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Duality of linear spaces of functions and nuclearity of solution spaces*

by

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In the previous paper [4] the author gave a duality theory of linear spaces of (generalized) functions on R^n, similar to the theory of Köthe spaces. But this duality theory were restrictive in two senses. First, we treated only such spaces whose elements have compact supports or duals of such spaces. Second, we imposed some artificial conditions (σ-invariance). So it is a problem to take these restrictions away.

In § 1 we explain briefly the theory of Köthe spaces. In § 2 we consider the duality theory without the condition of σ-invariance which are imposed in [4]. In § 3 we study linear spaces of functions without conditions on supports.

We should remark that in this paper the convolution $f*g$ means $\int_{R^n} f(x-y)g(y)\,dy$ as usual, not $\int_{R^n} f(x+y)g(y)\,dy$ as in [4]. But this does not give any essential change.

1. Köthe spaces. We shall sketch the duality theory of Köthe sequence spaces. Denote by ω the set of all sequences (x_n) and by φ the set of such sequences (x_n) that $x_n = 0$ for all but finite n's. The *α-dual* λ^* of a subspace $\lambda \subset \omega$ is defined by

$$\lambda^* = \{(x_n)\in\omega \,|\, \Sigma|x_n y_n| < \infty, \forall (y_n)\in\lambda\}.$$

Such a space λ that $\lambda = \lambda^{**}$ is called a *Köthe space* or a *perfect space*. The space ω is the largest perfect space. We define the *regular subspace* λ_r of a perfect space λ by

$$\lambda_r = \text{the } \beta(\lambda, \lambda^*)\text{-closure of } \varphi,$$

* In the conference the author gave a talk on duality of linear spaces of functions and on nuclearity of solution spaces of linear partial differential operators with constant coefficients. The content of the latter subject is found in [5] and [6], hence only the former is discussed here.

where $\beta(\lambda, \lambda^*)$ means the strong topology of λ with respect to λ^*. We denote by λ' the topological dual of λ with respect to $\beta(\lambda, \lambda^*)$. Many interesting properties of perfect spaces are known (see for example Köthe [2], Kōmura [3] and Pietsch [7]), but we cite only a part of them:

1° *A perfect space λ is complete with respect to the strong topology $\beta(\lambda, \lambda^*)$.*

2° $\lambda^* = \lambda_r'$ *for any perfect space* λ.

3° *A perfect space λ is separable with respect to $\beta(\lambda, \lambda^*)$ if and only if* $\lambda = \lambda_r$.

The theory of perfect sequence spaces was extended to the case of perfect function spaces by Dieudonné [1]. In this case the notion of regular subspace is not defined. In the following we shall sketch the theory of perfect spaces of functions on R^n.

Ω denotes the space of all locally integrable functions on R^n, and Φ the space of all essentially bounded measurable functions with compact supports on R^n. For a subspace $\Lambda \subset \Omega$

$$\Lambda^* = \Big\{f \in \Omega \,\Big|\, \int_{R^n} |fg|\,dx < \infty, \forall g \in \Lambda\Big\}$$

is called the *α-dual* of Λ. Λ is called a *Köthe space* or a *perfect space* if it is equal to the bidual Λ^{**}. We consider usually the strong topology $\beta(\Lambda, \Lambda^*)$. It should be noticed that this is even true for non-perfect spaces. Hence we call $\beta(\Lambda, \Lambda^*)$ the *natural topology* of Λ, and denote by Λ' the dual of Λ with respect to the natural topology $\beta(\Lambda, \Lambda^*)$. Evidently, we have $\Lambda^* \subset \Lambda'$.

Regular subspace is not defined. For instance, the largest perfect space Ω is not the topological dual of any subspace of Φ with respect to the natural topology. Thus main properties of perfect function spaces are a little different from properties of perfect sequence spaces:

1'. *A perfect space Λ is complete with respect to the natural topology* $\beta(\Lambda, \Lambda^*)$.

2'. *A perfect space Λ is separable with respect to the natural topology $\beta(\Lambda, \Lambda^*)$ if and only if* $\Lambda' = \Lambda^*$.

2. β-perfect spaces. We shall make use of two notations as in [4], τ- and σ-operations:

$$(\tau_x f)(y) = f(y+x), \qquad (\sigma_a f)(y) = f\left(\frac{y}{a}\right) = f\left(\frac{y_1}{a}, \frac{y_2}{a}, \ldots, \frac{y_n}{a}\right).$$

The convolution $f*g(x)$ of two functions $f, g \in \Omega$ is defined by

$$f*g(x) = \int_{R^n} f(x-y)g(y)\,dy = \langle \tau_x f, \sigma_{-1} g\rangle.$$

The convolution operator defined by $f\colon g \to f*g$ is identified with the function f itself. In [4] we defined a generalization of convolution operators and identified these operators with generalized functions. We studied only σ-invariant spaces E (i.e. $E = \sigma_a E$ for $a \neq 0$). This is mainly for the fact that our spaces contain sequences approximating the Dirac's measure δ. This fact is very useful to discuss the duality of linear spaces of functions. But the property of σ-invariance seems to be artificial, so we try to remove that in this chapter. We shall give often only outline of proofs, since there are little essential difference from [4].

For a linear operator $T\colon \Phi \to \Omega$, the transposed operator ${}^tT$ is defined always as an operator: $\Phi \to \Omega$.

Definition 1. A linear operator $T\colon \Phi \to \Omega$ is called an *operator of function type* if it satisfies

$$(1) \qquad \tau_x(Tf) = T(\tau_x f) \quad \text{for any } f \in D(T),$$

$$(2) \qquad {}^tT = \sigma_{-1}T,$$

where $\sigma_{-1}T$ is defined by $(\sigma_{-1}T)f = \sigma_{-1}(T\sigma_{-1}f)$.

The set of all operators of function type is denoted by $\mathfrak{F}$.

Proposition 1. *An operator $T \in \mathfrak{F}$ is closed and densely defined with respect to the weak topologies $\sigma(\Phi, \Omega)$ and $\sigma(\Omega, \Phi)$.*

Proof. It is clear that the operator $\sigma_{-1}T = {}^tT$ is closed with respect to the weak topologies. The closedness of T implies that the adjoint ${}^tT$ is densely defined with respect to the weak topology. Hence $T = \sigma_{-1}\sigma_{-1}T = {}^{tt}T$ is closed and densely defined.

Proposition 2. *If a linear operator $T\colon \Phi \to \Omega$ satisfies condition* (1) *and condition*

$$(2') \qquad \sigma_{-1}T \subset {}^tT \quad ({}^tT \text{ is single-valued and so } D(T) \text{ is } \sigma(\Phi, \Omega)\text{-dense}),$$

there exists a unique extension $\overline{T}$ of T which is in $\mathfrak{F}$.

Proof. It suffices to show ${}^tT = {}^{tt}(\sigma_{-1}T)$. Since $D(T)$ is dense with respect to the weak topology $\sigma(\Phi, \Omega)$, there exists a net $\{f_\lambda\} \subset D(T)$ which converges to the Dirac's measure δ with respect to the weak topology $\sigma\big((M)_0, (C)\big)$, where (C) means the space of all continuous functions on R^n, and $(M)_0$ the space of all measures with compact supports. Then for $f \in D({}^tT)$ we have for any $h \in \Phi$

$$\langle (Tf_\lambda)*\sigma_{-1}f, h\rangle = \langle Tf_\lambda, f*h\rangle = \langle f_\lambda, {}^tT(f*h)\rangle$$
$$\to \big(({}^tTf)*h\big)(0) = \langle h, \sigma_{-1}({}^tTf)\rangle.$$

Thus we may say that $(Tf_\lambda)*\sigma_{-1}$ approximates $\sigma_{-1}{}^tT$ with respect to $\sigma(\Omega, \Phi)$. In general, if a net of functions $\{g_\lambda\}$ approximates a linear operator L from Φ to Ω, i.e. $\sigma(\Omega, \Phi)\text{-}\lim_\lambda g_\lambda * f = Lf$ for $f \in D(L)$, then we have ${}^tLg = \sigma(\Omega, \Phi)\text{-}\lim_\lambda \sigma_{-1} g_\lambda * g$. Hence $D({}^tL) \supset \sigma_{-1}D(L)$. In our case it holds that $D({}^{tt}T) \supset \sigma_{-1}D({}^tT)$, or ${}^{tt}(\sigma_{-1}T) \supset {}^tT \supset \sigma_{-1}T$. Since ${}^{tt}(\sigma_{-1}T)$ is a minimal closed extension of $\sigma_{-1}T$, we have ${}^{tt}(\sigma_{-1}T) = {}^tT$, q.e.d.

If an operator T of function type satisfies

$$T\cdot D(T) \subset \Omega_0 \quad (= \{f \in \Omega \mid f \text{ has a compact support}\}),$$

then we say that T has a *compact support*.

Definition 2. Let S be an operator of function type and T an operator of function type with compact support. The operator U of function type which is the unique extension of the operator

$$f*g \to (Tf)*(Sg) \quad \text{for } f \in D(T),\ g \in D(S),$$

is called the *convolution* of T and S, and denoted by $T*S$. The existence of such an operator U follows from Proposition 2.

For the convolution the associative and commutative law holds good. Note that $Tf = T*f$ for $f \in D(T)$.

Definition 3. For a subspace $E \subset F$, the *β-dual* is defined by

$$E^\beta = \{T \in F \mid (\sigma_{-1}T)*S \in L^\infty_{\text{loc}} \text{ for any } S \in E\}, \tag{3}$$

where L^∞_{loc} means the space of all measurable locally essentially bounded functions.

Remark. If E contains an operator with non-compact support, all elements in E^β have compact support, since our convolution $T*S$ is defined only in case one of the pair T, S has a compact support.

We shall introduce a scalar product for a dual pair E and E^β. Let $\tilde\delta \in (L^\infty_{\text{loc}})'$ be an extension of the Dirac's measure δ. We define a scalar product

$$\langle T, S\rangle_{\tilde\delta} = \langle T*\sigma_{-1}S, \tilde\delta\rangle.$$

Definition 4. A subspace $E \subset F$ which is equal to $E^{\beta\beta}$ is said to be *β-perfect*. For a β-perfect space E the $\beta(E, E^\beta)$-closure of $E*\Phi$ is called the *regular subspace* of E, and denoted by E_r.

Though our scalar product depend on the choice of the duality measure $\tilde\delta$, almost all topological properties of β-perfect spaces or of regular subspaces are independent of it.

We do not know whether every β-perfect space E is total on E^β by the scalar product $\langle\ ,\ \rangle_{\tilde\delta}$. It is our conjecture that $\mathfrak{F}^\beta = \{0\}$. On the other hand, the β-dual $\mathfrak{F}_0^\beta$ of the space $\mathfrak{F}_0 = \{T \in \mathfrak{F} \mid T \text{ has a compact support}\}$ contains all entire functions of exponential type (see [4]).

Main properties of β-perfect spaces are as follows:

THEOREM. 1° *A β-perfect space E is complete with respect to the strong topology $\beta(E, E^\beta)$, if the β-dual E^β contains a total subspace whose elements have compact supports.*

2° *If a β-perfect space E is total on E^β, then $E^\beta = E_r'$.*

3° *If a β-perfect space E is separable with respect to the strong topology $\beta(E, E^\beta)$, then E is equal to the regular subspace E_r.*

Proof. 1° An operator $\sigma_{-1}T$ in $\sigma_{-1}(E^\beta)$ is considered as a linear operator from E to L^∞_{loc}: $S \to (\sigma_{-1}T)*S$ for $S \in E$. Hence if the space L^∞_{loc} is endowed with the topology of uniform convergence on every compact set in R^n, the space $\sigma_{-1}(E^\beta)$ determines the limit projective topology τ on the space E. Let S_0 be an element of the τ-completion of E. We can define the convolution of S_0 with any element $\sigma_{-1}T$ in $\sigma_{-1}(E^\beta)$ by

$$S_0 * \sigma_{-1}T = \lim_\alpha S_\alpha * \sigma_{-1}T \quad \text{in } L^\infty_{\text{loc}},$$

where $\{S_\alpha\}$ is a net in E satisfying $S_0 = \tau\text{-}\lim_\alpha S_\alpha$. Evidently, this convolution is well defined. Since by assumption the dual E^β contains a total subspace H whose all elements have compact supports and since for any $T \in H$ and $\varphi \in D(T)$ we have

$$T(\varphi*\varphi) = (T\varphi)*\varphi \in \Phi, \qquad S_\alpha * \sigma_{-1}T(\varphi*\varphi) = (S_\alpha * \sigma_{-1}T)*\sigma_{-1}\varphi * \sigma_{-1}\varphi \in L^\infty_{\text{loc}},$$

the set $\{f \in \Phi \cap E^\beta \mid S_0 * \sigma_{-1} f \in L^\infty_{\text{loc}}\}$ is total on E. Thus S_0* is densely defined as an operator from Φ to Ω. Since S_0* satisfies the conditions of Proposition 2, S_0 is identified to an element $\bar S_0$ of $\mathfrak{F}$: $S_0 * \sigma_{-1}T = \bar S_0 * \sigma_{-1}T$ for any $T \in E^\beta$. This implies $S_0 \in E$.

Since this topology τ is weaker than the strong topology $\beta(E, E^\beta)$ and stonger than the weak topology $\sigma(E, E^\beta)$, the τ-completeness of E implies the $\beta(E, E^\beta)$-completeness of E.

2° Here E_r' means the topological dual of the regular subspace E_r with respect to the topology induced by $\beta(E, E^\beta)$. Since E is total on E^β by assumption, $E*\Phi$ is also total on E^β. Hence $E_r' \supset E^\beta$. Let us verify the converse inclusion relation.

First we shall show that $E_r * \sigma_{-1}E^\beta \subset (C)$. Note that $(E*\Phi)*\sigma_{-1}E^\beta = (E*\sigma_{-1}E^\beta)*\Phi \subset (C)$, and that the set $\{\tau_x T \mid x \in K\}$ for an element $T \in E^\beta$ and for a compact set $K \subset R^n$ is bounded in E^β. If a net $\{S_\alpha\} \subset E*\Phi$ converges to $S \in E_r$ with respect to $\beta(E, E^\beta)$, then $(S_\alpha * \sigma_{-1}T)(x)$ converges to $(S*\sigma_{-1}T)(x)$ uniformly in $x \in K$. Since K is an arbitrary compact set in R^n, we have $S*\sigma_{-1}T \in (C)$ for $S \in E_r$, $T \in E^\beta$. More precisely, $\tau_x S$ is uniformly $\tau(E, E^\beta)$-continuous in $x \in K$, hence the set $\{\tau_x S \mid x \in K\}$ is $\tau(E, E^\beta)$-compact, where $\tau(E, E^\beta)$ means the Mackey topology of E with respect to E^β. From this it follows that $\langle \tau_x T, S\rangle \in (C)$ if a bounded net $\{T_\alpha\} \subset E^\beta$ converges to T with respect to the weak topology $\sigma(E_r', E_r)$, since a weakly convergent bounded net converges uniformly on a compact set. By Proposition 2, we easily see, that T is identified to an element $\tilde{T}$ in $\mathfrak{F}$, and that $\tilde{T}\in E^{\beta}$. Since any element T in E'_r is a weak limit of a bounded net $\{T_a\}$, the proof is completed.

3° Let E be separable with respect to $\beta(E, E^{\beta})$. For any $T\in E'$ there exists a sequence $\{T_k\}\subset E^{\beta}$ which converges weakly to T. That is, for any $S\in E$ and $x\in R^n$, the sequence $\langle\tau_x S, T_k\rangle$ converges to $\langle\tau_x S, T\rangle$. Hence $\langle\tau_x S, T\rangle$ is measurable in x. Local boundedness of $\langle\tau_x S, T\rangle$ is clear. By Proposition 2 there exists an element $\tilde{T}\in E^{\beta}$ which is identified with T: $\tilde{T}*\sigma_{-1}S = \langle\tau_x\tilde{T}, S\rangle_{\delta} = \langle\tau_x T, S\rangle$ for a. e. $x\in R^n$. But the separability of E implies that the both terms $\langle\tau_x\tilde{T}, S\rangle_{\delta}$ and $\langle\tau_x T, S\rangle$ are continuous in x (see [4]), and so we have $\langle\tau_x\tilde{T}, S\rangle = \langle\tau_x T, S\rangle$. This means $E^{\beta}/E^{\perp} = E'$. Since $E^{\perp} = E_r^{\perp}$ and since $E'_r = E^{\beta}/E_r^{\perp}$, we have $E' = E'_r$. The regular subspace E_r is by definition closed in E, hence $E = E_r$.

Remark. If a β-perfect space E contains a sequence which approximates the Dirac's measure δ, then the regular subspace E_r is necessarily separable (see [4] or the proof of 3° of the theorem in § 3), but we do not know if a regular subspace is separable in general case.

Example 1. The space L^p_{loc} for $1<p<\infty$ is the set of all locally p-power integrable functions on R^n. This space is β-perfect and equal to the regular subspace. The dual $(L^p_{\mathrm{loc}})^{\beta}$ is the space L^q_0 = the set of all q-power integrable functions with compact supports for $1/p+1/q = 1$. On the other hand, the space $L^{\infty}_{\mathrm{loc}}$ is β-perfect and its regular subspace is (C). The dual $(L^{\infty}_{\mathrm{loc}})^{\beta}$ is the space $(M)_0$ = the set of all measures with compact supports. The regular subspace of $(M)_0$ is equal to $\Omega_0(=L^1_0)$.

Example 2. Let E be the set of all linear partial differential operators $P(D)$ with constant coefficients. E is a subset of $\mathfrak{F}$. We have

$$E^{\beta} = (E^{\beta})_r = (\mathscr{E}),\qquad E^{\beta\beta} = (E^{\beta\beta})_r = (\mathscr{E}'),$$

where $(\mathscr{E})$ means the space of all infinitely differentiable functions and $(\mathscr{E}')$ the space of distributions with compact supports.

3. φ-perfect spaces. In this chapter we shall give a trial to remove the restriction on supports of functions. For this purpose we must define general convolutions of two operators of function type whose supports are not compact. We consider a condition on an operator of function type T:

(4) *There exists a function φ in $D(T)$ such that $\sigma_{1/k}\varphi\in D(T)$ for $k = 1, 2, \ldots$ and $\langle\varphi, 1\rangle = \int\varphi\,dx \neq 0$.*

The set of all elements of $\mathfrak{F}$ satisfying (4) is denoted by $\mathfrak{F}_{\sigma}$. In the following we treat only subspaces of $\mathfrak{F}_{\sigma}$.

Definition 5. If for a pair $T, S\in\mathfrak{F}_{\sigma}$ there exists a function $\varphi\in\Phi$ such that $\int\varphi\,dx\neq 0$ and

$$|T(\sigma_{1/k}\varphi)|*|S(\sigma_{1/k}\varphi)|\in\Omega,\qquad \sigma_{1/k}\varphi\in D(T)\cap D(S)\ \text{ for } k = 1, 2, \ldots,$$

we say that the pair T, S is *φ-convolutive*. The *convolution $T*S$ of a φ-convolutive pair* T, S is the extension $U\in\mathfrak{F}_{\sigma}$ of the operator

$$\sigma_{1/k}\varphi*\sigma_{1/k}\varphi\to(T\sigma_{1/k}\varphi)*(S\sigma_{1/k}\varphi)\qquad\text{for } k = 1, 2, \ldots$$

Making use of Proposition 2, we see without difficulty

Proposition 3. *For a φ-convolutive pair $T, S\in\mathfrak{F}_{\sigma}$, the convolution $T*S$ is uniquely determined (independently of the choice of φ).*

We consider the following condition on a subset $E\subset\mathfrak{F}_{\sigma}$:

(4′) *There exists a function $\varphi\in\Phi$ such that $\sigma_{1/k}\varphi\in\bigcap_{T\in E} D(T)$ for $k = 1, 2, \ldots$ and $\int\varphi\,dx\neq 0$.*

Definition 6. For a subset $E\subset\mathfrak{F}_{\sigma}$ satisfying condition (4′) the *φ-dual* of E is defined by

$$E^{\varphi} = \{T\in\mathfrak{F}_{\sigma}\,|\,\text{for any } S\in E \text{ the pair } S, \sigma_{-1}T \text{ is } \varphi\text{-convolutive, and } S*\sigma_{-1}T\in L^{\infty}_{\mathrm{loc}}\}.$$

Such a space E that $E = E^{\varphi\varphi}$ is said to be *φ-perfect.*

We fix an ultra-filter $\mathscr{K}$ in the set of all natural numbers k's such that $\lim_{\mathscr{K}} k = \infty$. The duality measure is defined by

$$\tilde{\delta} = \lim_{\mathscr{K}}\frac{k^n}{\langle\varphi, 1\rangle^2}\,\sigma_{1/k}\varphi*\sigma_{1/k}\varphi, \tag{5}$$

and the scalar product of $T\in E$ and $S\in E^{\varphi}$ by $\langle T, S\rangle_{\tilde{\sigma}} = \langle T*\sigma_{-1}\tilde{S}, \delta\rangle$.

Proposition 4. *A φ-perfect space E is total on the φ-dual E^{φ}.*

Proof. This follows immediately from relation (5) and the following

Lemma. *A φ-perfect space E contains $\sigma_{-1}(\sigma_{1/k}\varphi*\sigma_{1/k}\varphi)$ for $k = 1, 2, \ldots$*

Proof. The fact $\sigma_{1/k}\varphi\in D(S)$ for $S\in E^{\varphi}$ implies $S*\sigma_{1/k}\varphi\in\Omega$ hence $S*\sigma_{1/k}\varphi*\sigma_{1/k}\varphi\in\Omega*\sigma_{1/k}\varphi\in L^{\infty}_{\mathrm{loc}}$. Hence $\sigma_{-1}(\sigma_{1/k}\varphi*\sigma_{1/k}\varphi)\in E^{\varphi\varphi} = E$.

More precisely, we can prove without difficulty

Proposition 5. *For any element T in a φ-perfect space E and for any bounded set B in Φ, the set $N(T\sigma_{1/k}\varphi)*\sigma_{1/k}\varphi*B$ is bounded with respect to $\beta(E, E^{\varphi})$, where $N(T\sigma_{1/k}\varphi)$ is the normal hull of $T\sigma_{1/k}\varphi$: $\{f\in\Omega\,|\,|f|\leqslant|T\sigma_{1/k}\varphi|\}$.*

Definition 7. For a φ-perfect space E, the subspace E_r = the $\beta(E, E^{\varphi})$-closure of $E_0*\Phi$ is called the *regular subspace* of E, where $E_0 = \{T\in E\,|\,T$ has a compact support$\}$.

Our main results are:

THEOREM. 1° *A φ-perfect space E is complete with respect to the strong topology $\beta(E, E^\varphi)$.*

2° *$E^\beta = E_r'$ for a φ-perfect space E.*

3° *A φ-perfect space E is separable with respect to the strong topology $\beta(E, E^\varphi)$ if and only if $E = E_r$.*

Proof. 1° Let T_0 be an element of the $\beta(E, E^\varphi)$-completion of E, that is, $T_0 = \beta(E, E^\varphi)\text{-}\lim_a T_a$ for some net $\{T_a\} \subset E$. In the same way as in the proof 1° of the theorem in § 2, the element T_0 is identified to an operator in $\mathfrak{F}_\sigma$. We have only to show that T_0 is convolutive with any element S in E^φ. Since $N(S * \sigma_{1/k}\varphi) * \sigma_{1/k}\varphi * B$ is bounded in E^φ by Proposition 5, we obtain

$$\langle (T_a \sigma_{1/k}\varphi) * f, g\rangle \to \langle T_0 \sigma_{1/k}\varphi * f, g\rangle \text{ uniformly in } f \in N(S * \sigma_{1/k}\varphi), \quad g \in B.$$

Hence the net of functions $|T_a \sigma_{1/k}\varphi| * |S * \sigma_{1/k}\varphi|$ is convergent in Ω with respect to $\beta(\Omega, \Phi)$ and we have $|T_0 \sigma_{1/k}\varphi| * |S * \sigma_{1/k}\varphi| \in \Omega$.

2° Here E_r' means the topological dual of the regular subspace E_r with respect to the topology induced by $\beta(E, E^\varphi)$, and the proof is similar to the proof of the theorem in § 2.

3° We shall prove only the separability of E_r. Let $C(K) = \{f \in (C) | f(x) = 0 \text{ for } |x| > K\}$. $C(K)$ is a Banach space with respect to the topology induced by (C), and the inductive limit space (C_0) of $C(K)$, $K = 1, 2, \ldots$, is an (LF)-space. Note that $E_0 * \Phi * \sigma_{1/k}\varphi \subset (C_0)$ and that $(C_0) *_{1/k} \varphi * \sigma_{1/k}\varphi \subset E_0 * \Phi$. Since the set

$$\bigcup_k E_0 * \Phi * \sigma_{1/k}\varphi * \sigma_{1/k}\varphi$$

is dense in $E_0 * \Phi$ with respect to $\beta(E, E^\varphi)$, the $\beta(E, E^\varphi)$-closure of the set $\bigcup_k (C_0) * \sigma_{1/k}\varphi * \sigma_{1/k}\varphi$ is equal to the $\beta(E, E^\varphi)$-closure of $E_0 * \Phi$. But since the space (C_0) is separable and since the convolution operator $\sigma_{1/k}\varphi * : f \to \sigma_{1/k}\varphi * f$ is continuous from (C_0) to E with respect to $\beta(E, E^\varphi)$, the set $(C_0) * \sigma_{1/k}\varphi * \sigma_{1/k}\varphi$ is separable with respect to the topology induced by $\beta(E, E^\varphi)$. Hence the $\beta(E, E^\varphi)$-closure of $\bigcup_k (C_0) * \sigma_{1/k}\varphi * \sigma_{1/k}\varphi =$ the $\beta(E, E^\varphi)$-closure of $E_0 * \Phi$ is separable.

Example 1. Let φ be an arbitrary function in Φ such that $\int \varphi\, dx \neq 0$. Then the space L^p for $1 < p < \infty$ is φ-perfect and the φ-dual is the space L^q for $1/p + 1/q = 1$. The regular subspace L_r^q is equal to L^q itself. The space L^∞ is φ-perfect also, and the regular subspace L_r^∞ is the space $\{f \in (C) | \lim_{|x| \to \infty} f(x) = 0\}$. The dual $(L^\infty)^\varphi$ is the space M^1 of all bounded measures. The regular subspace of M^1 is the space L^1.

Example 2. Let φ be an arbitrary function in the space $(\mathscr{D})$ of Schwartz such that $\int \varphi\, dx \neq 0$. Then spaces $(\mathscr{E})$, $(\mathscr{D})$ and $(\mathscr{S})$ of Schwartz are all φ-perfect, and their φ-duals are respectively $(\mathscr{E}')$, $(\mathscr{D}')$ and $(\mathscr{S}')$. These six spaces are all separable, hence they are equal to their regular subspaces.

In these cases φ-duals do not depend on the choice of φ. It is interesting to introduce a notion of dual independent of φ. We may say that a pair of function type operators is *convolutive* if for some φ it is φ-convolutive. Let us define the *S-dual* of a space $E \subset \mathfrak{F}_\sigma$ by

$$E^s = \{T \in \mathfrak{F}_\sigma | \text{for any } S \in E, \text{ the pair } T, \sigma_{-1}S \text{ is convolutive and } T * \sigma_{-1}S \in L_{\mathrm{loc}}^\infty\}.$$

Each φ-perfect space E in the above examples satisfy that $E = E^{ss}$. A space E may be said to be *S-perfect* if $E = E^{ss}$. It seems more natural to consider S-duals and S-perfect spaces. But this leads us to complicated problems. We do not know for instance whether an S-perfect space E is complete with respect to the strong topology $\beta(E, E^s)$.

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