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On the continuous dependence of local analytic solutions of a functional equation on given functions

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We consider the problem of the continuous dependence on the given functions, for local analytic solutions of the equation

$$\Phi(z) = H_0\big(z, \Phi[f_0(z)]\big), \tag{1}$$

where H_0 and f_0 are given functions and Φ is unknown.

This problem was investigated in [2] for the equation

$$\Phi[f_0(z)] + g_0(z)\Phi(z) = h_0(z).$$

Together with equation (1) we consider the sequence of equations

$$\Phi(z) = H_n\big(z, \Phi[f_n(z)]\big) \quad (n = 1, 2, \ldots). \tag{2}$$

We shall assume that for $n = 0, 1, 2, \ldots$:

(I) *f_n is analytic in the disc $|z| \leqslant r_0$, $f_n(0) = 0$, and*

$$|f_n'(0)| \leqslant \vartheta < 1.$$

(II) *H_n is an analytic function of two complex variables (z, w) for $|z| \leqslant r_0$; $|w| \leqslant R_0$ and*

$$H_n(0, 0) = 0,$$

and

(III) *$f_n \to f_0$, $H_n \to H_0$ uniformly for $|z| \leqslant r_0$, $|w| \leqslant R_0$.*

By (I) and (III) there exists a positive integer p such that

$$\left| [f_n'(0)]^p \frac{\partial H_n}{\partial w}(0, 0) \right| < 1 \quad (n = 0, 1, 2, \ldots). \tag{3}$$

Further, we suppose that for $n = 0, 1, 2, \ldots$

$$\text{(IV)} \qquad [f_n'(0)]^k \frac{\partial H_n}{\partial w}(0, 0) \neq 1 \quad (k = 1, 2, \ldots, p-1).$$

It follows from W. Smajdor's theorem [4] (cf. also [1], p. 188, and [3]) that for every $n = 0, 1, 2, \ldots$ there exists exactly one solution Φ_n of equation (2) analytic in a neighbourhood of $z = 0$.

In this paper we give the proof of the following

THEOREM. *If hypotheses* (I)-(IV) *are fulfilled, then solutions* Φ_n *exist in a common neighbourhood of* $z = 0$ *and* Φ_n *tends to* Φ_0 *uniformly in this neighbourhood.*

First we prove two lemmas.

LEMMA 1. *Let* Φ_n *be a solution of equation* (2) *and let hypotheses* (I)-(IV) *be fulfilled. Then* $\Phi_n^{(k)}(0)$ *tends to* $\Phi_0^{(k)}(0)$ *as* $n \to \infty$ *for* $k = 1, 2, \ldots$

Proof. We define the functions $H_{n,k}(z, w, w_1, \ldots, w_k)$ by the recurrent relations

$$\begin{gathered} H_{n,1}(z, w, w_1) = \frac{\partial H_n}{\partial z} + f'_n(z)\frac{\partial H_n}{\partial w} w_1, \\ H_{n,k+1}(z, w, w_1, \ldots, w_{k+1}) = \frac{\partial H_{n,k}}{\partial z} + f'_n(z)\left(\frac{\partial H_{n,k}}{\partial w} w_1 + \ldots + \frac{\partial H_{n,k}}{\partial w_k} w_{k+1}\right) \end{gathered} \tag{4}$$

$(k = 1, 2, \ldots)$.

$H_{n,k}$ are analytic functions of variables $z, w, w_1, \ldots, w_k$ in the domain $D_k = \{(z, w, w_1, \ldots, w_k)\colon\ |z| \leqslant r_0;\ |w| \leqslant R_0;\ w_i \in C,\ i = 1, \ldots, k\}$, where C is a complex plane. Moreover, we have [4]

$$H_{n,k}(z, w, \ldots, w_k) = G_{n,k}(z, w, \ldots, w_{k-1}) + \frac{\partial H_n}{\partial w}[f'_n(z)]^k w_k, \tag{5}$$

where $G_{n,k}$ is analytic in D_{k-1}, and

$$\Phi_n^{(k)}(0) = H_{n,k}\big(0, 0, \Phi'_n(0), \ldots, \Phi_n^{(k)}(0)\big). \tag{6}$$

Hence and from (5), (3), (IV) we get

$$\Phi_n^{(k)}(0) = \frac{G_{n,k}\big(0, 0, \Phi'_n(0), \ldots, \Phi_n^{(k-1)}(0)\big)}{1 - [f'_n(0)]^k \dfrac{\partial H_n}{\partial w}(0, 0)}. \tag{7}$$

Since

$$\Phi'_n(0) = \frac{\dfrac{\partial H_n}{\partial z}(0, 0)}{1 - f'_n(0)\dfrac{\partial H_n}{\partial w}(0, 0)}$$

it follows from (III) and (IV) that $\Phi'_n(0)$ tends to

$$\Phi'_0(0) = \frac{\dfrac{\partial H_0}{\partial z}(0, 0)}{1 - f'_0(0)\dfrac{\partial H_0}{\partial w}(0, 0)}.$$

Thus Lemma 1 is true for $k = 1$.

It follows by induction from (III) that $H_{n,k}$ converges to $H_{0,k}$ and $G_{n,k}$ converges to $G_{0,k}$ uniformly on every compact $K \subset D_k$. The proof of Lemma 1 results hence by induction in view of (7).

LEMMA 2. *Suppose that*

1° *A is a compact metric space,*

2° *T_n is a continuous transformation of A into itself,*

3° *T_n converges to T_0 uniformly in A,*

4° *there exists exactly one fixpoint x_n of T_n in A for $n = 0, 1, 2, \ldots$*

Then x_n converges to x_0.

Proof. Suppose that Lemma is false. Then exists a subsequence x_{k_n} such that $\lim_{n\to\infty} x_{k_n} = y$ and $y \neq x_0$. From 2° and 3° we have

$$y = \lim_{n\to\infty} x_{k_n} = \lim_{n\to\infty} T_{k_n}[x_{k_n}] = T_0[y].$$

It is a contradiction with 4° and the lemma is proved.

Proof of the theorem. Let Φ_n be the analytic solution of equation (2). Evidently, we may write

$$\Phi_n(z) = P_n(z) + z^p \varphi_n(z); \qquad P_n(z) = \sum_{s=1}^{p} \frac{\Phi_n^{(s)}(0)}{s!} z^s, \tag{8}$$

φ_n is analytic at $z = 0$ and $\varphi_n(0) = 0$.

According to Lemma 1, for the proof our theorem it is enough to show that φ_n converges uniformly to φ_0 in a neighbourhood of $z = 0$.

Let us define the functions

$$h_n(z, v) = \frac{H_n(z, P_n[f_n(z)] + [f_n(z)]^p v) - P_n(z)}{z^p}. \tag{9}$$

By (I) and (II) the partial derivative

$$\frac{\partial h_n}{\partial v}(z, v) = \frac{\partial H_n}{\partial w}(z, w)\left[\frac{f_n(z)}{z}\right]^p, \qquad w = P_n[f_n(z)] + [f_n(z)]^p v \tag{10}$$

is an analytic function at $(z, v) = (0, 0)$. Next we put

$$g(z) = H_n(z, P_n[f_n(z)]) - P_n(z).$$

We shall show that

$$g^{(s)}z = H_{n,s}(z, P_n[f_n(z)], P_n'[f_n(z)], \ldots, P_n^{(s)}[f_n(z)]) - P_n^{(s)}(z). \tag{11}$$

In fact, we have by (4)

$$\begin{aligned} g'(z) &= \frac{\partial H_n}{\partial z}(z, P_n[f_n(z)]) + \frac{\partial H_n}{\partial w}(z, P_n[f_n(z)]) f_n'(z) P_n'[f_n(z)] - P_n'(z) \\ &= H_{n,1}(z, P_n[f_n(z)], P_n'[f_n(z)]) - P_n'(z). \end{aligned}$$

Thus (11) is true for $s = 1$. We assume that (11) holds for an $s \geqslant 1$. Hence we get

$$g^{(s+1)}(z)$$

$$= \frac{\partial H_{n,s}}{\partial z} + f_n'(z)\left[\frac{\partial H_{n,s}}{\partial w} P_n'[f_n(z)] + \ldots + \frac{\partial H_{n,s}}{\partial w_s} P_n^{(s+1)}[f_n(z)]\right] - P_n^{(s+1)}(z)$$

$$= H_{n,s+1}\big(z, P_n[f_n(z)], \ldots, P_n^{(s+1)}[f_n(z)]\big) - P_n^{(s+1)}(z)$$

and (11) is proved.

From (8) we obtain $P_n^{(s)}(0) = \Phi_n^{(s)}(0)$, $s = 1, 2, \ldots, p$. Now, putting $z = 0$ in (11), we have by (6)

$$g^{(s)}(0) = H_{n,s}\big(0, \Phi_n(0), \Phi_n'(0), \ldots, \Phi_n^{(s)}(0)\big) - \Phi_n^{(s)}(0) = 0, \quad s = 1, \ldots, p$$

so $h_n(z, 0)$ is an analytic function of z at the point $z = 0$. Hence and from (10) we conclude that h_n is analytic at $(z, v) = (0, 0)$. Moreover,

$$h_n(0, 0) = 0 \tag{12}$$

and φ_n defined by relation (8) satisfies the equation

$$\varphi(z) = h_n\big(z, \varphi[f_n(z)]\big). \tag{13}$$

Let us take an arbitrary $R_1 > 0$ and let $|v| \leqslant R_1$. Since $P_n(0) = f_n(0) = 0$, there is a $\sigma_1 > 0$, $\sigma_1 \leqslant r_0$, such that

$$|P_0[f_0(z)]| < \frac{R_0}{2} \quad \text{and} \quad |f_0(z)|^p < \frac{R_0}{2R_1} \quad \text{for } |z| \leqslant \sigma_1.$$

By Lemma 1 and (III) there is a positive integer N such that

$$|P_n[f_n(z)]| \leqslant \frac{R_0}{2} \quad \text{and} \quad |f_n(z)|^p \leqslant \frac{R_0}{2R_1} \quad \text{for } n \geqslant N \text{ and } |z| \leqslant \sigma_1.$$

From the continuity of $P_n[f_n(z)]$ and $f_n(z)$ there exists a $\sigma_2 > 0$ such that these inequalities are valid for $n = 1, \ldots, N-1$ and $|z| \leqslant \sigma_2$. Taking $r_1 = \min(\sigma_1, \sigma_2)$ we have

$$|P_n[f_n(z)] + [f_n(z)]^p v| \leqslant \frac{R_0}{2} + \frac{R_0}{2R_1} \cdot R_1 = R_0$$

for $(n = 1, 2, \ldots)$, and $|z| \leqslant r_1$.

Thus we see that h_n is analytic for $|z| \leqslant r_1$, $|v| \leqslant R_1$, $n = 0, 1, 2, \ldots$, and, moreover,

$$h_n \to h_0 \text{ uniformly for } |z| \leqslant r_1,\ |v| \leqslant R_1. \tag{14}$$

It follows from (3) that there exists a $\mu < 1$ such that

$$\left| f_n'(0)^p \frac{\partial H_n}{\partial w}(0, 0) \right| < \mu \quad (n = 0, 1, 2, \ldots). \tag{15}$$

By (10) we have

$$\frac{\partial h_n}{\partial v}(0,0) = f_n'(0)^p \frac{\partial H_n}{\partial w}(0,0).$$

Now, by (14) and (15) there exist numbers $r_2 > 0$ and $R_2 > 0$ such that

(16) $$|h_n(z, v_1) - h_n(z, v_2)| \leqslant \mu |v_1 - v_2|, \qquad |z| \leqslant r_2, \ |v_i| \leqslant R_2,$$

$i = 1, 2$ $(n = 0, 1, 2, \ldots)$.

Let us fix a K, $0 < K \leqslant R_2$. It follows from (12), (14) and from the continuity of h_0 that there exists an $r_3 > 0$ such that

(17) $$|h_n(z, 0)| \leqslant (1-\mu)K \quad \text{for } |z| \leqslant r_3 \ (n = 0, 1, 2, \ldots).$$

Moreover, by (I) and (III) there exists an $r_4 > 0$ such that

(18) $$|f_n(z)| \leqslant |z| \quad \text{for } |z| \leqslant r_4 \ (n = 0, 1, 2, \ldots).$$

Let us choose $r = \min(r_1, r_2, r_3, r_4)$. Define A as the set of analytic functions φ in the disc $|z| \leqslant r$ fulfilling the following condition

(19) $$|\varphi(z)| \leqslant K \quad \text{for } |z| \leqslant r \text{ and } \varphi(0) = 0.$$

Next, define the transformation $\psi = T_n[\varphi]$ by formula

(20) $$\psi(z) = h_n\big(z, \varphi[f_n(z)]\big).$$

We shall prove that the space A with the metric

$$\varrho(\varphi_1, \varphi_2) = \sup_{|z| \leqslant r} |\varphi_1(z) - \varphi_2(z)|$$

and the transformation T_n fulfils the conditions of Lemma 2.

1° By Vitali's theorem A is a compact metric space.

2° By (16) T_n is continuous. From (20), (18), (19), (16) and (17) we have

$$\begin{aligned}|\psi(z)| &\leqslant \big|h_n\big(z, \varphi[f_n(z)]\big) - h_n(z, 0)\big| + |h_n(z, 0)| \\ &\leqslant \mu |\varphi[f_n(z)]| + (1-\mu)K \leqslant K.\end{aligned}$$

Since $h_n(0, 0) = 0$, we have $\psi(0) = 0$. Thus $\psi \in A$ and this completes the proof of 2°.

3° Let us take an $\varepsilon > 0$. It follows from (14) that there exists an n_1 such that

(21) $$|h_n(z, v) - h_0(z, v)| \leqslant \varepsilon(1-\mu) \quad \text{for } |z| \leqslant r, \ |v| \leqslant R_2, \ n \geqslant n_1.$$

There is an n_2 such that for $n \geqslant n_2$, $|z| \leqslant r$ and every $\varphi \in A$

(22) $$|\varphi[f_n(z)] - \varphi[f_0(z)]| \leqslant \varepsilon.$$

Indeed

$$|\varphi[f_n(z)]-\varphi[f_0(z)]| \leqslant c\,|f_n(z)-f_0(z)|.$$

where $c \overset{\mathrm{df}}{=} \sup\limits_{\varphi\in A}\big(\sup\limits_{|\xi|\leqslant r}|\varphi'(\xi)|\big)$. The number c must be finite, for in the opposite case A cannot be compact.

Now from (22), (16), and (21) we get, for $n \geqslant \max(n_1, n_2)$,

$$\begin{aligned}&\big|h_n\big(z, \varphi[f_n(z)]\big)-h_0\big(z, \varphi[f_0(z)]\big)\big| \\ &\quad \leqslant \big|h_n\big(z, \varphi[f_n(z)]\big)-h_0\big(z, \varphi[f_n(z)]\big)\big|+\big|h_0\big(z, \varphi[f_n(z)]\big)-h_0\big(z, \varphi[f_0(z)]\big)\big| \\ &\quad \leqslant \varepsilon(1-\mu)+\mu\,|\varphi[f_n(z)]-\varphi[f_0(z)]| \leqslant \varepsilon(1-\mu)+\mu\varepsilon = \varepsilon.\end{aligned}$$

Taking supremum on the left-hand side we get

$$\varrho(T_n[\varphi], T_0[\varphi]) \leqslant \varepsilon \quad \text{for } n \geqslant \max(n_1, n_2) \text{ and } \varphi\in A.$$

This proves 3°.

Let $\varphi_1, \varphi_2 \in A$, $\psi_1 = T_n[\varphi_1]$, $\psi_2 = T_n[\varphi_2]$. From (16) and (18) we get

$$\begin{aligned}\varrho(\psi_1, \psi_2) &= \sup_{|z|\leqslant r}\big|h_n\big(z, \varphi_1[f_n(z)]\big)-h_n\big(z, \varphi_2[f_n(z)]\big)\big| \\ &\leqslant \mu \sup_{|z|\leqslant r}|\varphi_1[f_n(z)]-\varphi_2[f_n(z)]| \leqslant \mu\varrho(\varphi_1, \varphi_2).\end{aligned}$$

Since $\mu < 1$, T_n is a contraction and 4° follows from Banach's principle.

Now Lemma 2 completes the proof.

References

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