STUDIA MATHEMATICA, T. XXXVIII. (1970)

Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Fréchet spaces with a unique unconditional basis

by

B. MITJAGIN (Moscow, USSR)

The main purpose of my lecture is to discuss the various forms of the theorem on equivalence of unconditional bases in (linear) Fréchet spaces and related topics of the theory of linear operators, in particular, of the theory of Boolean algebras.

We recall that a system $(e_n, n = 0, 1, \ldots)$ in a Fréchet space E is called a *basis* if every vector $f \in E$ can be represented in a unique way in the form

$$f = \sum_{n=0}^{\infty} f_n e_n, \tag{1}$$

and the basis (e) is called *unconditional* (u.b.) if for every $f \in E$ its expansion (1) converges unconditionally, i.e. the series $\sum g_n f_n e_n$ converges for any choice of $g_n = \pm 1, n = 0, 1, \ldots$

Two bases (e) and (x) are called *equivalent* if for some automorphism $T: E \to E$

$$Te_n = x_n, \quad n = 0, 1, \ldots$$

We say that a Banach space E *has a unique u.b.* if any two normalised unconditional bases in E are equivalent.

The first theorem on equivalence of bases in infinite-dimensional space has been established[1] by E. Lorch in 1939. He proved [30] that

---

(1) After my lecture prof. W. Ruckle kindly paid my attention to the Köthe-Toeplitz's paper, 1934, where the following theorem has been proved:

Let S be a metric normal sequence space (i.e. $S = (x = (x_n)_{n=0}^{\infty})$ and $x \in S \Rightarrow gx = (g_n x_n)_{n=0}^{\infty} \in S$ for any g_n, $|g_n| \leqslant 1$) and let S be linearly homeomorphic to l^2. Then there exists such a sequence $d = (d_j)_{j=0}^{\infty}$ that

$$S = ((t_j d_j) : (t_j) \in l^2).$$

Lorch's theorem is clearly equivalent to this statement.

Some facts on normal sequence spaces are given in the report of Ruckle [43] in the present Conference.

in a Hilbert space H every u.v.b. $(x_\mu,\ \mu \in M)$, $\|x_\mu\| = 1$, is equivalent to an orthonormal basis in H.

The proof is based on a general idea of avareging. More precisely, the u.b. $(x_\mu,\ \mu \in M)$ induces the Boolean algebra of projections $(P_m,\ m \subset M)$, or the represantation of the compact group $Z_2^M = (g\colon g(\mu) = \pm 1)$ in E, namely for $f = \sum_{\mu \in M} f_\mu x_\mu$

$$(2)\qquad P_m f = \sum_{\mu \in m} f_\mu x_\mu \quad \text{and} \quad T(g) f = \sum_{\mu \in M} g(\mu) f_\mu x_\mu,$$

where Z_2^M denotes the Cartesian product of many copies of the dyadic group Z_2. The unconditional convergence of series (1) and the principle of uniform boundedness for F-spaces imply the uniform boundedness of operators P_m and $T(g)$, i.e. for some constant C the inequalities

$$(3)\qquad \|P_m\|,\ \|T(g)\| \leqslant C, \qquad \forall m \subset M,\ \forall g \in Z_2^M,$$

hold.

Then the new inner-product

$$(4)\qquad [x, y] = \int_{Z_2^M} n\big(T(g)x, T(g)y\big)\,dg$$

generates an equivalent norm in H, the system (x) is orthogonal with respect to the new inner-product and the new norms of (x_μ) are uniformly bounded from above and from below, i.e.

$$(5)\qquad 0 < 1/C \leqslant \|x_\mu\| \leqslant C < \infty,$$

so the expansions (1) and Pythagoras-Bessel theorem give the equivalence of the basis (x) to the canonical basis in the $l^2(M)$. Hence,

Every Hilbert space has a unique u.b.

This statement has been reproved by Gelfand [17] in 1951 and after 1951 many authors call it Bary-Gelfand [2] or Gelfand theorem.

It should be noticed that in every Hilbert space there exists a (non-un) conditional basis (Babenko [1], 1948; McCarthy and Schwartz [31], 1965; see also [16] and [36]). This means that there is a normalised basis (x_k) such that for some h in H its expansion $h = \sum_{k=1}^{\infty} h_k x_k$ is convergent but not unconditionally convergent. Such a basis has been constructed by Babenko in 1948; namely, the system

$$x_k(t) = t^\alpha \sin kt \qquad \text{in } L^2(0, \pi)$$

with $|\alpha| < 1/2$ provides the desired example. Various examples of this type of bases have been given by C. McCarthy and J. Schwartz and by other authors. Therefore,

There exist two bases — the orthogonal basis and Babenko's basis — which are not equivalent.

Further results are due to Dragilev [8], [9] who discovered in 1958-60 new phenomena in the basis theory. Now, disregarding the chronology, I will discuss the case of Banach spaces and I shall conclude with the recent results of J. Lindenstrauss, A. Pełczyński and M. Zippin.

The analogue of Lorch theorem is *false* in the space l^p for $p \neq 1, 2, \infty$. Indeed, Pełczyński [41] showed in 1960 that for any increasing sequence of integers n_k there is an isomorphism $l^p \xrightarrow{S} (l^2_{n_k})_{l^p}$, where $\dim l^2_n = n$. For an arbitrary countable family of Banach spaces E_k the symbol $(E_k)_{l^p}$ denotes the Banach space of all sequences $x = (x_k)$, $x_k \in E_k$, such that the series $\sum \|x_k\|^p_{E_k}$ converges and its sum defines the norm of x. Then for any tuple of complete orthonormal systems g_i^k, $i = N_{k-1}+1, \ldots, N_k$, $N_k = \sum_{j=1} k_{n_j}$, in $l^2_{n_k}$, *the system* $f_i = S^{-1} g_i^k$, $N_{k-1} < i \leqslant N_k$, *is an unconditional basis in* l^p *which is not equivalent to the canonical basis.*

An analogue of Babenko's basis has been shown by Pełczyński and Singer [42] in 1964 to exist in every infinite-dimensional Banach space with a basis. Their proof is not constructive but gives even a continuum of mutually non-equivalent normalised conditional bases. It is based on a carefull analysis of abstract Haar and Rademacher systems, Khintchin's inequality and block-systems of bases. The *block-system basis* (z_k) has by the definition the following form:

$$z_k = \sum_{i=n_k+1}^{n_{k+1}} a_i x_i, \qquad \|z_k\| = 1, \qquad n_k \to \infty.$$

Bohnenblust [6] has given in 1940 an axiomatic characterisation of c_0 and l^p as spaces X with such a u.n.b. (x) that for any block-basis (z) the operator

$$T_z\colon \sum_{k=1}^{n} c_k x_k \to \sum_{k=1}^{n} c_k z_k$$

is an isometry. (It should be mentioned that A. Kolmogoroff and M. Nagumo have proved in 1930 essentially the same result; their papers [21] and [37] are hardly known now.) Zippin [50], 1966 improved this result replacing the isometry hypothesis by an isomorphism, i.e. assuming that for every block-basis the following inequalities should hold:

$$(8)\qquad C_1(z)\Big\|\sum_1^n a_k z_k\Big\| \leqslant \Big\|\sum_1^n a_k x_k\Big\| \leqslant C_2(z)\Big\|\sum_1^n a_k z_k\Big\|,$$

i.e. the subspace Z spanned by (z_k) is isomorphic to X and $T\colon x_k \to z_k$ is an isomorphism.

As a corollary we infer that *if in a Banach space X all unconditional basis sequences are equivalent, then X is isomorphic to l^2*. This result is due to Pełczyński and Singer [42].

Furthermore, a very surprising fact has been discovered by Lindenstrauss and Pełczyński [27]; namely, they proved that

In l^1 and c_0 all u.n.b. are equivalent.

This result is based on the Grothendieck theorem: *every bounded operator from l^1 to a Hilbert space is absolutely-summing* (for details see Kwapień [26]). It should be noticed that the following fact has been also proved in [27]:

For every bounded Boolean algebra $\mathscr{P}$ of projections in L^1 (or L^∞) for some constants C_1 and C_2 depending only on $\mathscr{P}$ the following inequalities hold:

$$C_1 \sum_1^n \|P_k x\| \leqslant \Big\| \sum_1^n P_k x \Big\| \leqslant C_2 \sum_1^n \|P_k x\|, \tag{9}$$

$\forall x \in L^1$ and (P_k) is a disjoint system, or

$$C_1 \max_{1\leqslant k\leqslant n} \|P_k x\| \leqslant \Big\| \sum_1^n P_k x \Big\| \leqslant C_2 \max_{1\leqslant k\leqslant n} \|P_k x\|, \qquad \forall x \in L^\infty.$$

This series of papers has been completed by Lindenstrauss and Zippin [28], [29] who proved the following results:

If a Banach space X has a u.n.b. (x_k) and every u.n.b. (y_k) is equivalent to (x_k), that is the operator

$$T\colon x_k \to y_k \tag{10}$$

is an automorphism, then (x_k) has the property (8).

So by Bohnenblust-Zippin theorem X must be isomorphic to l^p, $1 \leqslant p < \infty$, or c_0 and, by the example of Pełczyński, p is necessarily 1 or 2. Finally, by Lorch theorem, p may be equal to 2 and, by Lindenstrauss-Pełczyński theorem, X may be isomorphic to l^1 or c_0.

Hence,

Only three Banach spaces (l^2, l^1 and c_0) have a unique u.n.b.

This is a very nice result and at first it seems to be the full solution of the problem of equivalence of u.n.b.'s in Banach spaces. But a simple example shows that the notion of equivalence is not natural for unconditional bases. If e_k^1 (e_k^2) is the canonical basis in l^1 (l^2), then for any monotone integer sequences n_k, m_k the system

$$e_1^1, e_2^1, \ldots, e_{n_1}^1, e_1^2, \ldots, e_{m_1}^2, e_{n_1+1}^1, \ldots, e_{n_2}^1, \ldots \tag{11}$$

is a u.n.b. in the direct product $X = l^1 \times l^2$ and varying n_k and m_k we may construct a continuum of non-equivalent u.n.b.'s in X though all of them are permutations of the same basis.

Now, we call two u.b.'s (x_k) and (y_k) *quasi-equivalent* (or *quasisimilar*) if there exist such a permutation s of integers, a sequence of positive numbers (r_n) and an automorphism $T\colon E \to E$ that $Tx_k = r_k y_{s(k)}$, $k = 0, 1, \ldots$

All bases (11) in $l^1 \times l^2$ are evidently quasi-equivalent.

PROBLEM 1. *Describe Banach spaces X with the following properties:*

1. *X has a u.b.* (e);
2. *every u.b. in X is quasisimilar to* (e);
3. *X is not isomorphic to l^2, l^1 or c_0.*

The spaces $c_0 \times l^2$ and $l^1 \times l^2$ have these properties as Edelstein [15] has recently proved.

It was M. Dragilev who suggested in 1958 the notion of quasi-equivalence and gave a non-trivial example of an F-space with a unique up to quasi-equivalence u.b. He investigated the space $H(D)$ of all holomorphic functions in the open unit disk $D = (z \in C^1\colon |z| < 1)$ with the topology of uniform convergence on compacts in D or, equivalently, with the countable system of norms

$$\|f\|_n = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(r_n e^{i\vartheta})|^2 \, d\vartheta\right)^{1/2}, \qquad r_n \uparrow 1, \tag{12}$$

and the canonical power basis $(z^k, k = 0, 1, \ldots)$ in $H(D)$.

It was discovered [8], [9] that

(a) *any basis in $H(D)$ is an unconditional* (*and even absolute*) *one*, and

(b) *quasi-equivalent to the power basis.*

At the same time, after the appearance of papers by Schwartz [44], Gelfand and Kostyuchenko [18] and essentially Grothendieck's memoir [19], the notion of *nuclear space* came to an analysis as well as the notion of *approximative dimensions* introduced by Kolmogoroff [22] and Pełczyński [40]. Then in 1960 Dynin and me [13] showed that

Any basis in a nuclear F-space is unconditional (*and absolute*).

The nuclearity of F-space is essential in this statement since, as has been recently proved by Wojtyński [46] and [47],

In any non-nuclear countably Hilbert space with a basis a non-unconditional basis may be constructed and *in any non-nuclear F-space with a basis a non-absolute basis can be constructed.*

Following these remarks I shall discuss the quasi-equivalence of *unconditional bases* only.

The space $H(D)$ is an example of a Hilbert scale. More precisely, let H be a Hilbert space and A be a positive operator, $A \geqslant 1$, and for $a > 0$ let H_a denote the Hilbert space

$$\left\{x \in H : x \in D_{A^a} \quad \text{i.e.} \quad \int \lambda^{2a} d(E_\lambda^A x, x) < \infty\right\}$$

with the inner-product

$$(x, x)_a = (A^a x, A^a x) = \int_0^\infty \lambda^{2a} d(E_\lambda^A x, x),$$

and let, for a negative a, H_a denote the completion of H with respect to the norm $(x, x)_{|a|}^{1/2}$ on H. The family H_a, $-\infty < a < \infty$, is a *Hilbert scale*, and we call the F-space $E = \bigcap_{a<a_0} H_a$ (for some $a_0 \leqslant \infty$) with the topology of convergence in each H_a, $a < a_0$, a *center of the Hilbert scale, finite* or *infinite* as is a_0.

If $H = l^2$ and A is the operator-multiplicator with the multiplies (a_n), i.e. $Ae_n = a_n e_n$, $n = 0, 1, \ldots$, and if $a_n = 2^n$, then

$$E = \bigcap_{a<a_0} H_a \simeq \begin{cases} H(D) & \text{if } a_0 < \infty, \\ H(C^1) & \text{if } a_0 = \infty. \end{cases}$$

This is a simple consequence of the Cauchy-Hadamard formula for the radius of convergence of power series and Taylor expansions of holomorphic functions.

In the general case the operator $A^s\colon H_a \rightleftarrows H_{a-s}$ is an isomorphism, and so all finite centers for the operator A are isomorphic.

By analysing Dragilev's proof [8], [9] of quasisimilarity of bases in $H(D)$ I showed [32], [33], §6, in 1961 that this theorem is *true for every nuclear center, finite or infinite, of a Hilbert scale*. The nuclearity of E means that the inverse of A, $A_1 = A^{-1}$, is a compact operator and the series $\sum \lambda_k^\delta(A_1)$ converges for any positive δ in the case of finite center, and for some positive δ in the case of infinite center. Here $\lambda_k(A_1)$ are the eigenvalues of the operator A_1.

It is evident that in this case the space E is isomorphic to the Köthe space of sequences with the Köthe matrix $M_{pn} = (a_n)^p$. All concrete spaces of holomorphic and infinitely differentiable functions are usually nuclear centers of Hilbert scales.

In 1965 Dragilev [10] found a wide class of nuclear spaces with an unconditional basis unique up to quasi-equivalence. In some sense this class is a generalization of centers of Hilbert scales. Employing some of my results [33], §5, on Hilbert scales, Zaharyuta [48] proved the uniqueness theorem for u. bases in centers of Hilbert scales in the case where the operator A_1 is compact. Just before this Conference Dragilev and Zaharyuta [12] proved that

Any direct product of finite and infinite (nuclear centers) of Hilbert scales has a unique u.b.

In particular, it is true for the space $H(D^n) \times H(C^k)$.

Recently I proved the analoguous uniqueness theorem for *every center of Hilbert scale without any assumption on the operator A but strict-positivity*.

The second part of my lecture is a sketch of the proof of this theorem (see also [34] and [35]).

At first it should be noted that without loss of generality I can assume the operator A to have a complete orthonormal system of eigenvectors in H, i.e. $Ae_\nu = e_\nu$, where $\nu \in N$ and N is a countable set. Indeed, otherwise I may consider the Hilbert scale generated by the operator

$$A' = [A] = \int_0^\infty [\lambda] dE_\lambda^A.$$

Then $(1/2)A \leqslant A' \leqslant A$, and the operator A' generates the same scale. The system $(e_\nu, \nu \in N)$ is a u.b. in the center $E = \bigcap_{a<a_0} H_a$; so in the sequel I shall compare an arbitrary u.b. $(x_\mu, \mu \in M)$ with the basis (e).

The first step of the proof is to construct, for a given u.b. (x), a new scale-representation of the F-space E, $E = \bigcap_{b<a_0} G_b$, with the generating operator $B \geqslant 1$ in G_0 such that

(a) *the system (x) is a complete orthogonal one in every space G_b of the new scale, i.e.* $[x_\mu, x_{\mu'}]_b = 0$, $\mu \neq \mu'$;

(b) *the vectors (x_μ) are eigenvectors of the operator B, i.e.* $Bx_\mu = \lambda'_\mu x_\mu$, $\mu \in M$.

For this purpose we first average, following E. Lorch [45], all inner products $(x, y)_a$ and get an equivalent system of inner products,

$$[x, y]'_a = \int_{Z_2^M} \big(T(g)x; T(g)y\big) dg,$$

and the corresponding scale of Hilbert spaces which may be not a Hilbert scale. Then, in the case of a finite center, the unit ball of H_0 is bounded in E and there exist constants D_a such that $D_a S(G'_a) \supset S(H_0)$, i.e. $D_a^2(x, x)_0 \geqslant [x, x]'_a$ for every $x \in H_0$. For the new inner product

$$[x, x]_0 = \sum \frac{1}{2^n} \frac{[x, x]'_{1/n}}{D_{1/n}^2} \leqslant (x, x)_0$$

holds and G_0 is contained in G'_b, H_a for $b, a < 0$.

Choosing a sufficiently small $b_0 < 0$ (cf. [33], §5) and taking the Hilbert scale for the spaces G'_{b_0} and G_0, we get the desired operator B and the new scale-representation of E. In the proof of the equivalence of the two systems of norms we essentially make use of the interpolation theorem for Hilbert scales.

In the case of an infinite center analogous interpolation arguments give the desired reconstruction of a new scale. In order to make the difference between these cases clear, now I give one statement and a question.

The complemented subspace X of a finite center of a Hilbert scale is isomorphic to a finite center of a Hilbert scale.

(Some closed results has been given by Bessaga [4] and [5].)

The proof of this statement is also based on an averaging with respect to the group $Z_2^2 = (\pm P \pm (1-P))$, where P is the projection onto X in E. I cannot prove this statement for infinite centers by means of the same method because the operators B_a, defined by

$$[x, y]'_a = [B_a x, B_a y]_0,$$

may be not commuting. When we consider a u.b. (x) and the corresponding representation $T(g)$ of the group Z_2^M, the operators B_a are commuting because they have a mutual complete system of eigenvectors (x) and we may not only interpolate but also extrapolate some inequalities concerning operator powers.

PROBLEM 2. *Is the complemented subspace of an infinite center of a Hilbert scale also an infinite center of a Hilbert scale?*

The second step is the comparison of spectral properties of operators A and B generating two scale representations of the space E. One can show that there exists a constant R such that for every interval $[u, v)$, $0 \leqslant u \leqslant v \leqslant \infty$, the equalities

$$\dim E^B_{r(u), r^{-1}(v)} \leqslant \dim E^A_{u,v} \leqslant \dim E^B_{r^{-1}(u), r(v)}$$

hold, where $r(u) = Ru^R$, r^{-1} is the inverse function and E^A is the spectral resolution of the indentity for A.

The main point is here a Paley-Wiener type statement; namely, if $x \in E$ and $E^A_{u,v} x = x$, i.e. the element x belongs to the interval $[u, v)$ in the A-representation, and $E^B_{t,w} x = 0$, $t = r^{-1}(u)$, $r(v) = w$, i.e. in the B-representation the element x belongs to the complement of the interval $[t, w)$, then $x = 0$.

In connection with this step I want to ask some questions on spectral resolutions of the identity for an operator with a cyclic vector or, more generally, on maximal Boolean algebras of projections.

PROBLEM 3. *Let $\mathscr{P}$ and $\mathscr{Q}$ be maximal Boolean algebras of operators in an infinite-dimensional Banach space X. Give some additional hypothesis* (²) *on $\mathscr{P}$ and $\mathscr{Q}$ under which the following condition is satisfied:*

(+) *for any positive ε there exist infinite-dimensional operators $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ such that $\|PQ\| < \varepsilon$.*

A positive answer to this question would help in the problem of contractibility of the linear group of concrete Banach spaces. Indeed,

The Kuiper and Neubauer arguments [25], [38] *may be formulated as a general scheme for a proof of contractibility of $GL(E)$,*

where essential analytic problems are connected with the existence of "good" families of projections. In this way I. Edelstein, E. Semenov and me have recently proved *the contractibility of the linear groups of* $L^1[0, 1]$, $L^\infty[0, 1]$, $C[0, 1]$ and other spaces of L^1- and C-type (see [14]).

The third step of the proof is purely combinatorial. Put $\gamma_0 = 1$ and $\gamma_{k+1} = r(\gamma_k)$, and put

$$N_k = \{\nu \in N : \gamma_k \leqslant \lambda_\nu < \gamma_{k+1}\},$$
$$M_k = \{\mu \in M : \gamma_k \leqslant \lambda'_\mu < \gamma_{k+1}\}.$$

Then the inequalities between the dimensions of the invariant subspaces $E^A_{u,v} H$ and $E^B_{t,w} H$ give the following inequalities for the powers of sets N_k and M_k and their unions:

$$\Big|\bigcup_{i=2}^{p-1} M_{m+i}\Big| \leqslant \Big|\bigcup_{i=1}^{p} N_{m+i}\Big| \leqslant \Big|\bigcup_{i=0}^{p+1} M_{m+i}\Big| \quad \text{for any } m \text{ and } p.$$

There exists a 1-1 correspondence $s: N \xrightarrow{\text{onto}} M$ such that, for every k, $s(N_k) \subset M_{k-1} \cup M_k \cup M_{k+1}$.

Then the operator $T: e_\nu \to x_{s(\nu)}$ is an automorphism of E, and thus the uniqueness theorem is proved.

In the proof of the existence of the desired 1-1 correspondence s, I essentially use Hall-König theorem ([20], Ch. 3) on a choice of different representatives from the system of finite sets. This theorem is used also in some economics-mathematical models, for instance, in the optimal assignment problem [39], and I have to note that it was my dealing with these topics, which led me to the idea of a construction of a suitable permutation s.

If the operator $A_1 = A^{-1}$ is compact, then the function $d(u) = \dim E^A_{(-\infty, u)}$ has only finite values, $\dim E^A_{u,v} = d(v) - d(u)$, and it is sufficient to consider only $d(u)$. Then the inverse function is exactly

(²) It should be noted that there are examples of $\mathscr{P}$ and $\mathscr{Q}$ which have not property (+). After my lecture, prof. C. A. McCarthy kindly informed me about such an example in the Hilbert space $L^2(0, 1)$.

$\lambda_k(A_1)$ and instead of the inequalities for dimensions of E^A and E^B we have to prove the inequalities between s-numbers; and this is a simple geometrical fact. The problem of permutation s in the compact case is degenerate also because a suitable correspondence is yielded by the fact that the sequences $\lambda_k(A_1)$ and $\lambda_k(B_1)$, $B_1 = B^{-1}$, are decreasing.

As a corollary to the uniqueness theorem for u.b.'s in arbitrary Hilbert scales, I give an example of four non-isomorphic non-Banach and non-Schwartz F-spaces.

Let $a_n = 2^{2^{n^2}}$, $b_n = 2^{2^{(2n)^2}}$, $H = (H_n)_{l^2}$, where H_n are infinite-dimensional Hilbert spaces, and $A: H \to H$, $(Ah)_n = a_n h_n$, $B: H \to H$, $(Bh)_n = b_n h_n$. Then the four spaces, namely, the finite and infinite centers of the Hilbert scales, generated by the operators A and B, are not isomorphic.

I am indebted to C. Bessaga, Z. Ciesielski, I. Edelstein, K. Gęba, G. Henkin, G. Neubauer, A. Pełczyński, W. Wojtyński and V. Zaharyuta for valuable discussions during the preparation of the present lecture.

References

[1] К. И. Бабенко, *О сопряжённых функциях*, Докл. АН СССР 62 (1948), p. 157-160.

[2] Н. К. Бари, *Биортогональные системы и базисы в гильбертовом пространстве*, Учён. Зап. МГУ, 148; Математика 4 (1951), p. 69-107.

[3] C. Bessaga, A. Pełczyński and S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Pol. Sci., Sér. math., 9:9 (1961), p. 677-683.

[4] C. Bessaga, *Some remarks on Dragilev's type theorem*, Studia Math. 31 (1968), p. 307-318.

[5] — *A theorem on complemented subspaces of nuclear spaces*, Present Conference, Afternoon Session, June 18.

[6] H. F. Bohnenblust, *An axiomatic characterization of L^p-spaces*, Duke Math. J. 6 (1940), p. 627-640.

[7] M. Day, *Normed linear spaces*, Berlin 1958.

[8] М. М. Драгилев, *О регулярной сходимости базисных разложений аналитических функций*, Науч. Докл. Высш. Школы, сер. физ. мат. наук, № 4 (1958), p. 27-32.

[9] — *Каноническая форма базиса пространства аналитических функций*, УМН 15:2 (1960), p. 181-188.

[10] — *О правильных базисах в ядерных пространствах*, Матем. Сборн. 68:2 (1965), p. 153-173.

[11] — В. П. Захарюта и М. Г. Хапланов, *О некоторых проблемах базиса аналитических функций*, Из-во Ростов. Госуниверситета, 1967.

[12] М. М. Драгилев, *О кратно правильных базисах в пространствах Кёте*, Докл. АН СССР, в печати.

[13] A. Dynin and B. Mitjagin, *Criterion for nuclearity in terms of approximative dimension*, Bull. Acad. Pol. Sci. 8:8 (1960), p. 534-540.

[14] I. Edelstein, B. Mitjagin and E. Semenov, *The linear groups of C and L^1 are contractible*, Bull. Acad. Pol. Sci. 18. 1 (1970), p. 27-33.

[15] И. С. Эдельштейн, *Дополняемые подпространства и безусловные базисы в $l^p \times l^2$*, Теория функций и функциональный анализ 10 (1970).

[16] В. М. Гапошкин, *Одно обобщение теоремы М. Рисса о сопряжённых функциях*, Матем. Сборн. 46 (1958), p. 359-372.

[17] И. М. Гельфанд, *Замечание к работе Н. К. Бари...*, Учён. Зап. МГУ, 148; Математика 4 (1951), p. 224-225.

[18] — и А. Г. Костюченко, *Разложение по собственным функциям дифференциальных и других операторов*, Докл. АН СССР 103 (1955), p. 349-352.

[19] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc. 16 (1955).

[20] M. Hall, *A survey of combinatorial analysis*, N. Y. 1958.

[21] A. Kolmogoroff, *Sur notion de la moyenne*, Rend. Accad. dei Lincei (6), 12 (1930), p. 388-391.

[22] А. Н. Колмогоров, *О линейной размерности топологических векторных пространств*, Докл. АН СССР 120 (1958), p. 239-241.

[23] С. Г. Крейн и Ю. И. Петунин, *Шкалы банаховых пространств*, УМН 21:2 (1966), p. 89-168.

[25] N. H. Kuiper, *The homotopy type of the unitary group of Hilbert space*, Topology, 3:1 (1965), p. 19-30.

[26] S. Kwapień, *Some applications of the theory of absolutely summing operators*, present Conference, Morning Session June 20 (Preprint Series 1968/69 No 37, June 1969, Aarhus Universitet).

[27] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*, Studia Math. 29 (1968), p. 275-326.

[28] J. Lindenstrauss and M. Zippin, *Banach spaces with a unique unconditional basis*, Journ. Funct. Anal. 3:1 (1969), p. 115-125.

[29] — *Banach spaces with sufficiently many Boolean algebras of projections*, Journ. Math. Anal. and Appl. 25:2 (1969), p. 309-320.

[30] E. R. Lorch, *Bicontinuous linear transformations in certain vector spaces*, Bull. Amer. Math. Soc. 45 (1939), p. 564-569.

[31] C. A. McCarthy and J. Schwartz, *On the norm of a finite Boolean algebras of projections, and applications to the theorem of Kreiss and Morton*, Comm. Pure Appl. Math. 18:1-2 (1965), p. 191-201.

[32] Б. С. Митягин, *Ядерные шкалы Рисса*, Докл. АН СССР 137 (1961), p. 519-522.

[33] — *Аппроксимативная размерность и базисы в ядерных пространствах*, УМН 16:4 (1961), p. 63-132.

[34] B. Mitjagin, *Sur l'équivalence des bases inconditionnelles dans les echelles de Hilbert*, C. R. Acad. Sc. Paris, sér. A. t. 269 (1969), p. 426-428.

[35] Б. С. Митягин, *Эквивалентность безусловных базисов в гильбертовых шкалах*, Studia Math. (in print).

[36] Л. А. Муравей, *Базисы Рисса в $L^2(-1, 1)$*, Труды Матем. Ин-та Стеклова 91 (1967), p. 113-131.

[37] M. Nagumo, *Ueber eine Klasse der Mittelwerte*, Jap. Journ. of Math. 7 (1930), p. 71-79.

[38] G. Neubauer, *Der Homotopietyp der Automorphismengruppe in der Räumen l^p und c_0*, Math. Ann. 174:1 (1968), p. 33-40

[39] J. von Neumann, *A certain zero-sum two person game equivalent to the optimal assignement problem*, Contributions to the theory of game 2 (1953), Princeton, p. 1-12.

[40] A. Pełczyński, *On the approximation of S-spaces by finite dimensional spaces*, Bull. Acad. Pol. Sci. 5:9 (1957), p. 879-881.
[41] — *Projections in certain Banach spaces*, Studia Math. 19 (1960), p. 209-228.
[42] — and I. Singer, *On non-equivalent bases and conditional bases in Banach spaces*, ibidem 25 (1965), p. 5-25.
[43] W. H. Ruckle, *Diagonal operators and the diagonal of a nuclear operator*, present Conference, afternoon session, June 18.
[44] L. Schwartz, *Théorie des noyaux*, Proc. Int. Cong. Math. 1952, v. 1, p. 220-230.
[45] W. Wojtyński, *On bases in certain countably-Hilbert spaces*, Bull. Acad. Pol. Sci., ser. math. 14 (1966), p. 681-684.
[46] — *On conditional bases in non-nuclear Fréchet spaces*, present Conference, afternoon session, June 18.
[47] — *On conditional bases in non-nuclear Fréchet spaces*, Studia Math. 35 (1970), p. 77-96.
[48] В. П. Захарюта, *О квазиэквивалентности базисов в конечных центрах гильбертовых шкал*, Докл. АН СССР 180:4 (1968), p. 786-788.
[49] M. Zippin, *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. 4 (1966), p. 265-272.

STUDIA MATHEMATICA, T. XXXVIII. (1970)
Colloquium on
Nuclear Spaces and Ideals in Operator Algebras

Small operators between Banach and Hilbert spaces

by

R. M. DUDLEY (Cambridge, Mass.)

1. Let B be a Banach space and let H be a Hilbert space with inner product $(\cdot,\cdot)$. We shall study operators from B into H which are sufficiently "small" to decompose continuous linear random processes on H.

Here are some definitions. Let P be a probability measure on a space Ω. Let $L^0(\Omega, P)$ be the space of all real P-measurable functions on Ω modulo functions vanishing P-almost everywhere. $L^0(\Omega, P)$ has its usual topology of convergence in measure. Given a real topological linear space S, a *continuous linear process* on S is a continuous linear map L from S into some $L^0(\Omega, P)$. S' denotes the (dual) space of continuous linear real-valued forms on S. L is called *decomposable* iff (i.e. if and only if) there is a mapping $\omega \to L_\omega$ from Ω into S' such that for every x in S, $L_\omega(x) = L(x)(\omega)$ for almost all ω. A continuous linear map C from another topological vector space X into S will be called L^0-*decomposing* iff $L\circ C$ is decomposable on X for every continuous linear process L on S.

The following result has been stated by S. Kwapień:

THEOREM 1. *An operator A from B into H is L^0-decomposing iff $A = J\circ C$ for some Hilbert-Schmidt operator J from H into itself and bounded operator C from B into H.*

The proof to be given here uses the following probabilistic result which may be of independent interest. (I do not know what would be the largest possible function of α in place of $\alpha^2/4$.)

LEMMA 1. *Let A_j, $j = 1, 2, \ldots$, be independent events, and $\alpha > 0$. Let $B_j \subset A_j$ for all j and $P(B_j) \geqslant \alpha P(A_j)$. Then*

$$P\Big(\bigcup_j B_j\Big) \geqslant \alpha^2 P\Big(\bigcup_j A_j\Big)/4.$$

Proof. If $P(A_j) = 0$ for all j, there is nothing to prove. Otherwise we have $\alpha \leqslant 1$. If, for some j, $P(A_j) > \alpha/2$, then

$$P(B_j) > \alpha^2/2 > \alpha^2 P\big(\bigcup A_j\big)/4.$$