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A note on the large sieve

by

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Dedicated to the memory of Professor Harold Davenport

I. The basic large sieve inequality as formulated by Davenport and Halberstam [3] states that if $x_1, \ldots, x_R$ are distinct real numbers such that

$$\min_{i\neq j} \|x_i - x_j\| \geqslant \delta,$$

where $\|x\| = \min_n |x-n|$, and if

$$S(x) = \sum_{n=M+1}^{M+N} a_n e^{2\pi i n x}$$

then

$$\sum_{i=1}^{R} |S(x_i)|^2 \leqslant C\left(N+\frac{1}{\delta}\right) \sum_{n=M+1}^{M+N} |a_n|^2 \tag{1}$$

where C is an absolute constant. It was pointed out in [1] that one could take $c = 1+\varepsilon$ for $N\delta$ large enough, and later Bombieri and Davenport [2] improved (1) showing that one could take $N+5\delta^{-1}$ instead of $C(N+1/\delta)$. This result, with the improved constant, has some interesting consequences for the small sieve; see [1] and H. L. Montgomery [4].

The proof given in [2] is rather complicated. We want to give here a short proof of the following

Theorem. *Let x_i, $S(x)$, δ be as before. Then we have*

$$\sum_{i=1}^{R} |S(x_i)|^2 \leqslant \left(N+\frac{2}{\delta}\right) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{2}$$

The arguments used in the proof of (2) are new and very simple and were found in conversations with Professor P. X. Gallagher of Columbia

University and with Professor Atle Selberg of the Institute for Advanced Studies; in particular, I owe to Gallagher the remark that inequality (1) was best understood in the framework of the theory of Hilbert spaces (this was also done independently by Selberg), and I owe to Selberg both the statement and proof of Proposition 1.

II. We give here the proof of the theorem.

PROPOSITION 1 (Selberg). *Let H be a Hilbert space with inner product $(\ ,\ )$ and let $\varphi_1, \ldots, \varphi_R, f \in H$. Then we have*

$$\sum_{i=1}^{R} \frac{|(f, \varphi_i)|^2}{\sum_{j=1}^{R} |(\varphi_i, \varphi_j)|} \leqslant \|f\|^2.$$

Proof. The proof is by a Bessel inequality argument. We have

$$\Big\| f - \sum_{i=1}^{R} \xi_i \varphi_i \Big\|^2 \geqslant 0$$

for every complex number ξ_i. Hence

$$\|f\|^2 - 2\,\mathrm{Re} \sum_{i=1}^{R} \xi_i \overline{(f, \varphi_i)} + \sum_{i,j=1}^{R} \xi_i \bar{\xi}_j (\varphi_i, \varphi_j) \geqslant 0. \tag{3}$$

Now

$$\sum_{i,j=1}^{R} \xi_i \bar{\xi}_j (\varphi_i, \varphi_j) \leqslant \sum_{i,j=1}^{R} \tfrac{1}{2}(|\xi_i|^2 + |\xi_j|^2) |(\varphi_i, \varphi_j)|; \tag{4}$$

substituting (4) into (3) and taking

$$\xi_i = \frac{(f, \varphi_i)}{\sum_{i,j=1}^{R} |(\varphi_i, \varphi_j)|^2}$$

we get what we want.

In order to prove (2) we may suppose, replacing N by $2N$ or $2N+1$ and making a translation, that

$$S(x) = \sum_{n=-N}^{N} a_n e^{2\pi i n x}.$$

Now we take $H = l^2$, the Hilbert space of sequences $\alpha = (a_n)$, $-\infty < n < +\infty$, with inner product

$$(\alpha, \beta) = \sum_{n=-\infty}^{\infty} a_n \bar{\beta}_n$$

and norm

$$\|\alpha\|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2.$$

We apply Proposition 1 with

$$f = (f_n) = \begin{cases} a_n & \text{if} \quad |n| \leqslant N, \\ 0 & \text{if} \quad |n| > N \end{cases}$$

and

$$\varphi_j = (\varphi_{j_n}) = \begin{cases} e^{-2\pi i n x_j} & \text{if} \quad |n| \leqslant N, \\ L^{-1/2}(N+L-|n|)^{1/2} e^{-2\pi i n x_j} & \text{if} \quad N < |n| \leqslant N+L, \\ 0 & \text{if} \quad |n| > N+L \end{cases}$$

where L is a positive integer to be chosen later. We have

$$|(f, \varphi_i)|^2 = |S(x_i)|^2,$$

$$\|f\|^2 = \sum_{n=-N}^{N} |a_n|^2,$$

hence by Proposition 1 the inequality (2) will be proved if we show that

$$\sum_{j=1}^{R} |(\varphi_i, \varphi_j)| \leqslant 2N + \frac{2}{\delta} \tag{5}$$

for every i.

If

$$K_M(x) = \sum_{n=-M}^{M} (M - |n|) e^{2\pi i n x} = \left(\frac{\sin \pi M x}{\sin \pi x} \right)^2$$

we have

$$|(\varphi_i, \varphi_j)| = \frac{1}{L} \{ K_{L+N}(x_j - x_i) - K_N(x_j - x_i) \}$$

hence

$$|(\varphi_i, \varphi_i)| = 2N + L,$$

while if $j \neq i$

$$|(\varphi_i, \varphi_j)| \leqslant \frac{1}{L} \frac{1}{(\sin \pi (x_j - x_i))^2}.$$

Now we use the condition

$$\min_{i \neq j} \|x_i - x_j\| \geqslant \delta.$$

This implies that the intervals I_m given by

$$I_m:\ m\delta \leqslant |x_i - x| < (m+1)\delta,$$

where $(m+1)\delta \leqslant \frac{1}{2}$, contain at most two points x_j, while every x_j falls in some I_m. Using the inequality $\left|\sin\frac{\pi}{2}y\right| \geqslant |y|$ for $|y| \leqslant 1$ we get

$$\sum_{\substack{j=1\\ j\neq i}}^{R} \frac{1}{(\sin\pi(x_j - x_i))^2} = \sum_{\substack{m\neq 0\\ x_j\in I_m}} \frac{1}{(\sin\pi(x_j - x_i))^2} \leqslant 2\sum_{m=1}^{\infty}\frac{1}{4m^2\delta^2} = \frac{\pi^2}{12}\delta^{-2}.$$

It follows that

$$\sum_{j=1}^{R} |(\varphi_i, \varphi_j)| \leqslant 2N + L + \frac{1}{L}\,\frac{\pi^2}{12}\delta^{-2},$$

and (5) follows taking L the nearest integer to $\frac{\pi}{\sqrt{12}}\delta^{-1}$. This completes the proof.

References

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On inequalities of Large Sieve type

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In memory of my friend and teacher
Professor H. Davenport

1. Introduction. The inequality of the Large Sieve of Yu. V. Linnik, in a general form, can be expressed by:

Let $a_1, \ldots, a_N$ be N complex numbers, then

$$\sum_{q\leqslant Q}\ \sum_{\chi(\mathrm{mod}\,q)}^{*} \left|\sum_{n=1}^{N} a_n\chi(n)\right|^2 \leqslant \sum_{q\leqslant Q}\ \sum_{\substack{b=1\\(b,q)=1}}^{q} \left|\sum_{n=1}^{N} a_n \exp\left(2\pi i\frac{b}{q}n\right)\right|^2 \leqslant \delta(N,Q)\sum_{n=1}^{N}|a_n|^2$$

where q runs through all the rational integers not exceeding Q, * denotes summation over all primitive Dirichlet characters, and $\delta(N,Q)$ is a function of N and Q alone.

That such a function $\delta(N,Q)$ exists, which is in some sense 'not too large', was first proved by Linnik [13]. His result was successively refined by Roth [18], Bombieri [1], Davenport and Halberstam [5], and Davenport and Bombieri [3], [4]. We also mention the papers of Montgomery [15], and Wolke [22], the first of which in particular combines the large sieve with a method of Halász [10], and the second of which further refines the function $\delta(N,Q)$ under certain conditions. We state this last result presently. We mention in particular the inequalities

$$\delta(N,Q) \leqslant N + c_1Q^2, \qquad \text{Davenport and Bombieri [4],}$$

and

$$\delta(N,Q) \leqslant Q^2 + \pi N, \qquad \text{Gallagher [8],}$$

the final inequality having a very simple proof.

In their paper [5] Davenport and Halberstam prove that one can take $\delta(N,Q) = 2.2\max(N,Q^2)$ and ask whether there exists any