(x) < 0\}$. Finally, for any given $f\colon X \to [-\infty, \infty]$ we write

$$U_f = \{y \in X\colon f(y) < 0\}, \quad F_f = \{y \in X\colon f(y) \leqslant 0\}.$$

LEMMA 15. *Let G be an open subset of X, let $x \in G \cap Z$, let $f \in \mathscr{K}$ with $f(x) < 0$. Then there is a function $g \in \mathscr{D}(x)$ such that $g > f$ and $F_g \subseteq G$.*

Let $f = f_1 \vee f_2 \vee \ldots \vee f_n$, where the f_r are in $\mathscr{A}(X)$. Since $x \in G$, we can choose $p \geqslant 1$ and $f_{n+1}, \ldots, f_{n+p}$ in $\mathscr{A}(X)$ so that $x \in U_u \subseteq G$, where $u = f_{n+1} \vee \ldots \vee f_{n+p}$. Writing $v = f \vee u$ we have, for $r = 1, 2, \ldots, n+p$,

$$\hat{f}_r(x) = f_r(x) \leqslant v(x) < 0.$$

For each r we can therefore choose $g_r \in -\mathscr{E}$ so that $g_r > f_r$ and $g_r(x) < 0$. Taking $g = g_1 \vee g_2 \vee \ldots \vee g_{n+p}$, we have $g \in \mathscr{W}$, $g > v$ and $g(x) < 0$. Evidently, $g > f$; and also $g > u$, which implies that $F_g \subseteq G$.

For the proof of Theorem 13, consider a sequence $\{V_n : n \geqslant 1\}$ of relatively open dense subsets of Z. Let $V_0 \neq \emptyset$ be open in Z. We shall show that $\bigcap_{n=0}^{\infty} V_n \neq \emptyset$.

By Lemma 15 we can suppose that $V_0 \supseteq F_{f_0} \cap Z$ for some $f_0 \in \mathscr{D} \equiv \bigcup \{\mathscr{D}(x) : x \in Z\}$. For each $n \geqslant 1$ there is an open subset G_n of X such that $V_n = G_n \cap Z$. We shall choose a sequence $\{f_n : n \geqslant 1\}$ in $\mathscr{D}$ so that, for $n = 1, 2, \ldots$,

$$f_{n-1} < f_n \quad \text{and} \quad F_{f_n} \subseteq G_n.$$

Suppose that $f_0, f_1, \ldots, f_n$ have been chosen, where $n \geqslant 0$. Evidently, $U_{f_n} \cap Z$ is non-empty and open in Z. It therefore meets V_{n+1} in some point y. Thus $f_n \in \mathscr{D}(y)$ for some $y \in G_{n+1} \cap Z$. By Lemma 15 we can choose $f_{n+1} \in \mathscr{D}(y)$ so that $f_{n+1} > f_n$ and $F_{f_{n+1}} \subseteq G_{n+1}$. A sequence of the required type therefore exists.

Now let $f = \lim_n f_n$. Then f is a lower semicontinuous, decreasing (convex) function. It therefore attains its X-minimum at some point z of Z. Since the F_{f_n} have the finite intersection property and are closed,

$$F_f = \bigcap_{n=0}^{\infty} F_{f_n} \neq \emptyset.$$

Thus

$$z \in F_f \cap Z = \bigcap_{n=0}^{\infty} (F_{f_n} \cap Z),$$

and a fortiori $\bigcap_{n=0}^{\infty} V_n \neq \emptyset$.

6. Uniqueness. By Corollary 3 every element x of X is majorized by a maximal element, i.e. R_x always meets Z. Adapting a theorem of [9], we arrive at

THEOREM 16. *For each $x \in X$ the following statements are equivalent:*
(i) R_x meets Z in just one point;
(ii) for each $f \in \mathscr{E}$, $\hat{f}$ is constant on R_x;
(iii) if $f \in \mathscr{E}$ and $x \prec z \in Z$, then $\hat{f}(x) = f(z)$;
(iv) $\hat{x}$ is additive on $\mathscr{E}$.

Suppose that (i) is true, that $f \in \mathscr{E}$, and that $x \prec z \in Z$. Then by Corollary 6 we have $f(x) = f(z)$. If now $x \prec y$, then $R_y \subseteq R_x$ and so $R_y \cap Z = \{z\}$. By the same reasoning $\hat{f}(y) = f(z)$, and (ii) is clear.

Next if (ii) is true and $f \in \mathscr{E}$, and if $x \prec z \in Z$, then, by Theorem 12, $f(z) = \hat{f}(z) = \hat{f}(x)$, i.e. (iii) follows from (ii).

That (iii) implies (iv) is obvious. To show that (iv) implies (i), assume (iv) and consider $u, v \in \mathscr{E}$. Writing $\Phi(u-v) = \hat{u}(x) - \hat{v}(x)$ we obtain a well-defined linear form Φ on $\mathscr{E} - \mathscr{E}$. It is obviously positive and $\Phi(1) = 1$. By the extension theorem for positive linear forms, Φ extends uniquely to a positive linear form on $\mathscr{A}(X)$. The extension is a state ε_z of $\mathscr{A}(X)$ and we have, whenever $u \in \mathscr{E}$ and $y \succ x$,

$$u(z) = \Phi(z) = \hat{u}(x) \geqslant \hat{u}(y) \geqslant u(y),$$

which shows that $z \succ y$. Thus z is the final element in R_x, and so we have deduced (i) from (iv).

When condition (i) holds for every $x \in X$, we shall call the pair $(X, \mathscr{E})$ *simplicial*. By an *atonic function* on X we shall mean a function f on X such that $f(x) = f(y)$ whenever $x, y \in X$ and $x \prec y$. The set of all atonic functions in $\mathscr{A}(X)$ will be denoted by $\mathscr{B}(X)$. By Proposition 10 we have $\mathscr{B}(X) = \bar{\mathscr{E}} \cap (-\bar{\mathscr{E}})$. In addition to the characterizations of simplicial pairs given by Theorem 16 there is another, which is a weak analogue of the separation theorem in [12], though the following formulation and proof is closer to [7].

PROPOSITION 17. *The following statements are equivalent:*
(i) $(X, \mathscr{E})$ is a simplicial pair;
(ii) if $f, -g \in \mathscr{E}$, $f < g$, $x \in X$, and $\varepsilon > 0$, then there exist $u, -v \in \mathscr{E}$ such that $f < u < v < g$ and $v(x) - u(x) < \varepsilon$;
(iii) if $f, -g \in \mathscr{E}$, $f < g$, and $\varepsilon > 0$, then there exist $u, -v \in \mathscr{E}$ such that $f < u < v < g$ and $v - u < \varepsilon$;
(iv) if $f, -g \in \mathscr{E}$ and $f < g$, then there is an $h \in \mathscr{B}(X)$ such that $f < h < g$.

If f, g, ε, x are as in (ii), then we can find $v \in -\mathscr{E}$ such that $f < v < g$ and $v(x) < \hat{f}(x) + \varepsilon$. Given that (i) is true, $\hat{f}$ is atonic, by Theorem 16, and hence increasing, and $f < v$. By Proposition 11 we can find $u \in \mathscr{E}$ such that $\hat{f} < u < v$. Then $f < u < v < g$ and $v(x) - u(x) < \varepsilon$. Thus (i) implies (ii).

To show that (ii) implies (iii), consider the set $\mathcal{O}$ of all functions $v-u$, where $u, -v \in \mathcal{E}$ and $f < u < v < g$. Evidently $\mathcal{O}$ is convex. If $\mathcal{O}$ is disjoint from the open convex set $\{h \in \mathcal{A}(X): h < \varepsilon\}$, then by the Hahn-Banach theorem there is a state ε_x of $\mathcal{A}(X)$ such that $w(x) \geqslant \varepsilon$ for all $w \in \mathcal{O}$, contradicting (ii). Thus (ii) implies (iii).

Next, if (iii) is true then we can choose two sequences $\{u_n\}$, $\{-v_n\}$ in $\mathcal{E}$ such that

$$f < u_n < u_{n+1} < v_{n+1} < v_n < g, \qquad v_n - u_n < \frac{1}{n},$$

for all n. Then u_n and v_n tend uniformly to a common limit $h \in \mathcal{B}(X)$, and $f < h < g$, so (iv) is clear.

Finally, to prove that (iv) implies (i) note that if $f \in \mathcal{E}$ and $h \in \mathcal{B}(X)$ with $h > f$, then $\hat{f}(x) \leqslant \hat{h}(x) = h(x)$. Hence, using (iv), we have, when $y \succ x$,

$$\begin{aligned}\hat{f}(x) \geqslant \hat{f}(y) &= \inf\{g(y): g \in -\mathcal{E}, g > f\}\\ &\geqslant \inf\{h(y): h \in \mathcal{B}(X), h > f\}\\ &= \inf\{h(x): h \in \mathcal{B}(X), h > f\} \geqslant \hat{f}(x).\end{aligned}$$

Thus all terms here are equal and so $\hat{f}$ is atonic for all $f \in \mathcal{E}$, and (i) now follows by Theorem 16.

PROPOSITION 18. *The pair $(X, \mathcal{E})$ is simplicial if and only if the functions of $\mathcal{B}(X)$ separate the points of Z.*

This was suggested by Corollary 3.5 of [7].

Suppose that $(X, \mathcal{E})$ is simplicial and that $x, y \in Z$ with $h(x) = h(y)$ for all $h \in \mathcal{B}(X)$. If $f \in \mathcal{E}$, then by part (iv) of Proposition 17 we have

$$\begin{aligned}f(x) = \hat{f}(x) &= \inf\{h(x): h \in \mathcal{B}(X), h > f\}\\ &= \inf\{h(y): h \in \mathcal{B}(X), h > f\} = \hat{f}(y) = f(y).\end{aligned}$$

Thus $f(x) = f(y)$ for all $f \in \mathcal{E}$, and by Theorem 4 we have $x = y$.

If, conversely, $\mathcal{B}(X)$ separates the points of Z, then not two elements of Z can belong to the same set R_x for $x \in X$. That is, for each $x \in X$, R_x meets Z in just one point.

It does not seem possible to sharpen Proposition 17 to make it look like the main theorem of [12] without further hypotheses. A similar remark applies to Proposition 11.

7. Relationship to standard Choquet boundary theory. We enlarge here on some of the comments of § 1.

Let Ω and $\mathcal{P}(\Omega)$ be as in § 1, and let $\mathcal{G}$ be a wedge of continuous real-valued functions on Ω that contains the constant functions, separates

the points of Ω and is such that $f \vee g \in \mathcal{G}$ whenever $f, g \in \mathcal{G}$. If $\mathcal{S}$ is the space of upper semicontinuous maps $f: \Omega \to [-\infty, \infty)$, then there is a natural order-preserving bijection $\alpha: \mathcal{S} \to \mathcal{U}(X)$ defined by $(\alpha f)(\mu) = \int f d\mu$, which maps $\mathcal{C}(\Omega)$ linearly onto $\mathcal{A}(X)$. It is easy to show that, on taking $\mathcal{E} = \alpha\mathcal{G}$, $X = \mathcal{P}(\Omega)$, all the conditions of Theorem 5 are met, so that the preceding theory applies. The ordering $\gg$ defined in X by writing $\mu \gg \nu$ if $\int g d\mu \geqslant \int g d\nu$ for all $g \in \mathcal{G}$ coincides with the $\mathcal{E}$-ordering $\succ$ used above. The Choquet boundary $\mathrm{Ch}_{\mathcal{G}}\Omega$ of Ω relative to $\mathcal{G}$ is, by definition,

$$\{\omega \in \Omega: \mu \succ \varepsilon_\omega \text{ implies that } \mu = \varepsilon_\omega\}.$$

Since we know that $X_e = \{\varepsilon_\omega: \omega \in \Omega\}$, it follows that

$$\partial_{\mathcal{E}} X = \{\varepsilon_\omega: \omega \in \mathrm{Ch}_{\mathcal{G}}\Omega\}.$$

One can easily show moreover that if $\hat{f}$ is defined for $f \in \mathcal{S}$ by

$$\hat{f}(\omega) = \inf\{g(\omega): g \in -\mathcal{G}, g > f\} \qquad (\omega \in \Omega),$$

then $\alpha(\hat{f}) = \widehat{\alpha f}$ for all $f \in \mathcal{S}$. Similarly, α maps $\mathcal{G}$-convex and $\mathcal{G}$-concave functions in $\mathcal{S}$ onto increasing and decreasing functions, in $\mathcal{U}(X)$, respectively. These considerations allow one to regard much of Choquet boundary theory as formulated in [6, 7, 8, 11, 13, 18, 21] as a special case of the preceding theory.

We can also, however, relate the work of §§ 4-6 to standard Choquet theory in a different way by means of a construction of Alfsen and Skau [1], based on a special case treated by Vincent-Smith [23]. For this one takes $\mathcal{I}$ to be the set of all functions on X of the form $f_1 \vee f_2 \vee \ldots \vee f_n$, where $n \geqslant 1$ is a natural number and the f_r are in $\mathcal{E}$. Taking $\Omega = X$ and $\mathcal{G} = \mathcal{I}$, we find that the basic hypotheses for Choquet boundary theory as just described are met. The $\mathcal{I}$-ordering $\gg$ of measures X is known from that theory to be a partial ordering.

PROPOSITION 19. *If $\mu \in \mathcal{P}(X)$ and $x \in X$, then $\mu \gg \varepsilon_x$ if and only if $c_\mu \succ x$. If μ is an $\mathcal{I}$-maximal measure of $\mathcal{P}(X)$, then $c_\mu \in Z_{\mathcal{E}}(X)$.*

If $\mu \gg \varepsilon_x$, then $c_\mu \succ x$ follows immediately from the fact that $\mathcal{E} \subseteq \mathcal{I}$ (as remarked in [1]). If, conversely, $c_\mu \succ x$, then when $f_1, f_2, \ldots, f_n \in \mathcal{E}$ we have

$$\mu(\max_r f_r) \geqslant \max_r \mu(f_r) = \max_r f_r(c_\mu) \geqslant \max_r f_r(x),$$

so that $\mu \gg \varepsilon_x$.

If μ is $\mathcal{I}$-maximal, then, by standard Choquet boundary theory, $\mu(\hat{f}) = \mu(f)$ for all $f \in \mathcal{A}(X)$. In other words, $\hat{f}(c_\mu) = f(c_\mu)$ for all such f, so that, by Theorem 12, $c_\mu \in Z_{\mathcal{E}}(X)$.

PROPOSITION 20. (Alfsen and Skau). $\partial_{\mathcal{E}} X = \mathrm{Ch}_{\mathcal{I}} X$.

This result is based on a special case considered by Vincent-Smith [23]. In fact, Alfsen and Skau prove this result without the filtering condition on $\mathscr{E}$. In the present context a very simple proof is possible.

Suppose that $x \in \partial_{\mathscr{E}} X$ and $\mu \gg \varepsilon_x$. Then $c_\mu \succ x \in Z$, hence $c_\mu = x$ $(\in X_e)$, and hence $\mu = \varepsilon_x$. Therefore $x \in \mathrm{Ch}_{\mathscr{I}} X$.

Conversely, if $x \in \mathrm{Ch}_{\mathscr{I}} X$ and $y \succ x$, then $\varepsilon_y \gg \varepsilon_x$ and hence $y = x$, which shows that $x \in Z$. If $c_\mu = x$, then $\mu \gg \varepsilon_x$ and hence $\mu = \varepsilon_x$, which shows that $x \in X_e$. Thus $\mathrm{Ch}_{\mathscr{I}} X \equiv \partial_{\mathscr{E}} X$, and the proof is complete.

COROLLARY 21. *$\partial_{\mathscr{E}} X$ is a Baire space.*

This follows from Proposition 20 by the theorem of [14].

It is possible to develop these considerations so as to deduce the results of §§ 4-6 from the standard Choquet theory associated with $\mathscr{I}$. The proofs given in §§ 4-6 are, however, much more direct.

We have seen that the centroid of every $\mathscr{I}$-maximal measure lies in Z. Proposition 20 allows us, by standard Choquet theory, to state the converse: every point of Z is the centroid of an $\mathscr{I}$-maximal measure. By Proposition 20 such measures are carried, in the appropriate sense, by $\partial_{\mathscr{E}} X$. In fact, we can write, without inconsistency,

$$\hat{f}(x) = \inf\{g(x):\ g \in -\mathscr{I},\ g > f\},$$

whenever $f \in \mathscr{C}(X)$ and $x \in X$. Defining B_f, as before, as $\{x:\ \hat{f}(x) = f(x)\}$, we have, by Proposition 20 and standard Choquet theory,

$$\partial_{\mathscr{E}} X = \bigcap \{B_f:\ f \in \mathscr{C}(X)\}.$$

A measure $\mu \in \mathscr{P}(X)$ is $\mathscr{I}$-maximal if and only if $\mu(B_f) = 0$ for all $f \in \mathscr{C}(X)$. We thus have

PROPOSITION 22. *For each $z \in Z$ there exists a measure μ in $\mathscr{P}(X)$ with barycentre z, that is carried by $\partial_{\mathscr{E}} X$ in the sense that $\mu(B_f) = 0$ for all $f \in \mathscr{C}(X)$.*

In the circumstances of Proposition 22 we shall say that μ is a boundary measure representing z.

We shall say that a semicontinuous function on X is *affine* on Z if $\mu(f) = f(x)$ whenever $x \in Z$ and $\mu \in \mathscr{P}(X)$ with $c_\mu = x$. By a straightforward adaptation of the proof of Theorem 16 we arrive at the following uniqueness theorem:

PROPOSITION 23. *The following statements are equivalent*:

(i) *for each $x \in Z$ there is a unique boundary measure representing x;*

(ii) *for each $f \in \mathscr{I}$ the function $\hat{f}$ is affine on Z;*

(iii) *if $x \in Z$ and μ is a boundary measure representing x, then $\mu(f) = \hat{f}(x)$ for all $f \in \mathscr{I}$;*

(iv) *the map $f \to \hat{f}(x)$ is additive on $\mathscr{I}$ for each $x \in Z$.*

8. Stable subsets. Consider a non-empty compact convex $\mathscr{E}$-stable subset Y of X and let $\mathscr{E}_1$ denote the set of restrictions $\{f|Y:\ f \in \mathscr{E}\}$. We shall show that the pair $(Y, \mathscr{E}_1)$ meets the conditions of § 3, so that the preceding theory applies to it. The only non-trivial agendum here is the proof of the filtering condition.

PROPOSITION 24. *For each $f \in \mathscr{A}(Y)$ the family $\{g \in -\mathscr{E}_1:\ g > f\}$ is downward filtering.*

By the Hahn-Banach theorem the set of restrictions $\{h|Y:\ h \in \mathscr{A}(X)\}$ is dense in $\mathscr{A}(Y)$. It will therefore be enough to prove the filtering property for f of the form $f = h|Y$ with $h \in \mathscr{A}(X)$. Let $g_1, g_2 \in -\mathscr{E}_1$ be such that $g_1 \wedge g_2 > f$. Then for some $\varepsilon > 0$ we have $g_1 \wedge g_2 \geqslant f + \varepsilon$. For each $x \in Y$ we can find $y \succ x$ such that $\hat{h}(x) = h(y)$, so that, for $r = 1, 2$,

$$\hat{h}(x) = h(y) \leqslant g_r(y) - \varepsilon \leqslant g_r(x) - \varepsilon.$$

This shows that

$$\hat{h}\,|\,Y < g_1 \wedge g_2.$$

By the filtering condition on $\mathscr{E}$ and a standard compactness argument we can therefore find $g \in -\mathscr{E}$ such that $g > h$ and $g|Y < g_1 \wedge g_2$. This completes the proof.

The theory for the pair $(X, \mathscr{E})$ can be related to that for $(Y, \mathscr{E}_1)$ in various ways. It is obvious for instance that

$$Z_{\mathscr{E}_1}(Y) = Y \cap Z_{\mathscr{E}}(X), \qquad \partial_{\mathscr{E}_1} Y = Y_e \cap Z_{\mathscr{E}}(X).$$

A special case is of interest. If $x \in X$, we may take $Y = R_x$. In this way we obtain existence of extreme points of $R_x \cap Z$, the theory of §§ 4-6 for R_x, and the representation (Proposition 22) of each maximal element of Z that majorizes x as a weighted mean of extreme elements of $R_x \cap Z$. The last remark generalizes to the present situation a theorem of Vincent-Smith [23] (also treated by Alfsen and Skau [1]). In addition we also now have (Proposition 23) criteria for the uniqueness of such a decomposition.

The existence of extreme points of $R_x \cap Z$ (see Corollary 3) generalizes, as Vincent-Smith [23] has shown, the theorem of Carathéodory which states that each point of a compact convex subset K of R^n is representable as a convex combination of affinely independent points of K_e.

9. Universal caps. Let C be a cone in V that has a compact universal cap X (see [21]). We can partially order X by writing $x \prec y$ whenever $x, y \in X$ with $y - x \in C$. We shall now take $\mathscr{E}$ to be the class of all functions in $\mathscr{A}(X)$ that are increasing for this partial order.

THEOREM 25. *When X and $\mathscr{E}$ satisfy the above conditions, $\{g \in \mathscr{E}:\ g < f\}$ is an upward filtering family for each $f \in \mathscr{A}(X)$.*

It follows that all the conditions of § 3 are satisfied, so that the preceding theory applies. For this special case some parts of that theory are elementary (e.g. the density of $\mathscr{E}-\mathscr{E}$ in $\mathscr{A}(X)$), or have been treated by other methods (see e.g. [21]).

Results in somewhat the same spirit as Theorem 25 have been given by Kung-Fu Ng [20] and Asimow [3]. Writing

$$\mathscr{A}_0(X) = \{f \in \mathscr{A}(X): f(0) = 0\},$$

we can state Ng's result as follows: the set of functions $\{g \in \mathscr{A}_0(X): 0 \leqslant g < 1\}$ is upward filtering. Asimow's theorem generalizes this, and both authors state converse theorems.

For the proof of Theorem 25 we can suppose without loss of generality that $V = C-C$ and that the topology of V is the weak topology $\sigma(V, \mathscr{A}_0(X))$. This implies that V is the dual of the Banach space $\mathscr{A}_0(X)$, and that X is just the intersection of C with the unit ball of V, so that, by the Kreĭn-Šmulian theorem, C is a closed set.

LEMMA 26. *Let L^* be the dual of a Banach space L, and let K, F be compact convex and closed convex subsets, respectively, of L^* (for the topology $\sigma(L^*, L)$). Then $K+F$ is a closed set for $\sigma(L^*, L)$.*

We can suppose that K, F are non-empty, and write $W = K+F$. We can also suppose that $\|x\| \leqslant 1$ for all $x \in K$. For each $r \geqslant 0$ we write

$$W_r = W \cap \Sigma_r, \quad F_r = F \cap \Sigma_r,$$

where $\Sigma_r = \{x \in L: \|x\| \leqslant r\}$. Since $K \subseteq \Sigma_1$ we have, for all $r \geqslant 0$,

$$W_r \subseteq K + F_{r+1}.$$

Consequently,

$$W_r = (K+F_{r+1}) \cap \Sigma_r.$$

Both terms in this intersection are $\sigma(L^*, L)$-compact, and hence so is W_r. Since W is obviously convex, it follows now by the Kreĭn-Šmulian theorem that W is closed.

Now let g_1, g_2 be non-negative elements of $\mathscr{E}$ and consider, in the product space $V \times R$, the sets

$$K_r = \{(x, t): x \in X, 0 \leqslant t \leqslant g_r(x)\} \quad (r = 1, 2).$$

These are compact convex, and so then is K, the convex hull of $K_1 \cup K_2$.

LEMMA 27. *If $x, y \in X$, $x \prec y$, and if $(x, t) \in K$, then $(y, t) \in K$.*

We can find $(x_1, t_1) \in K_1$, $(x_2, t_2) \in K_2$ and real numbers $\lambda_1, \lambda_2 \geqslant 0$ such that

$$\lambda_1 + \lambda_2 = 1, \quad \lambda_1 x_1 + \lambda_2 x_2 = x, \quad \lambda_1 t_1 + \lambda_2 t_2 = t.$$

Now let $z = y - x$ and write

$$s = \max\{\alpha: x + \alpha z \in X\}, \quad s_r = \max\{\alpha: x_r + \alpha z \in X\} \quad (r = 1, 2).$$

Evidently $s \geqslant 1$. We write $x + sz = w$, $x_r + s_r z = w_r$ and claim that $\lambda_1 w_1 + \lambda_2 w_2 = w$. In fact, since X is convex we have

$$x + (\lambda_1 s_1 + \lambda_2 s_2) z = \lambda_1 w_1 + \lambda_2 w_2 \in X,$$

which shows that $\lambda_1 s_1 + \lambda_2 s_2 \leqslant s$. On the other hand, since $C \setminus X$ is convex, the same reasoning applied to $\lambda_1(w_1 + \varepsilon z) + \lambda_2(w_2 + \varepsilon z)$ shows that $\lambda_1(s_1 + \varepsilon) + \lambda_2(s_2 + \varepsilon) \geqslant s$ for all $\varepsilon > 0$. Consequently, $\lambda_1 w_1 + \lambda_2 w_2 = w$.

It follows that the convex hull of the parallel closed linear segments $[x_1, w_1]$, $[x_2, w_2]$ contains the segment $[x, w]$, and in consequence the point y. In fact, since $\lambda_1 s_1 + \lambda_2 s_2 = s \geqslant 1$ we can choose η_r so that $0 \leqslant \eta_r \leqslant s_r$ and $\lambda_1 \eta_1 + \lambda_2 \eta_2 = 1$. We then have, writing $y_r = x + \eta_r z$, $\lambda_1 y_1 + \lambda_2 y_2 = y$. Now, since the functions g_1, g_2 are increasing, we have $(y_r, t_r) \in K_r$ for $r = 1, 2$. Consequently,

$$(y, t) = \lambda_1(y_1, t_1) + \lambda_2(y_2, t_2)$$

belongs to K.

Now we suppose that $f \in \mathscr{A}(X)$ with $f > g_1 \vee g_2$, and we define

$$F = \{(x, f(x)): x \in X\}, \quad F_1 = F - C.$$

LEMMA 28. *The sets K and F_1 are disjoint.*

Suppose, if possible, that $(x, t) \in F_1 \cap K$. Then for some $y \in X$ with $y \succ x$ we have $(y, t) \in F$, that is, $t = f(y)$. By the preceding lemma $(y, t) \in K$, and so we can find $(y_r, t_r) \in K_r$ such that (y, t) is a convex combination $\lambda_1(y_1, t_1) + \lambda_2(y_2, t_2)$. We now have $t_r \leqslant g_r(y_r) < f(y_r)$ for $r = 1, 2$, and hence

$$t = \lambda_1 t_1 + \lambda_2 t_2 < \lambda_1 f(y_1) + \lambda_2 f(y_2) = f(y),$$

which contradicts $t = f(y)$. The lemma is therefore proved.

We can now prove Theorem 25. By the Hahn-Banach theorem there is a closed hyperplane H in $V \times R$ that separates the closed convex set F_1 from the compact convex set K. This hyperplane must be of the form $\{(x, h(x)): x \in V\}$, where h is a affine functional on V whose restriction to X is continuous. We clearly have

$$g_1 \vee g_2 < h|X < f,$$

and it remains only to show that h is increasing. If not, then for some $x \in X$ we have $h(x) < h(0)$. Since $h - h(0)$ is a linear functional we have

$$h(-nx) = h(0) + n\big(h(0) - h(x)\big),$$

so that, for large positive n, $h(-nx) > f(0)$. But that contradicts the assumption that H separates K from F_1.

10. Note on the filtering condition. For some parts of the preceding theory the conditions of § 3 can be replaced by the following: $\mathscr{E}$ is a wedge in $\mathscr{A}(X)$ that contains the constant functions, separates the points of X, and is such that $\mathscr{E}-\mathscr{E}$ is dense in $\mathscr{A}(X)$. In effect, these are the hypotheses used by Alfsen and Skau [1]. The reader will find that the omission of the filtering condition from the revised hypotheses complicates the previous theory in two ways: (i) the functions f, where $f \in \mathscr{U}(X)$, are now concave instead of affine, (ii) the sets B_f (where $f \in \mathscr{A}(X)$) and Z are in consequence not faces, but only unions of faces. The effect is to make the argument more measure-theoretic and to weaken many of the conclusions.

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