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THE CONJUNCTION OF A PARTICULAR FORM OF MAJORITY LOGIC FUNCTIONS AND SOME OF ITS PROPERTIES

1. Formulation of the problem. Many practical problems, met in engineering and in natural sciences and, particularly, in information theory, theory of finite dynamic systems and system reliability theory, can be reduced to the solution of a problem which may be formulated in the logical function language as follows.

Let $f(x)$ denote a function of the class P_2 of binary logic functions determined on a set of arguments X, the elements of which are the vectors $x = (x_1, x_2, \ldots, x_r)$, such that $x_i \epsilon \{0, 1\}$ for $i = 1, 2, \ldots, r$, and $f(x) \epsilon \{0, 1\}$.

Definition 1. A function $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_r) \subset P_2$ *does not depend substantially upon the variable* x_i if

$$f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_r) \equiv f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_r).$$

Let us assume that $f(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$, $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant r$, with $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, r\}$ is a function determined on a set of arguments X whereas the variables not belonging to the set $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ are the variables upon which the function does not depend substantially.

Definition 2. $f(x) = f(x_1, x_2, \ldots, x_r)$ is called a *majority logic function* if there exists such a k, $1 \leqslant k \leqslant r$, that

$$f(x) = \begin{cases} 1 & \text{if } S(x) \geqslant k, \\ 0 & \text{if } S(x) < k, \end{cases}$$

where $S(x)$ denotes the number of the components of $x = (x_1, x_2, \ldots, x_r)$ assuming the value 1. The number k is called the *degree* of the function.

Let us denote by W the class of majority logic binary functions.

Example 1. The function $f_1(x)$, defined in Table 1, is not a majority function, whereas the function $f_2(x)$ is a majority function of degree $k = 2$.

TABLE 1.

x_1	x_2	x_3	$S(x)$	$f_1(x)$	$f_2(x)$
0	0	0	0	0	0
0	0	1	1	0	0
0	1	0	1	1	0
0	1	1	2	1	1
1	0	0	1	0	0
1	0	1	2	1	1
1	1	0	2	1	1
1	1	1	3	1	1

Let $f_{i,n}(x) \subset P_2$, $x = (x_1, x_2, \ldots, x_r)$, $i \leqslant n \leqslant r$, $i = 1, 2, \ldots, n+m-1$, $m = 1, 2, \ldots, a < \infty$, denote a function depending substantially on n variables $x_i, x_{i+1}, \ldots, x_{i+n-1}$.

Consider the functions

$$f(x) = \bigcap_{i=1}^{m} f_{i,n}(x), \tag{1}$$

where $x = (x_1, x_2, \ldots, x_{n+m-1})$, $n \geqslant 1$, and

$$f_{i,n}(x) = \begin{cases} 1 \text{ for } x \text{ such that } S(x_i, x_{i+1}, \ldots, x_{i+n-1}) \geqslant k, \\ 0 \text{ for } x \text{ such that } S(x_i, x_{i+1}, \ldots, x_{i+n-1}) < k, \end{cases}$$
$$i = 1, 2, \ldots, m, \quad 1 \leqslant k \leqslant n.$$

The functions $f_{i,n}(x)$ are majority functions defined on sets of n arguments the elements of which are vectors $(x_i, x_{i+1}, \ldots, x_{i+n-1})$; therefore these functions are majority functions defined on those variables on which they depend substantially.

Having in mind the remark stated after definition 1, we may present function (1) in the form

$$f(x_1, x_2, \ldots, x_{n+m-1}) = f_{1,n}(x_1, x_2, \ldots, x_n) \wedge f_{2,n}(x_2, x_3, \ldots, x_{n+1}) \wedge \ldots \wedge f_{m,n}(x_m, x_{m+1}, \ldots, x_{n+m-1}),$$

where $f_{i,n}(x_i, x_{i+1}, \ldots, x_{i+n-1}) \subset W$, $i = 1, 2, \ldots, m$.

Let X_f^1 denote a subset of the argument $x = (x_1, x_2, \ldots, x_{n+m-1})$ for which the function $f(x)$ assumes the value 1, i.e.

$$X_f^1 = \{x : f(x) = 1\},$$

and let μ_p denote the power of the subset of elements of X_f^1 for which $S(x) = p$, that is let

$$\mu_p = \mathrm{Card}\{x : [S(x) = p] \wedge [x \in X_f^1]\}.$$

The problem to be solved involves the determination of the vector $\mu = (\mu_0, \mu_1, \ldots, \mu_{n+m-1})$ for the function $f(x)$ defined by (1) with fixed k, n, m.

Example 2. Determine μ for $k = 1$, $n = 2$, $m = 2$, i.e. for

$$f(x_1, x_2, x_3) = f_{1,2}(x_1, x_2) \wedge f_{2,2}(x_2, x_3),$$

where $f_{1,2}$ and $f_{2,2}$ are majority functions of degree $k = 1$.

The results of subsequent calculations are shown in Table 2. Thus $\mu = (0, 1, 3, 1)$.

TABLE 2.

x_1	x_2	x_3	$S(x_1, x_2)$	$S(x_2, x_3)$	$f_{1,2}(x)$	$f_{2,2}(x)$	$f(x)$	$S(x)$
0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	1	0	1
0	1	0	1	1	1	1	1	1
0	1	1	1	2	1	1	1	2
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	2
1	1	0	2	1	1	1	1	2
1	1	1	2	2	1	1	1	3

Fig. 1 illustrates the result by black circles indicating the edges in binary three-dimensional space in which the function $f(x)$ takes the value 1.

Although the above algorithm is simple, the use of it for large values of $n+m-1$ is rather troublesome (even if a computer is applied).

We examine in the present paper certain general properties of the considered function that will be helpful while developing the algorithm of the formulated problem. Afterwards a solution algorithm will be given for the case $m \leqslant n+1$.

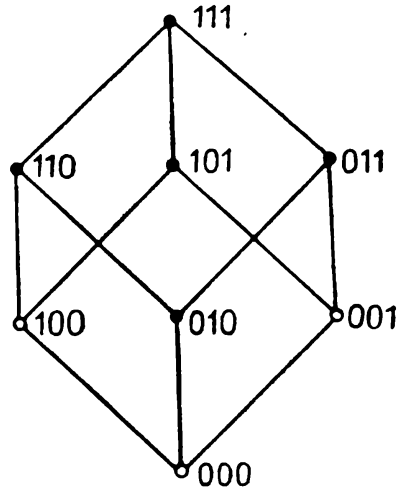

Fig. 1

2. Certain general properties of the considered function. Write $x_i = (x_{1i}, x_{2i}, \ldots, x_{ri})$. Let $x_i \succcurlyeq x_j$ mean that $x_{1i} \geqslant x_{1j}$, $x_{2i} \geqslant x_{2j}, \ldots, x_{ri} \geqslant x_{rj}$, and that $x_i \not\equiv x_j$ for $i \neq j$.

Definition 3. The function $f(x) \subset P_2$ is called *monotonic* if for two arbitrary elements of the set X, such that $x_i \in\!\!- x_j$, the relation $f(x_i) \geqslant f(x_j)$ holds.

We denote by M the class of monotonic functions defined above.

LEMMA 1. *Function $f(x)$ defined by* (1) *belongs to the class M.*

Proof. Assume, contrary to the statement, that there exist such $x_i, x_j \in X$ $(i \neq j)$ for which $x_i \in\!\!- x_j$ and $f(x_i) < f(x_j)$, i.e. $f(x_j) = 1$ and $f(x_i) = 0$.

From the asumption $f(x_i) = 0$ it follows the existence of some p, $1 \leqslant p \leqslant m$, for which $f_{p,n}(x_i) = 0$. From the assumption $x_i \in\!\!- x_j$ we have, however, that for every p

$$S(x_{p,i}, x_{p+1,i}, \ldots, x_{p+n-1,i}) > S(x_{p,j}, x_{p+1,j}, \ldots, x_{p+n-1,j}),$$

and, since $f_{p,n} \subset W$ and $f_{p,n}(x_i) = 0$, we have $f_{p,n}(x_j) = 0$. However, from the assumption $f(x_j) = 1$ it follows that $f_{p,n}(x_j) = 1$ for every p, which contradicts the previously said.

This completes the proof.

From lemma 1 the two following properties of the function (1) follow directly:

PROPERTY 1. *If for the function $f(x)$, defined by* (1), *there exists a p, $0 \leqslant p \leqslant n+m-1$, such that $\mu_p = \binom{n+m-1}{p}$, then $\mu_s = \binom{n+m-1}{s}$ for every $s > p$, i.e. we have*

$$\Big[\bigvee_p \mu_p = \binom{n+m-1}{p}\Big] \Rightarrow \Big[\bigwedge_{s>p} \mu_s = \binom{n+m-1}{s}\Big].$$

PROPERTY 2. *If for the function $f(x)$, defined by* (1), *there exists a p, $0 \leqslant p \leqslant n+m-1$, such that $\mu_p = 0$, then $\mu_s = 0$ for every $s < p$, i.e. we have*

$$\big[\bigvee_p \mu_p = 0\big] \Rightarrow \big[\bigwedge_{s<p} \mu_s = 0\big].$$

Note that in the function (1) there is no variable on which it would not depend substantially. Note, too, that class W is contained in the class M.

Let us write

$$X^1_{\min_f} = \big\{x_i : [x_i \in X^1_f] \wedge \big[\bigwedge_{x_j \dashv x_i} x_j \notin X^1_f\big]\big\}.$$

It is easy to see that if $f(x) \subset M$, then for every $x_j \in\!\!- x_i$, where $x_i \in X^1_{\min_f}$, we have $f(x_j) = 1$.

If $f(x_1, x_2, \ldots, x_r) \subset W$, then

$$\text{Card } X^1_{\min_f} = \binom{r}{k},$$

where k is the degree of the given majority function. For any arbitrary argument pair x_i, x_j ($x_i \not\equiv x_j$) of the set $X^1_{\min_f}$ we have $S(x_i) = S(x_j) = k$, and if $x_i \in X^1_{\min_f}$, then every element x_j such that $S(x_j) = S(x_i)$ is also an element of the set $X^1_{\min_f}$.

If, however, $f(x) \subset P_2$ belongs to class M and does not belong to class W, then there exist x_i, x_j in the set $X^1_{\min_f}$ such that $S(x_i) \neq S(x_j)$.

Lemma 2. *The function $f(x)$ defined by* (1) *does not belong to the class W with the exception of the trivial case $k = n$ or $m = 1$.*

Proof. Note that every element $x \in X$ such that

$$S(x_1, x_2, \ldots, x_n) = k, \tag{2}$$

$$x_{n+1} = x_1, x_{n+2} = x_2, \ldots, x_{2n} = x_n, x_{2n+1} = x_1, \ldots$$

is the element of the set $X^1_{\min_f}$ of the function defined by (1).

Among the elements x satisfying (2) there exists always an element x_i such that $x_{p,i} = 1$ and $x_{p+1,i} = 0$, $1 \leqslant p \leqslant n+m-2$. Let x_j be an element of the set X such that for every l, $1 \leqslant l \leqslant n+m-1$, $l \neq p$, $l \neq p+1$ is $x_{li} = x_{lj}\, x_{pj} = 0$, and $x_{p+1,j} = 1$, so that $S(x_i) = S(x_j)$. Because of the assumption that x_i satisfies (2) it follows that $S(x_{p-n+1,i}, x_{p-n+2,i}, \ldots, x_{p,i}) = k$ and, since $x_{pj} = 0$, we have, $S(x_{p-n+1,j}, x_{p-n+2,j}, \ldots, x_{p,j}) = k-1$.

As $x_j \notin X^1_{\min_f}$ and because x_i is by assumption an element of $X^1_{\min_f}$ and $S(x_i) = S(x_j)$, the function $f(x)$ does not belong to the class W. Hence the lemma is proved.

From lemma 2 the following property of the considered function results:

Property 3. *For the function $f(x)$ defined by* (1) *there exists always, with the exception of the trivial cases $k = n$ and $m = i$, a p $(0 < p < n+m-1)$ such that*

$$0 < \mu_p < \binom{n+m-1}{p}.$$

Let ϱ and η denote such minimum numbers that $\mu_\varrho \neq 0$ and $\mu_\eta = \binom{n+m-1}{\eta}$, respectively.

Since

$$\varrho = \min_{x_i \in X^1_{\min_f}} S(x_i)$$

and since every element of the set $X^1_{\min_f}$ satisfies condition (2), we have

$$\min_{x_i \in X^1_{\min_f}} S(x_i) = S(x_1 = 0, \ldots, x_{n-k} = 0, x_{n-k+1} = 1, \ldots, x_n = 1, x_{n+1} = x_1, \ldots, x_{2n} = x_n, x_{2n+1} = x_1, \ldots, x_{n+m-1} = x_\omega),$$

where

$$\omega = \begin{cases} n+m-1-E\left[\dfrac{n+m-1}{n}\right]n & \text{if } m \neq n(c-1)+1, \\ n & \text{if } m = n(c-1)+1, \end{cases} \quad c = 1, 2, \ldots$$

Thus

$$\varrho = E\left[\frac{n+m-1}{n}\right]k + [\omega - (n-k)]\,\delta^*(m, n),$$

where

$$\delta^*(m, n) = \begin{cases} 1 & \text{if } m \neq n(c-1)+1, \\ 0 & \text{if } m = n(c-1)+1, \end{cases} \quad c = 1, 2, \ldots,$$

or

$$\varrho = E\left[\frac{n+m-1}{n}\right]k + z\delta(z), \tag{3}$$

where

$$z = m+k-1-E\left[\frac{n+m-1}{n}\right]n, \qquad \delta(z) = \begin{cases} 1 & \text{for } z \geqslant 0, \\ 0 & \text{for } z < 0. \end{cases}$$

From

$$\eta = S(x_1 = 0, \ldots, x_{n-k} = 0, x_{n-k+1} = 1, \ldots, x_{n+m-1} = 1)$$

it follows

$$\eta = m+k-1. \tag{4}$$

Thus, the difficulties connected with the calculation of the vector μ consist in the determination of the components μ_s for $\varrho \leqslant s < \eta$.

3. Solution of the problem for the case $m \leqslant n+1$. The solution will be sought with the help of combinatorial methods. At first we shall show that the following combinatorial problem is equivalent to the given one.

Let N denote a set with $n+m-1$ numbers $1, 2, \ldots, n+m-1$. From the set $P(N)$ of all subsets of the set N we choose m following subsets with n elements each:

$$N_1 = \{1, 2, \ldots, n\},$$
$$N_2 = \{2, 3, \ldots, n+1\},$$
$$\ldots$$
$$N_m = \{m, m+1, \ldots, n+m-1\}.$$

Let $D = (a_1, a_2, \ldots, a_p)$, $a_1 < a_2 < \ldots < a_p$, $1 \leqslant p \leqslant n+m-1$, denote an ordered selection of p elements of the set N and $K_i(D)$ — the number of the elements in the selection D, which simultaneously are elements of the subset N_i,

$$K_i(D) = \mathrm{Card}\{a : (a \in D) \wedge (a \in N_i)\}.$$

We want to know the number of ways a selection can be made in such a manner that the following conditions will be satisfied:

$$K_i \geqslant k, \quad i = 1, 2, \ldots, m. \tag{5}$$

The answer to this question is equivalent to answering the question what is the value of μ_p, since for any x such that the components x_{a_1}, $x_{a_2}, \ldots, x_{a_p}$ take the value i and the remaining ones are zero we have $S(x) = p$ and $f(x) = 1$.

Divide the set N into three following subsets:

$$A = \{1, 2, \ldots, m-1\},$$
$$B = \{m, m+1, \ldots, n\},$$
$$C = \{n+1, n+2, \ldots, n+m-1\}.$$

Let $M_A(D)$, $M_B(D)$, $M_C(D)$ denote the numbers of elements of D which are elements of the sets A,B,C, respectively.

Thus, for any arbitrary selection $D = (a_1, a_2, \ldots, a_p)$ we have

$$0 \leqslant M_A(D) \leqslant m-1, \quad 0 \leqslant M_B(D) \leqslant n-m+1, \quad 0 \leqslant M_C(D) \leqslant m-1,$$

$$M_A(D) + M_B(D) + M_C(D) = p.$$

A necessary condition for the selection D to satisfy condition (5) is that it fulfills the following system of inequalities:

$$\begin{aligned} M_A(D) + M_B(D) &\geqslant k, \\ M_C(D) + M_B(D) &\geqslant k. \end{aligned} \tag{6}$$

Let $M = (M_A, M_B, M_C)$ denote any arbitrary combination of the values M_A, M_B, M_C satisfying condition (6), and $\{M\}$ — the set of all such combinations. Let $\mu_p(M)$ denote the number of selections D satisfying (5) for the combination M satisfying (6).

Thus

$$\mu_p = \sum_{\{M\}} \mu_p(M). \tag{7}$$

If $M_B \geqslant k$, then every selection $D = (a_1, a_2, \ldots, a_p)$ satisfying the necessary condition (6) satisfies also condition (5). Therefore

$$\mu_p(M_A, M_B \geqslant k, M_C) = \binom{m-1}{M_A}\binom{n-m+1}{M_B}\binom{m-1}{M_C}. \tag{8}$$

If $M_B < k$, then

$$\mu_p(M_A, M_B < k, M_C) \leqslant \mu_p(M_A, M_B \geqslant k, M_C),$$

since not all the selections D satisfying (6) satisfy (5).

Write $\varepsilon = k - M_B$, $R_A = M_A - \varepsilon = M_A + M_B - k$, $R_C = M_C - \varepsilon = M_C + M_B - k$.

Note that from the selections

$$D = (a_1, a_2, \ldots, a_{R_A}, a_{R_A+1}, \ldots, a_{R_A+\varepsilon}, a_{M_A+1}, \ldots, a_{M_A+M_B},$$
$$a_{M_A+M_B+1}, \ldots, a_{M_A+M_B+\varepsilon}, a_{k+M_B+1}, \ldots, a_{M_A+M_B+M_C}),$$

satisfying the necessary condition (6), condition (5) satisfy only those or which

$$a_{M_A+M_B+l} - a_{R_A+l} \leqslant n, \quad l = 1, 2, \ldots, \varepsilon. \tag{9}$$

Let

$$D^* = (a^*_{R_A+1}, \ldots, a^*_{R_A+\varepsilon}, a^*_{M_A+M_B+1}, \ldots, a^*_{M_A+M_B+\varepsilon})$$

denote an ordered selection of 2ε elements with $a^*_{R_A+1}, \ldots, a^*_{R_A+\varepsilon}$ belonging to the set $A = \{1, 2, \ldots, m-1\}$ and $a^*_{M_A+M_B+1}, \ldots, a^*_{M_A+M_B+\varepsilon}$ belonging to the set $C = \{n+1, \ldots, n+m-1\}$, and let

$$a^*_{M_A+M_B+l} - a^*_{R_A+l} \leqslant n, \quad l = 1, 2, \ldots, \varepsilon.$$

Let $\Phi(a^*_{R_A+1}, a^*_{M_A+M_B+\varepsilon})$ denote the number of possible different selections D^* for $a^*_{R_A+1}$ fixed and for $a^*_{M_A+M_B+\varepsilon}$.

Since every selection with

$$a_1 < a_2 < \ldots < a_{R_A} < a^*_{R_A+1},$$
$$a_{R_A+1} = a^*_{R_A+1}, \ldots, a_{R_A+\varepsilon} = a^*_{R_A+\varepsilon},$$
$$m \leqslant a_{M_A+1} < \ldots < a_{M_A+M_B} \leqslant n,$$
$$a_{M_A+M_B+1} = a^*_{M_A+M_A+1}, \ldots, a_{M_A+M_B+\varepsilon} = a^*_{M_A+M_B+\varepsilon},$$
$$a^*_{M_A+M_B+\varepsilon} < a_{k+M_A+1} < \ldots < a_{M_A+M_B+M_C} \leqslant n+m-1,$$

satisfies (5), then for the case $M_B < k$ we obtain

(10) $\mu_p(M)$

$$= \sum_{\substack{1\leqslant a^*_{R_A+1}\leqslant m-1 \\ n+1\leqslant a^*_{M_A+M_B+\varepsilon}\leqslant n+m-1}} \binom{a^*_{R_A+1}-1}{R_A}\Phi(a^*_{R_A+1}, a^*_{M_A+M_B+\varepsilon})\times$$
$$\times\binom{n-m+1}{M_B}\binom{n+m-1-a^*_{M_A+M_B+\varepsilon}}{R_C}.$$

Denoting by $N = (x_1, x_2, \ldots, x_\varepsilon, y_1, y_2, \ldots, y_\varepsilon)$ a selection of the set of numbers $\{1, 2, \ldots, m-1\}$ with

$$\text{(11)}\qquad \begin{cases} x_1 < x_2 < \ldots < x_\varepsilon, \\ y_1 < y_2 < \ldots < y_\varepsilon, \\ y_1 \leqslant x_1,\ y_2 \leqslant x_2, \ldots, y_\varepsilon \leqslant x_\varepsilon, \end{cases}$$

and by $\varphi(x, y, \varepsilon, m-1)$ the number of such selections for fixed $x_1 = x$ and $y_\varepsilon = y$, we may present (10) in the form

(12) $\mu_p(M_A, M_B < k, M_C)$

$$= \binom{n-m+1}{M_B}\sum_{\substack{1\leqslant x\leqslant m-1 \\ 1\leqslant y\leqslant m-1}}\varphi(x, y, \varepsilon, m-1)\binom{x-1}{R_A}\binom{m-1-y}{R_C}.$$

Let $\Psi^*(x_1, x_2, \ldots, x_\varepsilon)$ denote the number of selections of type N for fixed $x_1, x_2, \ldots, x_\varepsilon$.

Since every selection with

$$0 < y_1 < y_2 < \ldots < y_{a_1} \leqslant x_1,$$
$$x_1 < y_{a_1+1} < y_{a_1+2} < \ldots < y_{a_1+a_2} \leqslant x_2,$$

$$x_{\varepsilon-1} < y_{a_1+\ldots+a_{\varepsilon-1}+1} < \ldots < y_{a_1+\ldots+a_\varepsilon} \leqslant x_\varepsilon,$$

where $a_1 = 0, 1, \ldots, i = 1, 2, \ldots)$ satisfy the conditions

$$1 \leqslant a_1 \leqslant \varepsilon,$$
$$2 \leqslant a_1 + a_2 \leqslant \varepsilon,$$

(13)

satisfies the sufficient condition for being a selection of type N, we have

$$(14)\qquad \Psi^*(x_1, x_2, \ldots, x_\varepsilon) = \sum_{\{a\}_\varepsilon} \binom{x_1}{a_1}\binom{x_2-x_1}{a_2}\cdots\binom{x_\varepsilon-x_{\varepsilon-1}}{a_\varepsilon},$$

where $\{a\}_\varepsilon$ denotes the set $a = (a_1, a_2, \ldots, a_\varepsilon)$ satisfying (13).

Let $\varphi^*(x_1, x_2, \ldots, x_\varepsilon, y_\varepsilon)$ denote the number of selections of type N for fixed $x_1, x_2, \ldots, x_\varepsilon, y_\varepsilon$ $(x_\varepsilon \geqslant y_\varepsilon)$ (of course, for $x_\varepsilon < y_\varepsilon$ we have $\varphi^* = 0$).

Similarly, it can be easily shown that

$$(15)\qquad \varphi^*(x_1, x_2, \ldots, x_\varepsilon, y_\varepsilon)|_{x_1<y_\varepsilon} = \sum_{\{\beta\}_{\varepsilon-1,r}} \binom{x_1}{\beta_1}\binom{x_2-x_1}{\beta_2}\cdots\binom{x_r-x_{r-1}}{\beta_r}\binom{y_\varepsilon-x_r-1}{\beta_{r+1}},$$

where r is the maximum number such that $x_r < y_\varepsilon$ and where $\{\beta\}_{\varepsilon-1,r}$ denotes the set of vectors $\beta = (\beta_1, \beta_2, \ldots, \beta_{r+1})$ $(\beta_i = 0, 1, \ldots; i = 1, 2, \ldots \ldots, r+1)$ satisfying the conditions

$$(16)\qquad \begin{aligned} 1 &\leqslant \beta_1 \leqslant \varepsilon-1, \\ 2 &\leqslant \beta_1+\beta_2 \leqslant \varepsilon-1, \\ &\ \ \vdots \\ r &\leqslant \beta_1+\beta_2+\ldots+\beta_r \leqslant \varepsilon-1, \\ &\beta_1+\beta_2+\ldots+\beta_{r+1} = \varepsilon-1, \end{aligned}$$

and

$$(17)\qquad \varphi^*(x_1, x_2, \ldots, x_\varepsilon, y_\varepsilon)|_{x_1\geqslant y_\varepsilon} = \binom{y_\varepsilon-1}{\varepsilon-1}.$$

Therefore

$$(18)\qquad \varphi(x, y, \varepsilon, m-1)|_{x<y-\varepsilon+1} = \sum_{x=x_1<\ldots<x_\varepsilon\leqslant m-1} \varphi^*(x_1, x_2, \ldots, x_\varepsilon, y)$$

and

$$(19)\qquad \varphi(x, y, \varepsilon, m-1)|_{x\geqslant y-\varepsilon+1} = \binom{m-1-x}{\varepsilon-1}\binom{y-1}{\varepsilon-1}.$$

4. Applications. Let us now point out the connections of the problem considered with certain application areas.

From the point of view of reliability theory the functions considered in this paper describe the so called *coherent reliability structures* ([1]-[7]).

From the theory of finite automata as well as from the point of view of information theory, the functions considered describe a certain class of so called *sequential majority decision systems* used e.g. for signal detection with noisy background and for the correction of errors under information transmission ([6]-[8]). Finally, from the point of view of logical function theory the functions considered belong to the class of monotonic logical functions [4].

An application of the results obtained in the present work can be illustrated by the following example:

Example 3. Assume that $x_1, x_2, \ldots, x_{n+m-1}$ are two-point, mutually independent random variables with identical probability distributions

$$\Pr\,[x_i = 1] = P, \quad i = 1, 2, \ldots, n+m-1.$$

We search the probability P_d of the event that the function (1) will assume the value 1, i.e.

$$P_d = \Pr[f(x) = 1].$$

Note that

$$P_d = \sum_{j=1}^{n+m-1} \mu_j P^j (1-P)^{n+m-1-j}.$$

Thus, the problem is reduced to the determination of the vector

$$\mu = (\mu_0, \mu_1, \ldots, \mu_{n+m-1}).$$

The vector μ can be determined for arbitrary, finite parameters k, n, m $(k \leqslant n)$ in a "trivial" manner by determining the value of the function $f(x)$ for every $x_i \in X$ and by finding

$$\mu_p = \mathrm{Card}\{x : [S(x) = p] \wedge [f(x) = 1]\}.$$

Example 4. Determine the values of the function $\varphi(x, y, \varepsilon = 3, m-1 = 5)$ for all values (x, y).

The values of φ for (x, y) such that $x \geqslant y-2$ may be determined directly from (19). The values of the function for (x, y) such that $x < y-2$ are determined from (18) and (15) by taking into consideration $\varphi^*(x_1, x_2, \ldots, x_\varepsilon < y_\varepsilon, y_\varepsilon) = 0$.

Therefore

By analogy, we obtain

$$\varphi(1, 4, 3, 5) = 8.$$

Numerical results are presented in Table 3.

TABLE 3. Values of the function $\varphi(x, y, \varepsilon = 3, m-1 = 5)$.

$y \backslash x$	1	2	3	4	5
1	0	0	0	0	0
2	0	0	0	0	0
3	6	3	1	0	0
4	8	9	3	0	0
5	6	8	6	0	0

Example 5. Determine the vector μ for the function (1) in the case of $k = 3$, $n = 5$, $m = 6$ $(m \leqslant n+1)$.

From (3) and (6) we obtain $\varrho = 6$, $\eta = 8$. Thus

$$\mu_0 = \mu_1 = \ldots = \mu_5 = 0, \ \mu_8 = \binom{10}{8}, \ \mu_9 = \binom{10}{9}, \ \mu_{10} = 1.$$

Now we determine the value μ_6. Condition (6) is satisfied by one vector only, $M = (M_A = 3, M_B = 0, M_C = 3)$, for which $\varepsilon = 3$, $R_A = 0$, $R_C = 0$, so that from (12) we obtain

$$\mu_6 = \sum_{x,y} \varphi(x, y, \varepsilon = 3, m-1 = 5).$$

By the use of table 3 from the previous example, we obtain $\mu_6 = 50$.

Now we will try to determine the value μ_7. Condition (6) is satisfied by $M_1 = (4, 0, 3)$ and by $M_2 = (3, 0, 4)$ for which $\varepsilon_1 = \varepsilon_2 = 3$, $R_{A_1} = 1$, $R_{A_2} = 0$, $R_{C_1} = 0$, $R_{C_2} = 1$, respectively.

By the use of (7), (12) and from table 3 we obtain

$$\mu_7 = \sum_{x,y} \varphi(x, y, 3, 5)(x-1) + \sum_{x,y} \varphi(x, y, 3, 5)(5-y)$$
$$= [(3+9+8)+2(1+3+6)]+[(8+9+3)+2(6+3+1)] = 80.$$

Therefore $\mu = (0, 0, 0, 0, 0, 0, 50, 80, 45, 10, 1)$.

TABLE 4. Values of μ for some functions with $n+m-1 = 10$ and $n+1 \geqslant m$.

k	n	m	ϱ	η	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}
2	5	6	4	7	0	0	0	0	50	150	185	120	45	10	1
3	5	6	6	8	0	0	0	0	0	0	50	80	45	10	1
4	7	4	4	7	0	0	0	0	1	30	115	120	45	10	1
3	8	3	3	5	0	0	0	20	140	252	210	120	45	10	1
5	8	3	5	7	0	0	0	0	0	6	70	120	45	10	1
5	9	2	5	6	0	0	0	0	0	56	210	120	45	10	1

Applying the method developed in the present paper, values of μ for six selected functions with $n+m-1 = 10$, $n+1 \geqslant m$, were determined. The results of the calculations are presented in Table 4.

References

[1] R. E. Barlow and F. Proschan, *Mathematical theory of reliability*, J. Wiley, New York 1965.

[2] J. D. Esary and F. Proschan, *The reliability of coherent systems*, Boeing Scientific Research Laboratories, 1962.

[3] — *Coherent structures of non-identical components*, Boeing Scientific Research Laboratories, 1963.

[4] В. К. Коробков, *О монотонных функциях алгебры логики*, Проблемы кибернетики, вып. 13, Наука, Москва 1965, pp. 5-28.

[5] R. Kulesza i B. Korzan, *O problemie uwzględniania w procesie syntezy wpływu losowych zmian odwracalnych własności elementów funkcjonalnych*, Arch. Automatyki i Telemechaniki 9 (1964), pp. 443-452.

[6] R. Kulesza, *Badanie własności pewnego binarnego systemu dynamicznego w aspekcie niezawodności*, Prace IV Krajowej Konferencji Automatyki, zeszyt 3, Akademia Górniczo-Hutnicza, Kraków 1967, pp. 25-28.

[7] — *Zagadnienia niezawodności w wielkich systemach*, in *Optymalne wydobywanie informacji i sterowanie w sytuacjach niedeterministycznych*, Ossolineum 1968, pp. 414-425.

[8] С. Э. Кузьмин, *Цифровая обработка радиолокационной информации*, Советское Радио, Москва 1967.

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KONIUNKCJA WIĘKSZOŚCIOWYCH FUNKCJI LOGICZNYCH SZCZEGÓLNEJ POSTACI I PEWNE JEJ WŁASNOŚCI

STRESZCZENIE

Wiele zagadnień praktycznych z zakresu techniki i przyrody, spotykanych w szczególności w teorii informacji, teorii skończonych systemów dynamicznych i teorii niezawodności systemów, sprowadza się do rozwiązania problemu, który w języku dwuwartościowych funkcji logicznych można sformułować następująco. Dana jest funkcja

$$f(x_1, x_2, \ldots, x_{n+m-1}) =$$
$$= f_{1,n}(x_1, x_2, \ldots, x_n) \wedge f_{2,n}(x_2, x_3, \ldots, x_{n+1}) \wedge \ldots \wedge f_{m,n}(x_m, x_{m+1}, \ldots, x_{n+m-1}),$$

gdzie $f_{i,n}(x_i, x_{i+1}, \ldots, x_{i+n-1})$, $i = 1, 2, \ldots, m$, są funkcjami większościowymi stopnia k, $1 \leqslant k < n$.

Niech X_f^1 oznacza podzbiór zbioru argumentów, dla którego funkcja $f(x) = f(x_1, x_2, \ldots, x_{n+m-1})$ przyjmuje wartość 1, tj. niech

$$X_f^1 = \{x : f(x) = 1\},$$

a μ_p niech oznacza liczność podzbioru takich elementów zbioru X_f^1, dla których $S(x) = p$, czyli

$$\mu_p = \mathrm{Card}\{x : [x \in X_f^1] \wedge [S(x) = p]\},$$

gdzie $S(x)$ oznacza ilość składowych wektora x przyjmujących wartość 1.

Zadanie polega na wyznaczeniu wektora $\mu = (\mu_0, \mu_1, \ldots, \mu_{n+m-1})$ przy ustalonych k, n, m.

W pracy rozpatrzono niektóre ogólne własności badanej funkcji oraz podano algorytm rozwiązania zadania dla przypadku $m < n+1$.

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ОБЪЕДИНЕНИЕ МАЖОРИТАРНЫХ ЛОГИЧЕСКИХ ФУНКЦИЙ ОСОБОГО ВИДА И НЕКОТОРЫЕ ЕГО СВОЙСТВА

РЕЗЮМЕ

Много практических задач из техники и естествознания, встречающихся в теории информации, теории конечных динамических систем и теории надежности систем, сводится к решению проблемы, которая в языке двухзначных логических функции может быть сформулирована следующим образом.

Имеем функцию

$$f(x_1, x_2, \ldots, x_{n+m-1}) =$$
$$= f_{1,n}(x_1, x_2, \ldots, x_n) \wedge f_{2,n}(x_2, x_3, \ldots, x_{n+1}) \wedge \ldots \wedge f_{m,n}(x_m, x_{m+1}, \ldots, x_{n+m-1}),$$

где $f_{i,n}(x_i, x_{i+1}, \ldots, x_{i+n-1})$, $i = 1, 2, \ldots, m$, являются мажоритарными функциями ранга $k (1 \leqslant k < n)$.

Пусть X_f^1 обозначает подмножество множества аргументов, на котором функция $f(x) = f(x_1, x_2, \ldots, x_{n+m-1})$ принимает значение 1, т.е.

$$X_f^1 \{x : f(x) = 1\},$$

и пусть μ_p обозначает мощность подмножества таких элементов множества X_f^1, для которых $S(x) = p$, т.е.

$$\mu_p = \mathrm{Card}\{x : [x \in X_f^1] \wedge [S(x) = p]\},$$

где $S(x)$ является количеством компонентов вектора x принимающих значение 1. Задача состоит в определении вектора $\mu = (\mu_0, \mu_1, \ldots, \mu_{n+m-1})$ при фиксированных k, n, m.

В работе рассмотрены некоторые свойства исследуемой функции и определен алгоритм решения задачи для случая $m \leqslant n+1$.
